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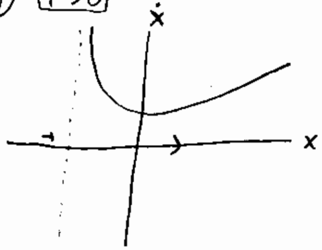
APM147
10/7/04

Problem Set 2 Solutions

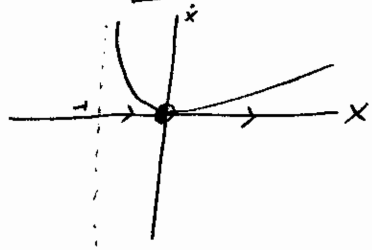
1. The map will be contracting for $|\mathbf{J}| < 1$. $|\mathbf{J}| = \left|\frac{\partial x_{n+1}}{\partial x_n}\right| = |r(1-2x_n)| = |r|\,|(1-2x_n)|$. We'd like the map to be contracting for all $x \in [0,1]$, so we need $|r| < 1$ for $|\mathbf{J}| < 1$ at x=0 or 1. Negative r makes no sense physically, so the map is dissipative for $0 \le r < 1$.

(2) $\boxed{\dot{x} = r + x - \ln(1+x)}$ Saddle-node bifurcation

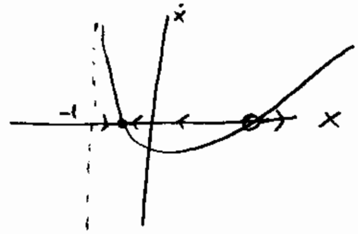
(i) $r>0$



$r=0$

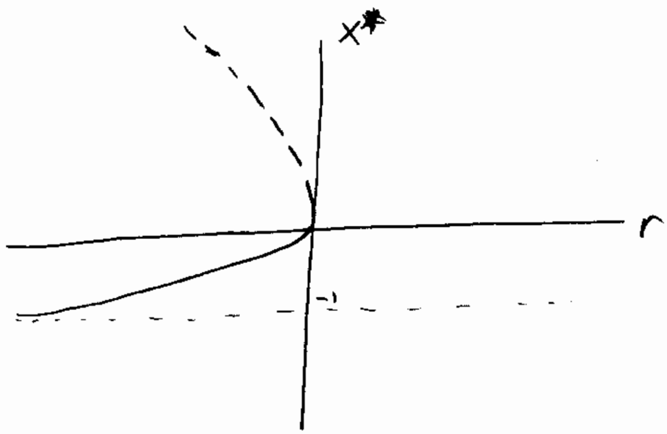


$r<0$



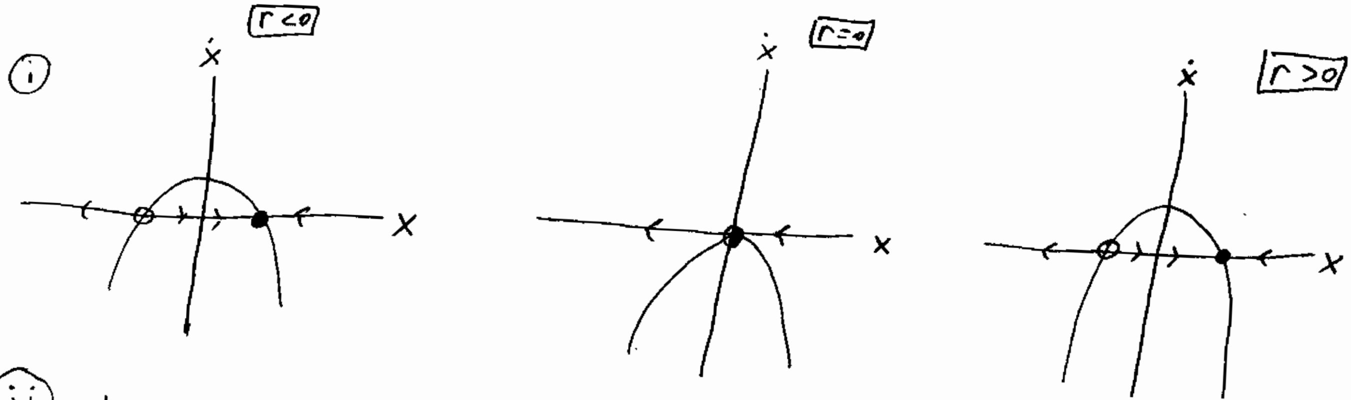
$x \approx 0$ near bifurcation $\rightarrow$ $\dot{x} \approx r + x - (x - \frac{1}{2}x^2) = r + \frac{1}{2}x^2$

looks like a saddle-node!



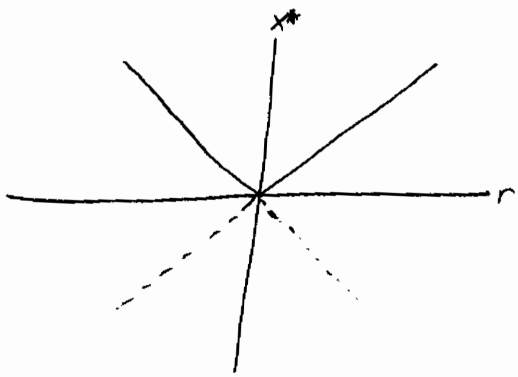
$\boxed{\dot{x} = r^2 - x^2}$ Transcritical bifurcation

(i)



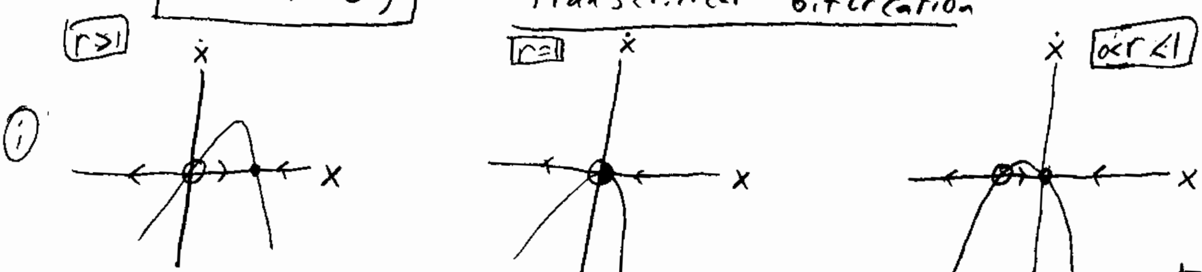
(ii) let $y = x + r \rightarrow \dot{x} = \dot{y}, \quad x^2 = y^2 - 2yr + r^2$

$\dot{y} = (2r)y - y^2 \leftarrow$ normal form for transcritical

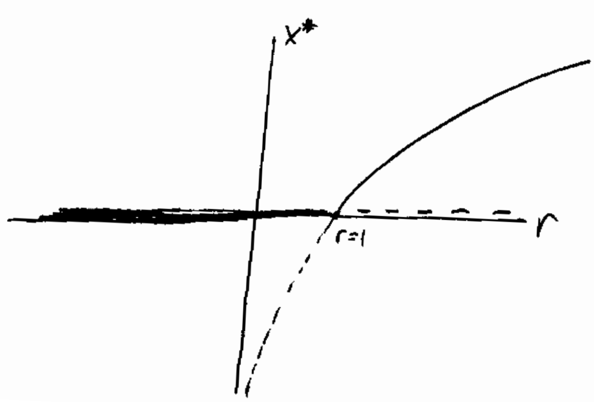


$\boxed{\dot{x} = x(r - e^x)}$ transcritical bifurcation

(i)

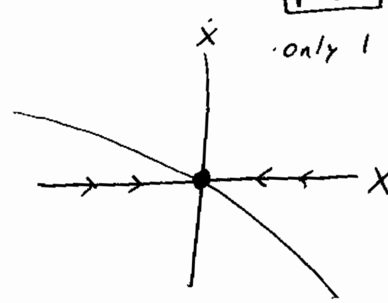


(ii) $\dot{x} \approx x(r - (1+x)) \approx (r-1)x - x^2$



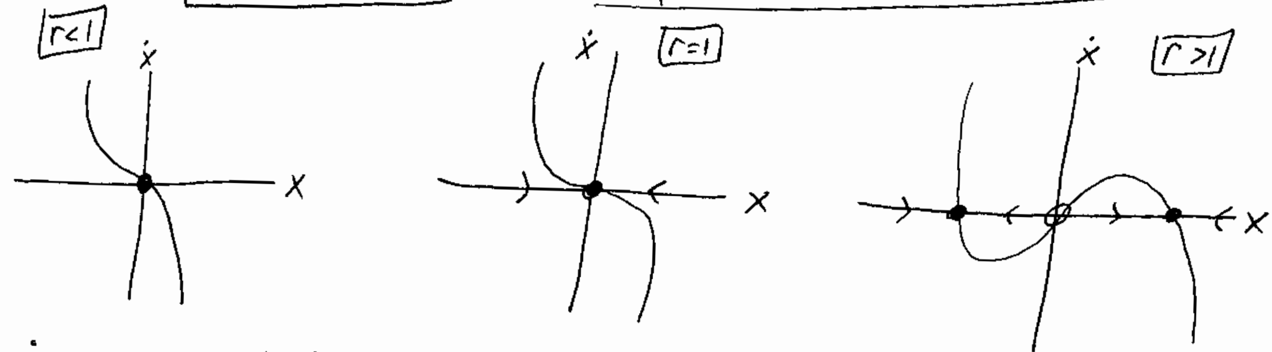
$\boxed{r<0}$

only 1 f.p.

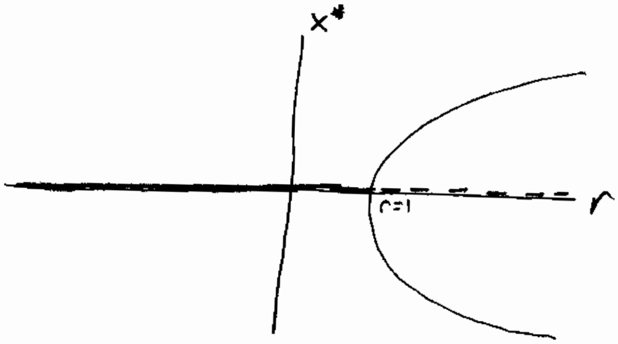


$\boxed{\dot{x} = rx - \sinh x}$ Supercritical Pitchfork Bifurcation

(i)

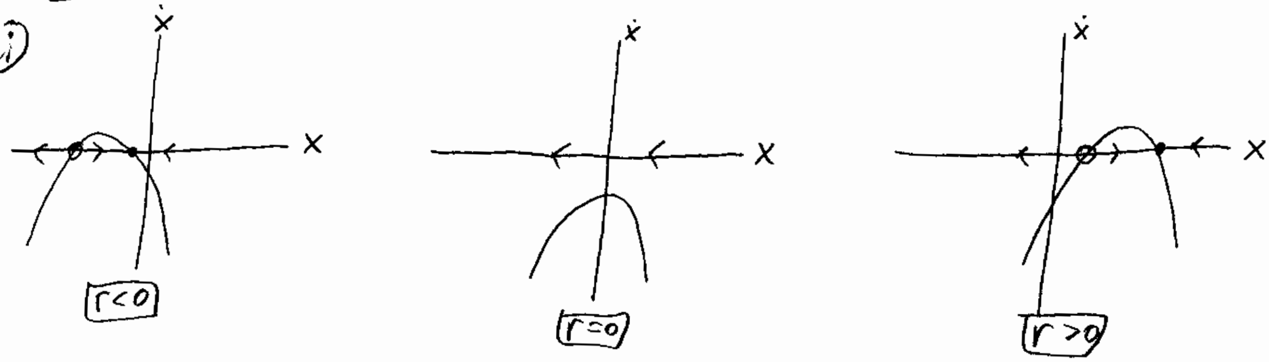


(ii) $\dot{x} \approx rx - (x + \frac{1}{6}x^3) = (r-1)x - \frac{1}{6}x^3$

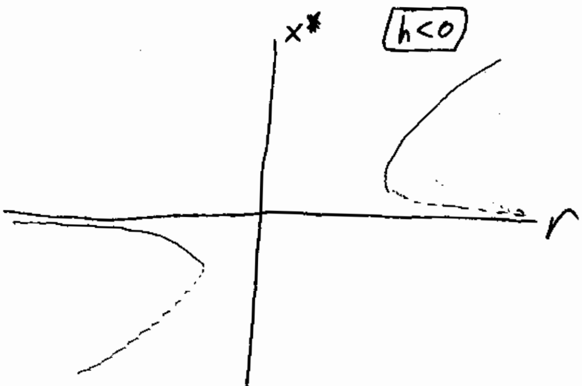


(3) $\boxed{h<0}$

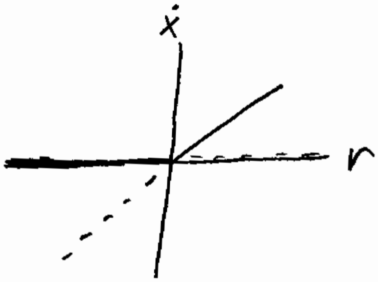
i)



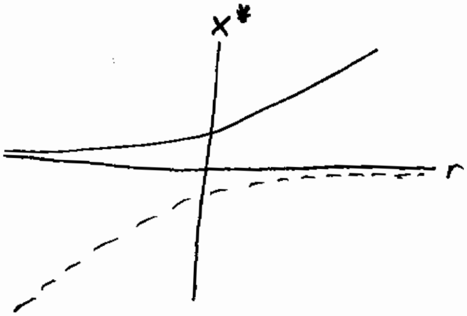
- So we have two saddle-node bifurcations.
- $x^* = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} + h}$
- So the saddle-nodes occur @ $h = -\frac{r^2}{4}$



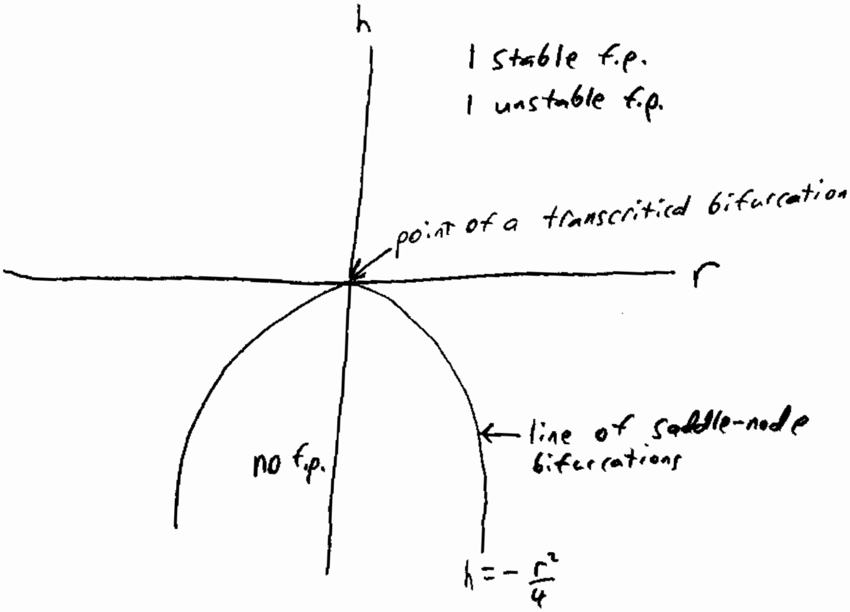
$\boxed{h=0}$



$\boxed{h>0}$ $x^* = \frac{r}{2} \pm \sqrt{\left(\frac{r}{2}\right)^2 + h}$ ← no bifurcation will occur, one stable and one unstable f.p. for all r



(ii)



(iii) $\dot{x} = h + r + x^2 = \alpha + x^2$ w/ $\alpha \equiv h + r$

- so a perturbation to the saddle-node bifurcation does nothing but shift the bifurcation point. The normal form is unchanged by such a perturbation.

4. Substituting our ansatz into the discretized equation we get:

$$(B^{(n+1)\Delta t})e^{i\mu m\Delta x} - B^{(n-1)\Delta t})e^{i\mu m\Delta x})/\Delta t = -\frac{c}{\Delta x}(B^{n\Delta t})e^{i\mu(m+1)\Delta x} - B^{n\Delta t})e^{i\mu(m-1)\Delta x}) \tag{1}$$

Which can be simplified to:

$$B^{\Delta t} - B^{-\Delta t} = -2i\sigma \tag{2}$$

With $\sigma = \frac{c\Delta t}{\Delta x} sin(\mu\Delta x)$. This has the solution:

$$B^{\Delta t} = -i\sigma \pm \sqrt{1-\sigma^2} \tag{3}$$

Since $F \propto (B^{\Delta t})^n$, the scheme will be unstable if $\mid B^{\Delta t} \mid > 1$ for one of the two roots. We have two cases to consider.

First, $\mid \sigma \mid > 1$, so $B^{\Delta t}$ is pure imaginary. In this case:

$$\mid B^{\Delta t} \mid = \mid -\sigma \pm \sqrt{\sigma^2 - 1} \mid \tag{4}$$

Clearly we will have one unstable root and the scheme will be unstable.

Second, consider $\mid \sigma \mid \leq 1$, so $B^{\Delta t}$ is complex. In this case:

$$\mid B^{\Delta t} \mid = \sigma^2 + 1 - \sigma^2 = 1 \tag{5}$$

And both roots are stable.

We find that if we wish to use the leapfrog scheme without a Robert filter we must choose Δx and Δt carefully so that $\mid \sigma \mid \leq 1$ to avoid exponential divergence of the numerical solution. This behavior has nothing to do with the original equation (anything involving Δx and Δt cannot).