Dorian Abbot
APM147
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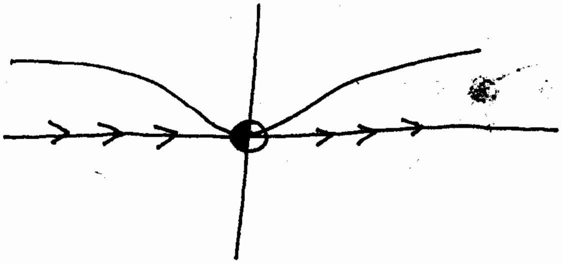
Problem Set 1 Solutions

A.

1. $\dot{x} = F(x) = 1 - e^{-x^2} \Rightarrow x^* = 0$ $\quad\quad$ $\frac{dF}{dx} = 2xe^{-x^2} \Rightarrow \frac{dF}{dx}|_{x^*} = 0$

Linearized stability fails.

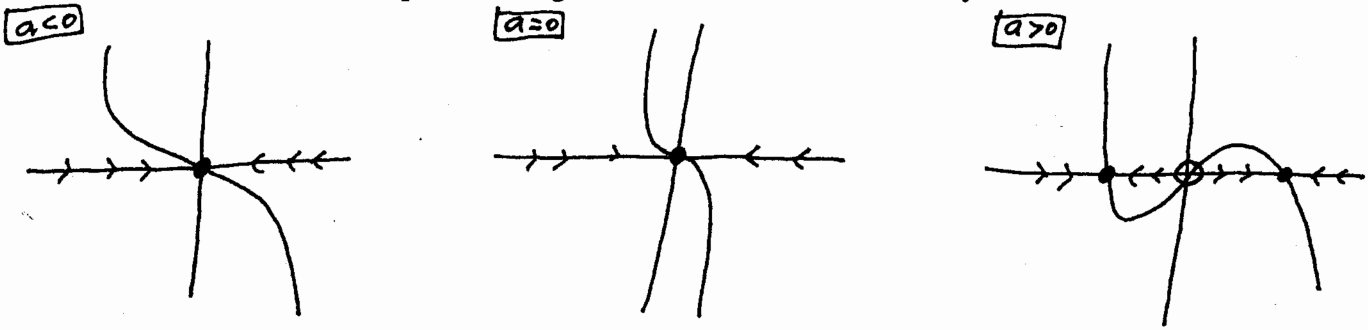


The graphical approach shows that the fixed point is half-stable.

2. $\dot{x} = F(x) = ax - x^3 \Rightarrow x^* = 0, \pm\sqrt{a}$ $\quad\quad$ $\frac{dF}{dx} = a - 3x^2$, so:

$$\begin{aligned} \frac{dF}{dx}|_{x^*=0} &= a \Rightarrow unstable \\ \frac{dF}{dx}|_{x^*=\sqrt{a}} &= -2a \Rightarrow stable \\ \frac{dF}{dx}|_{x^*=-\sqrt{a}} &= -2a \Rightarrow stable \end{aligned}$$

Notice that the $x^* = \pm\sqrt{a}$ roots only exist as distinct roots for a>0. A supercritical pitchfork bifurcation occurs as a passes through zero. Notice that linear stability fails at a=0.



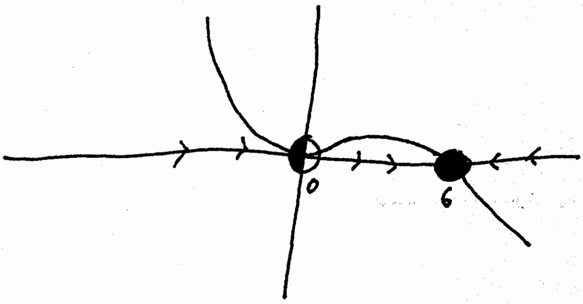
3. $\dot{x} = F(x) = x(1-x)(2-x) \Rightarrow x^* = 0, 1, 2$ $\quad\quad$ $\frac{dF}{dx} = 3x^2 - 6x + 2$, so:

$$\begin{aligned} \frac{dF}{dx}|_{x^*=0} &= 2 \Rightarrow unstable \\ \frac{dF}{dx}|_{x^*=1} &= -1 \Rightarrow stable \\ \frac{dF}{dx}|_{x^*=2} &= 2 \Rightarrow unstable \end{aligned}$$

4. $\dot{x} = F(x) = x^2(6-x) \Rightarrow x^* = 0, 6$ $\frac{dF}{dx} = 12x - 3x^2$, so:

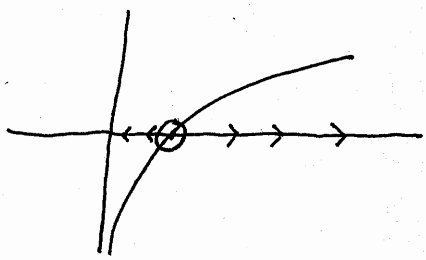
$$\frac{dF}{dx}|_{x^*=0} = 0 \Rightarrow ?$$
$$\frac{dF}{dx}|_{x^*=6} = -36 \Rightarrow stable$$

Using the graphical approach we find that the fixed point at x=0 is half-stable.



5. $\dot{x} = F(x) = ln(x) \Rightarrow x^* = 1$ $\frac{dF}{dx} = \frac{1}{x} \Rightarrow \frac{dF}{dx}|_{x^*} = 1$

The fixed point at x=1 is unstable.



B.

1. $x^* = rx^*(1-x^*) \Rightarrow x^*(rx^* + (1-r)) = 0 \Rightarrow x^* = 0, 1 - \frac{1}{r}$

$\frac{dF}{dx} = r(1-2x) \Rightarrow \frac{dF}{dx}|_{x^*=0} = r$ and $\frac{dF}{dx}|_{x^*=1-\frac{1}{r}} = 2 - r$

A fixed point is stable if $|\frac{dF}{dx}|_{x^*}| < 1$ and unstable if $|\frac{dF}{dx}|_{x^*}| > 1$. We'll only consider r>0 on physical grounds.

The fixed point $x^* = 0$ is stable for $0 \leq r < 1$ and unstable for $r > 1$. At r=1 we'll need to use graphical methods.

The fixed point $x^* = 1 - \frac{1}{r}$ is stable for $1 < r < 3$ and unstable for and $r > 3$. For $0 \leq r < 1$ this fixed point is unphysical. At r=1 both fixed points overlap. At r=3 we'll need to use graphical methods.

It's hard to see what's happening at r=3 with a cobweb diagram, so we'll use a timeseries. Fig. 1 shows that at r=3 the fixed point at $x^* = 1 - \frac{1}{r}$ is very slightly stable. We would have to expand to $O(\eta^2)$ to see this analytically. The fixed point at $x^* = 0$ is clearly stable for r=1.

2. Fig. 2 shows the timeseries plots for the different values of r. At r=0.4 the system approaches the fixed point at x=0. At r=2 the system monotonically approaches the fixed point at $x^* = 1 - \frac{1}{r}$.

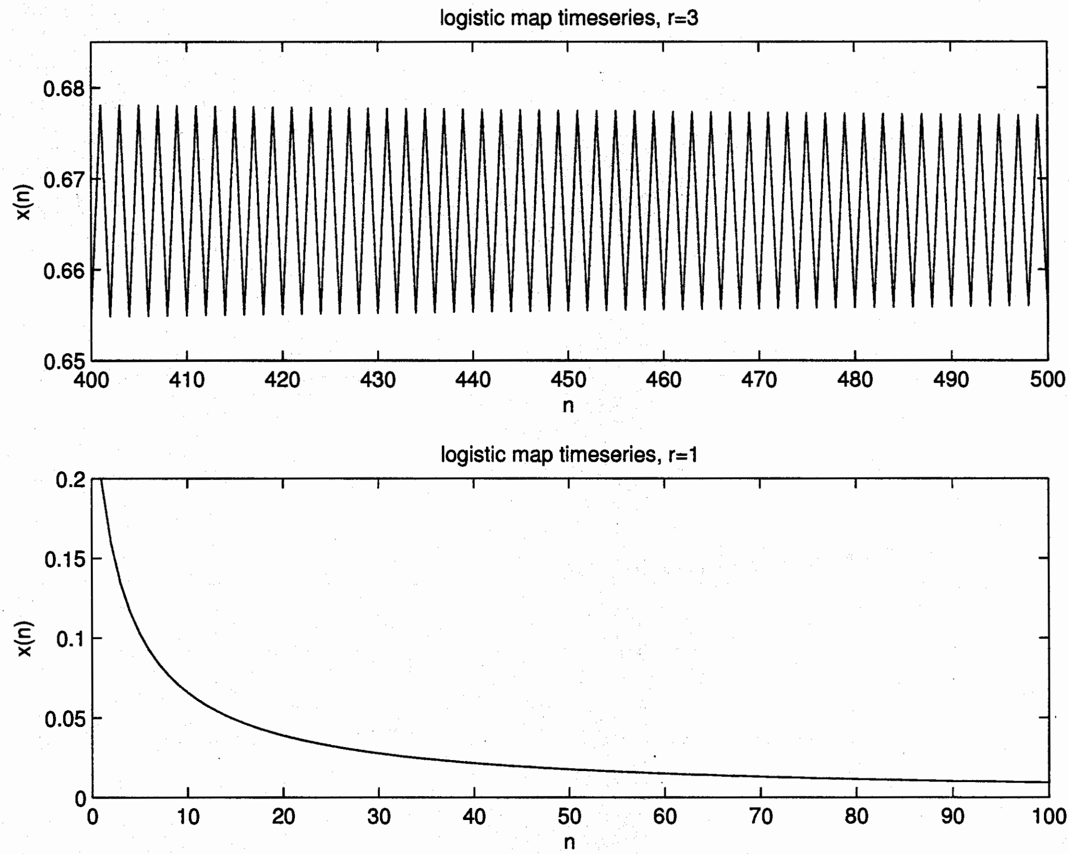
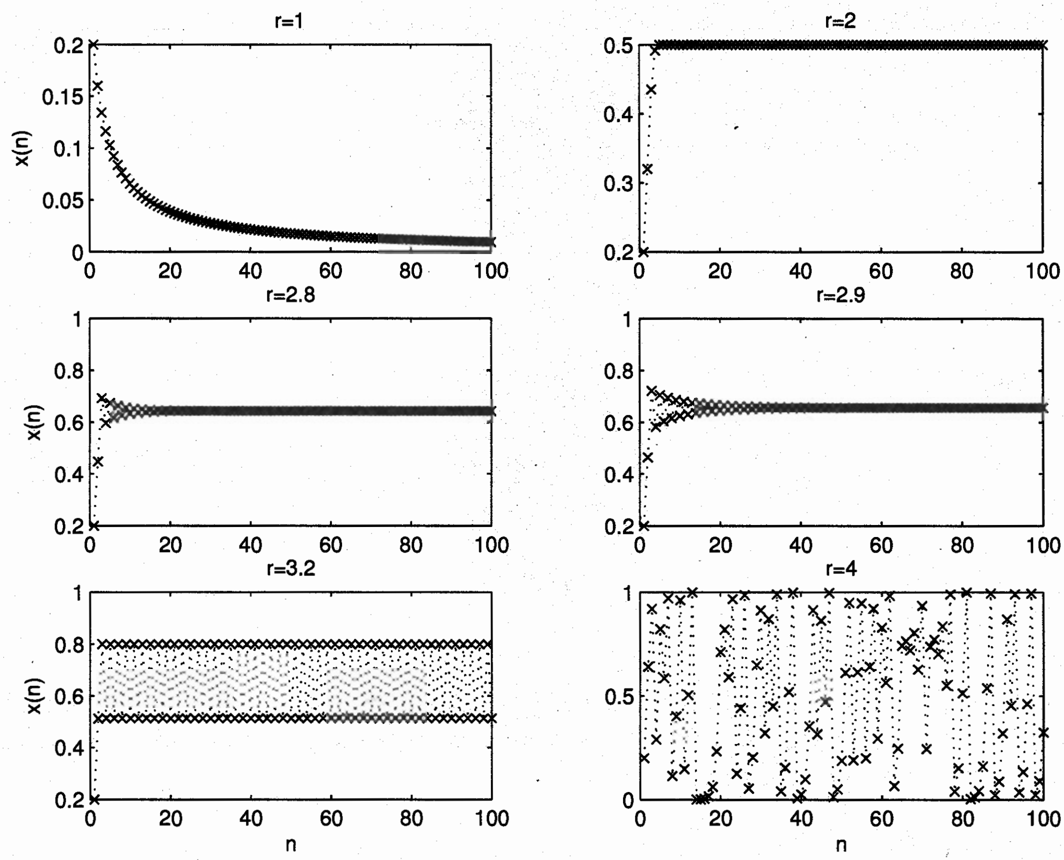


Figure 1:

At r=2.8 and r=2.9 this fixed point is approached in an oscillatory manner. At r=3 a flip bifurcation occurs and for r=3.2 we have a 2-cycle. The system undergoes the period-doubling route to chaos and at r=4 there is chaotic behavior.

C. For large n we expect the system to be very near the fixed point at x=0. This means $x_n \ll 1$ so we have: $x_{n+1} = rx_n(1-x_n) \approx rx_n$. The solution to this is $x_n = (const.)r^n$. The constant is determined by when the system enters this regime.



r=1
0.2
0.15
0.1
0.05
0
x(n)
0
20
40
60
80
100
r=2
0.5
0.4
0.3
0.2
r=2.8
1
0.8
0.6
0.4
r=2.9
r=3.2
r=4
n

Figure 2: