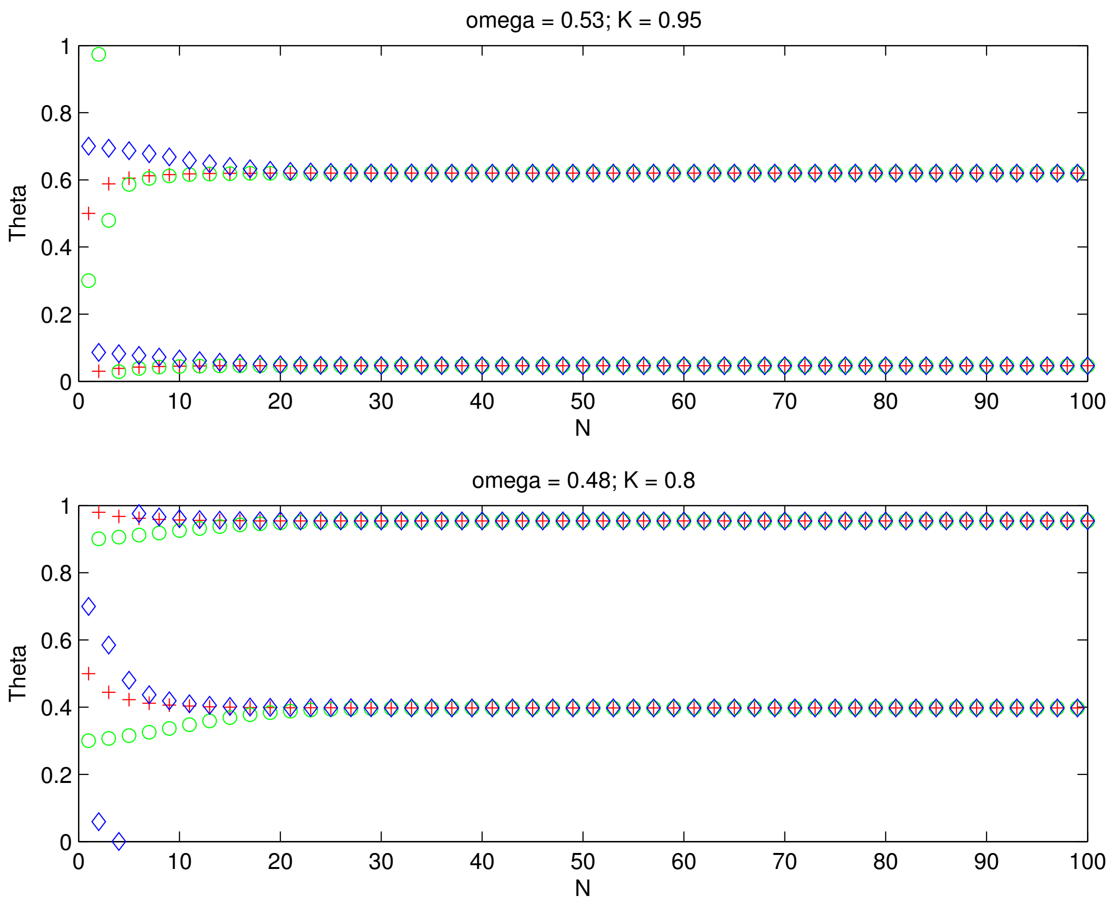
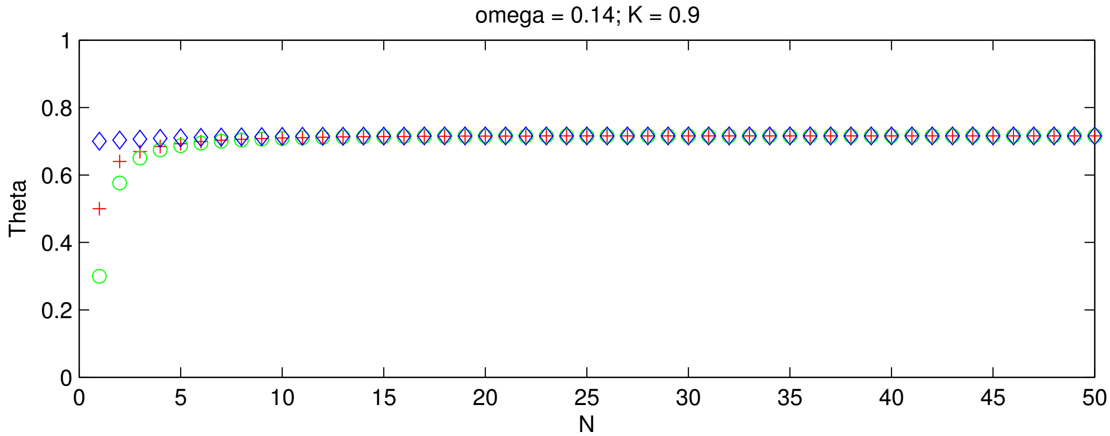
Dorian Abbot
APM147
10/21/04

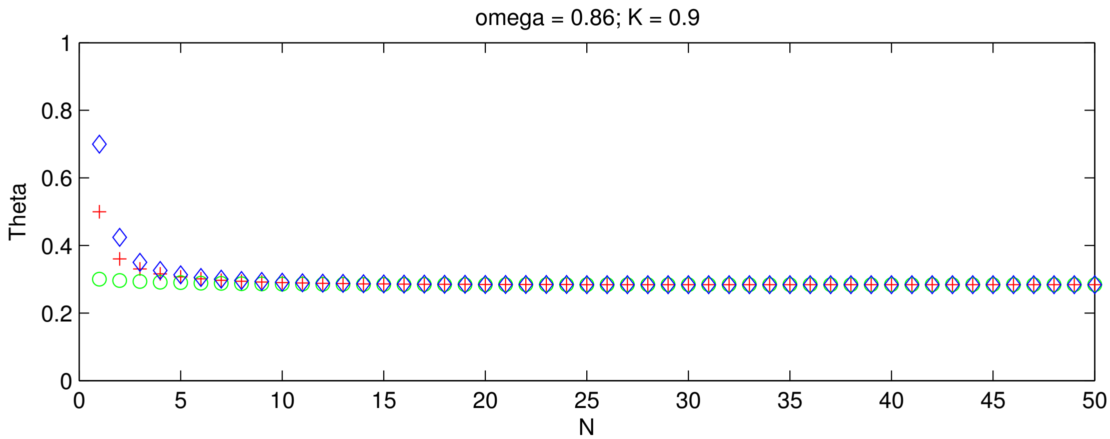
Problem Set 4 Solutions

1) a) Here the circle map is integrated at two different K, Ω combinations that both produce p/q=1/2 for three initial conditions each.

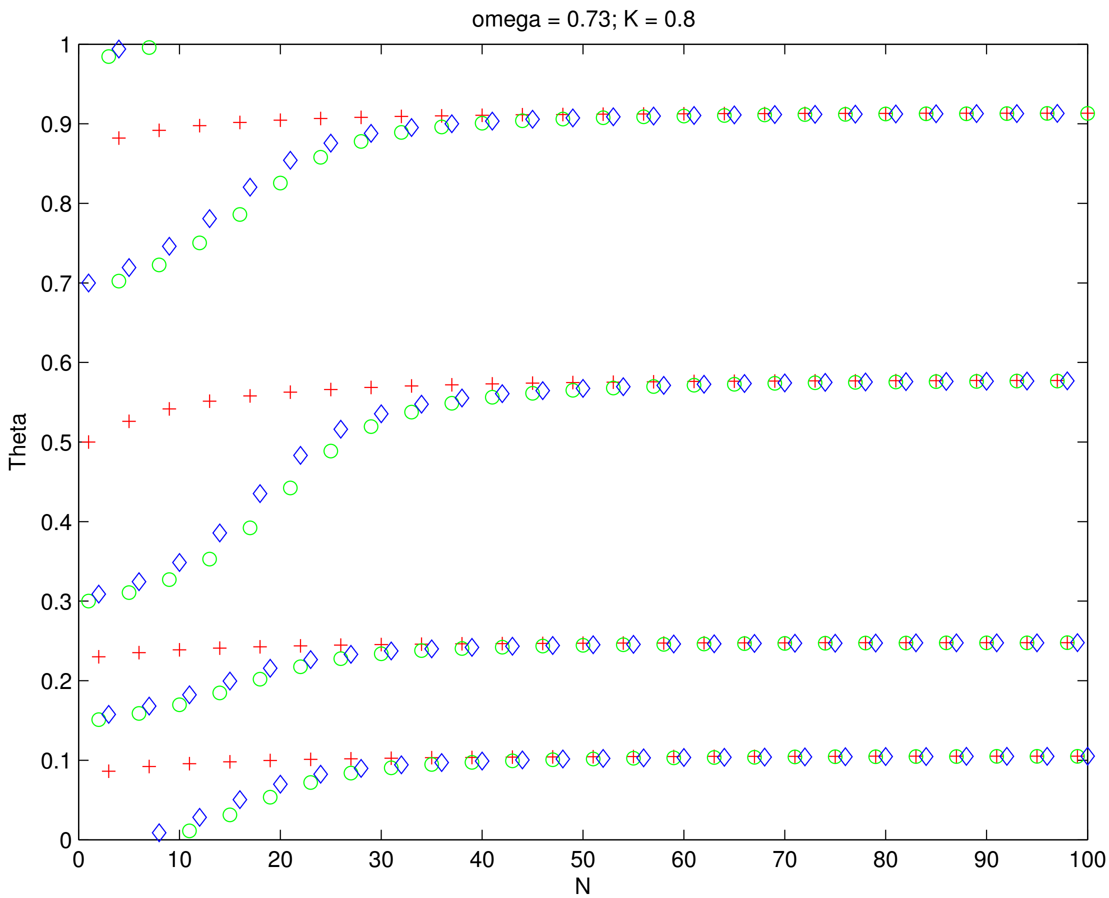


b) The p/q=0/1 plot is above and the p/q=1/1 plot is below. They look similar - each settles into a period one pattern after some initial behavior.

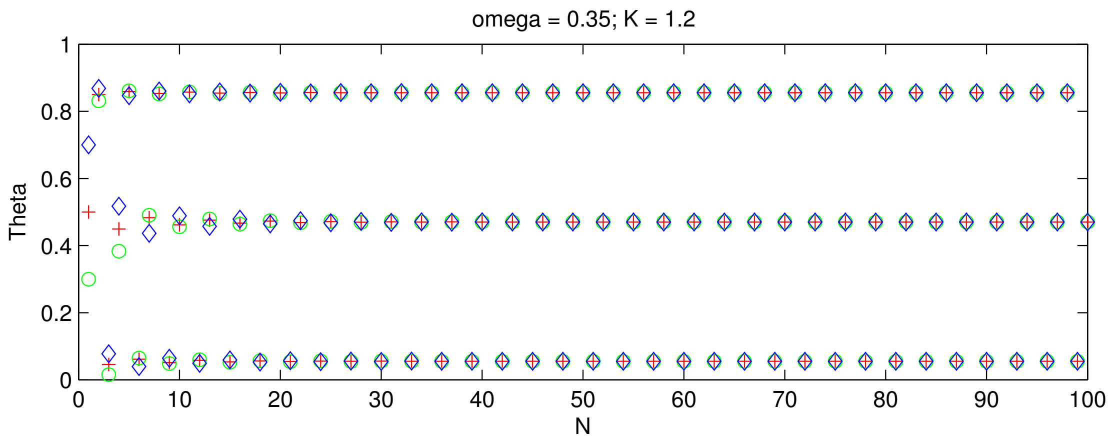


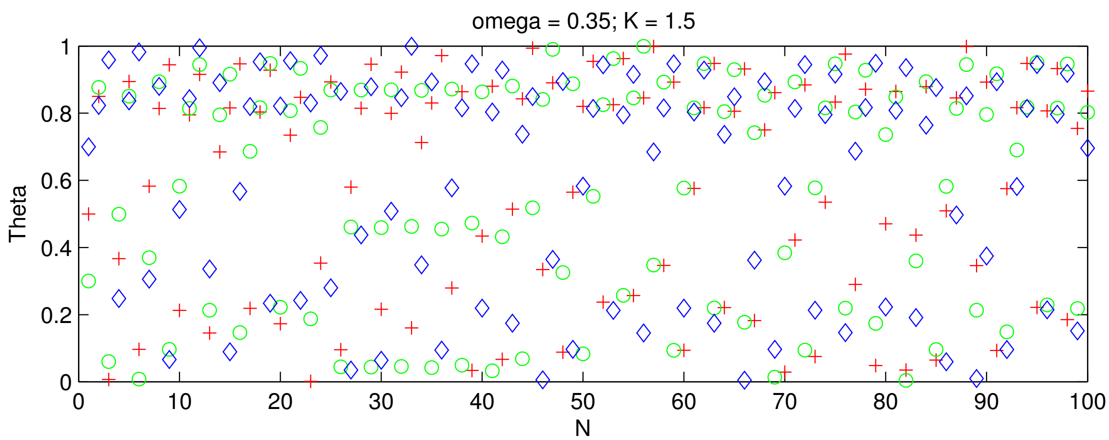


c) A p/q=3/4 solution.

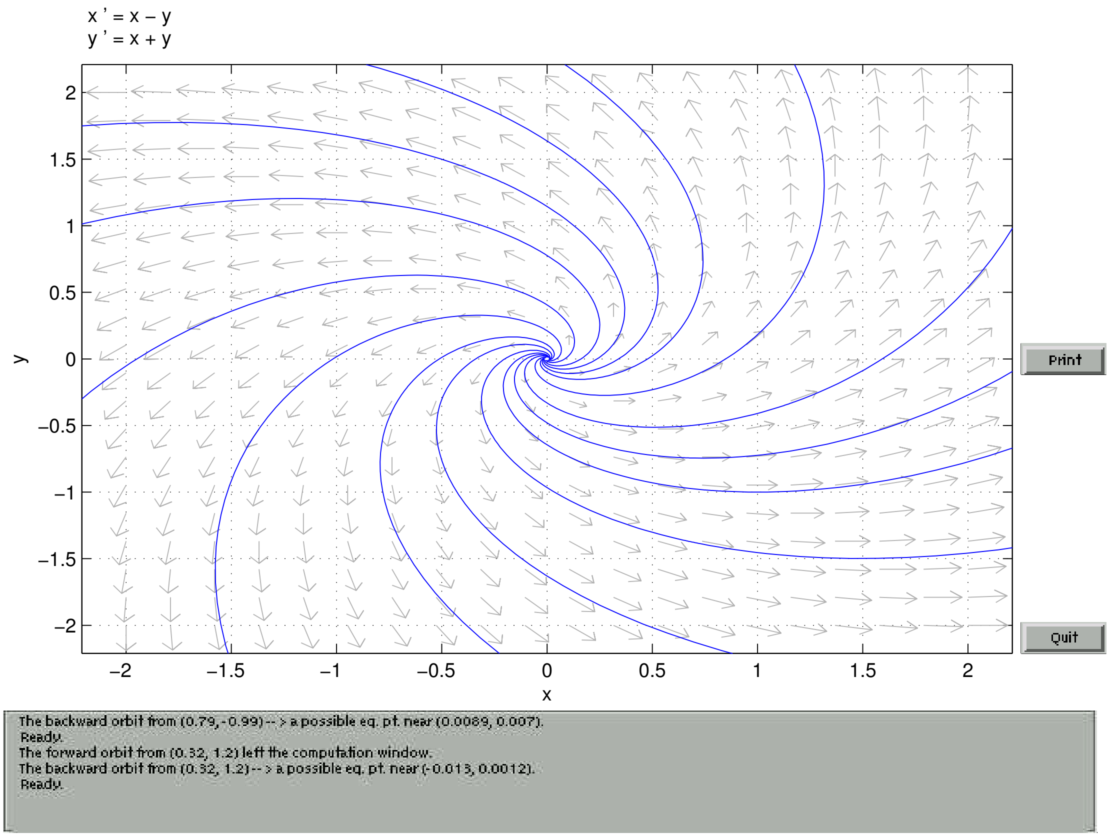


d) For K>1 chaotic and nonchaotic regions are densely interwoven in the K-Ω plane.

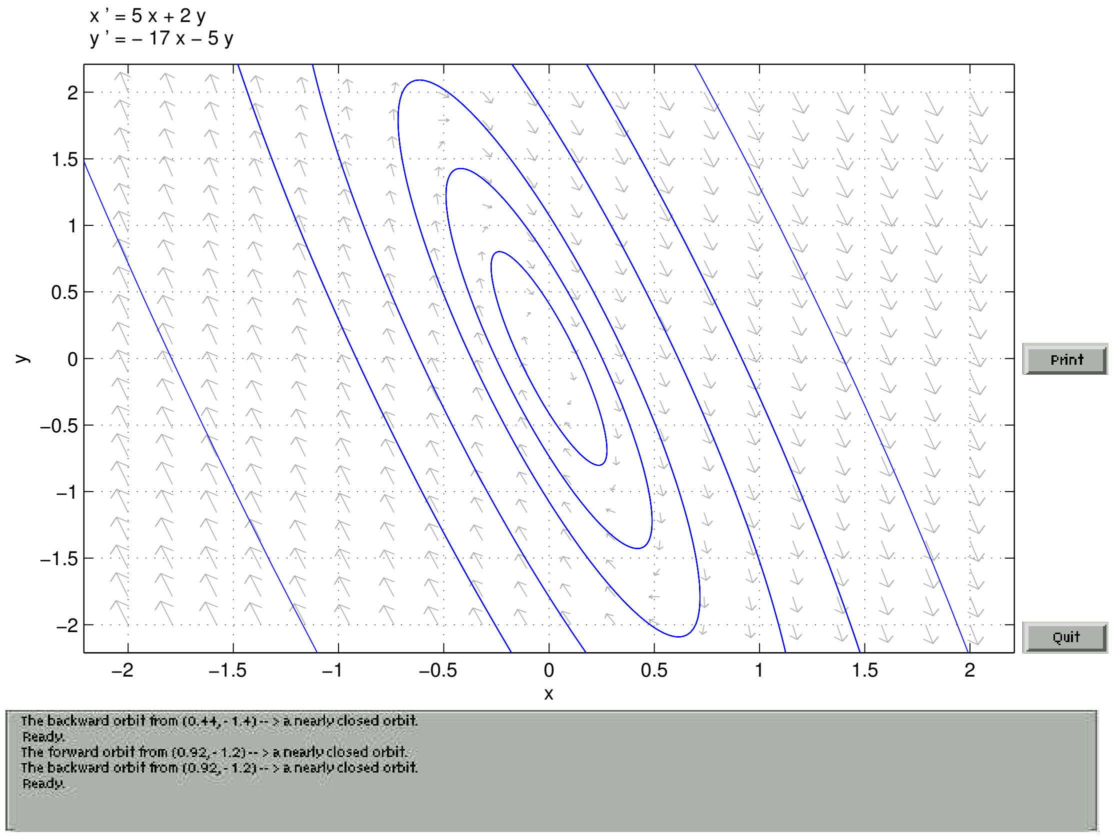




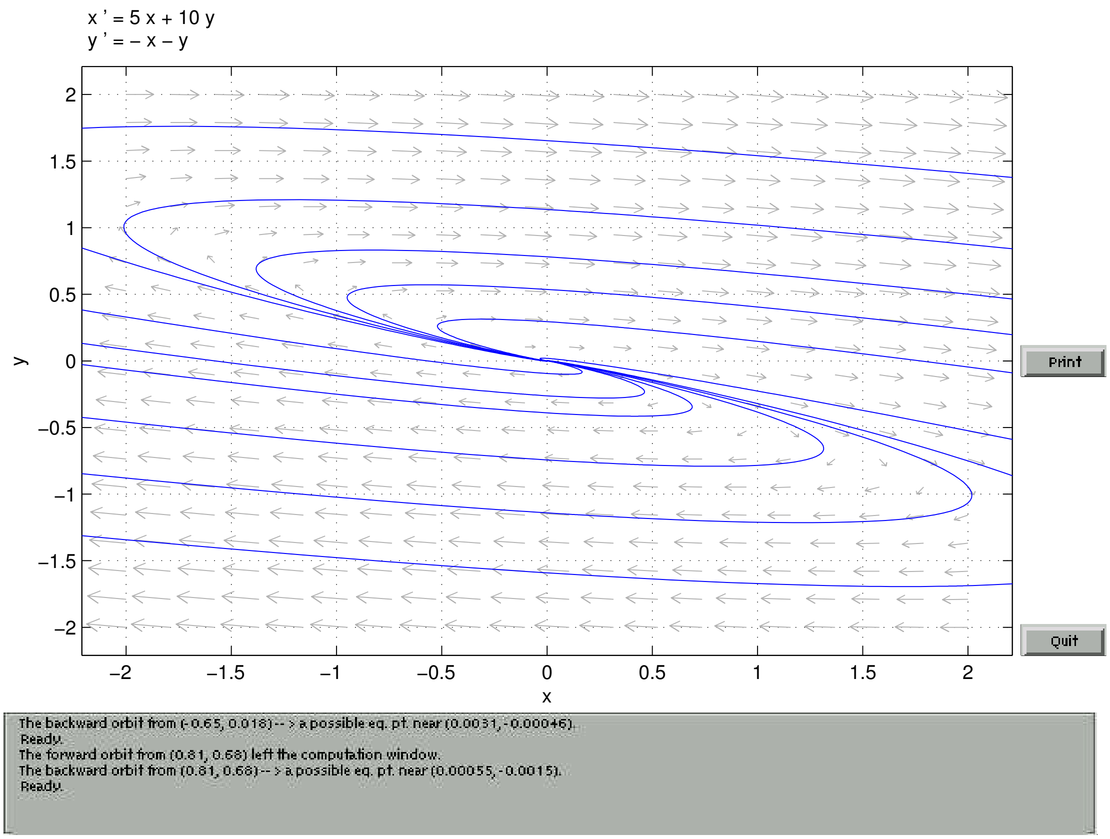
2) a) The f.p. is at (x,y)=(0,0). A=[1 -1; 1 1], λ_1=1+i, v_1=[1 -i], λ_2=1-i, v_2=[1 i]. An unstable spiral.



b) The f.p. is at (x,y)=(0,0). A=[5 2; -17 -5], λ_1=3i, v_1=[2 -5+3i], λ_2=-3i, v_2=[2 -5-3i]. A center.



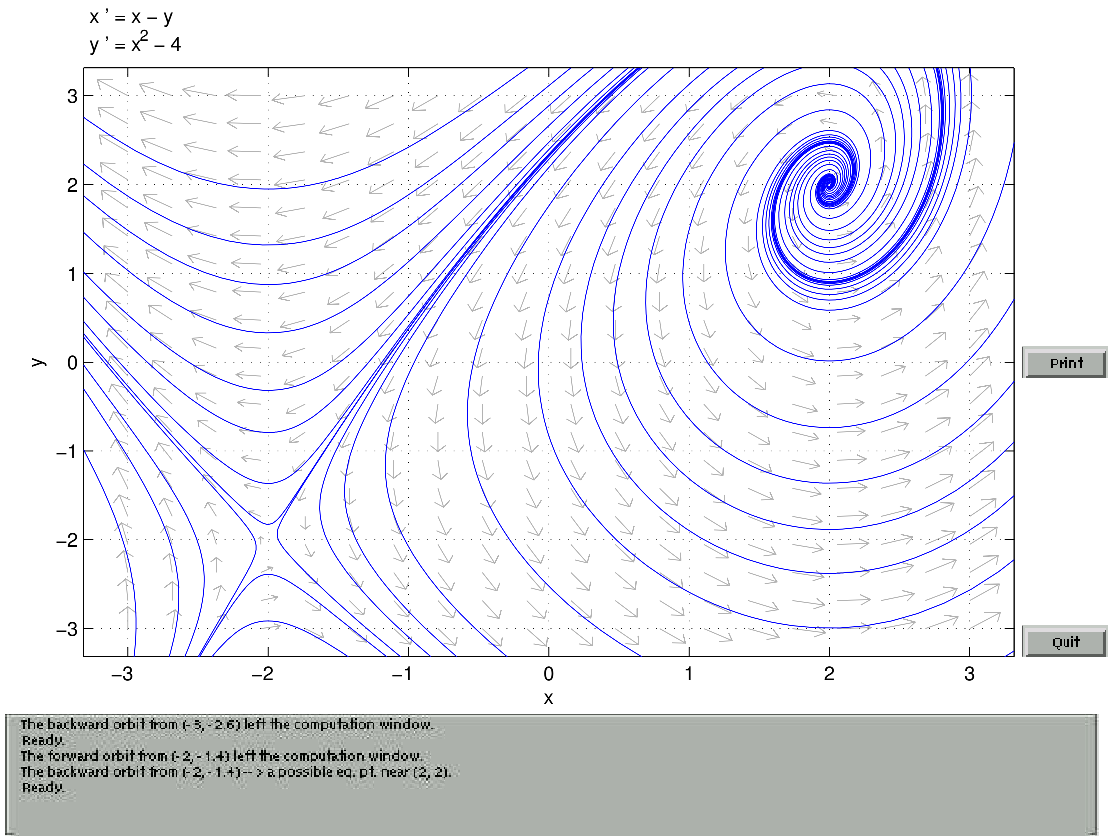
c) The f.p. is at (x,y)=(0,0). A=[5 10; -1 -1], λ_1=2+i, v_1=[10 -3+i], λ_2=2-i, v_2=[10 -3-i]. An unstable spiral.



3) a) The f.p. are at (x,y)=(2,2) and (-2,-2). The Jacobian is J=[1 -1; 2x 0].

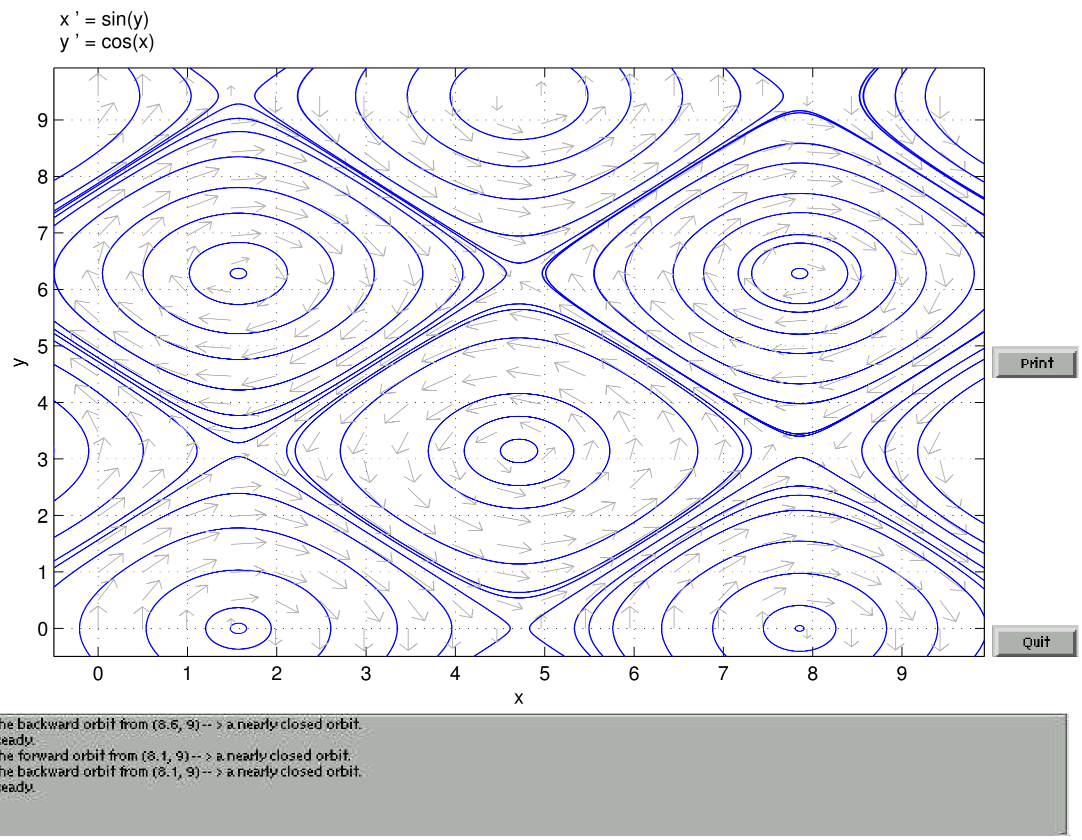
$A_1 = J|_{(-2,-2)} = [1\ -1; -4\ 0] \Rightarrow\ \tau = 1,\ \Delta = -4\ \Rightarrow\ \lambda_1 = \frac{1}{2} + \sqrt{\frac{17}{2}},\ v_1 = [2; 1 - \sqrt{17}],\ \lambda_2 = \frac{1}{2} - \sqrt{\frac{17}{2}},\ v_2 = [2; 1 + \sqrt{17}]$. A saddle point.

$A_2 = J|_{(2,2)} = [1\ -1; 4\ 0] \Rightarrow\ \tau = 1,\ \Delta = 4\ \Rightarrow\ \lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{15}i}{2}$. An unstable spiral.



b) The f.p. are at (x,y)=((n+$\frac{1}{2}$)π,mπ) for integer n and m. The Jacobian is J=[0 cos(y); -sin(y) 0]. $A = J|_{((n+\frac{1}{2})\pi, m\pi)} = [0\ (-1)^m; (-1)^{n+1}\ 0] \Rightarrow \tau = 0,\ \Delta = (-1)^{n+m} \Rightarrow \lambda_{1,2} = \pm(-1)^{\frac{n+m+1}{2}}$

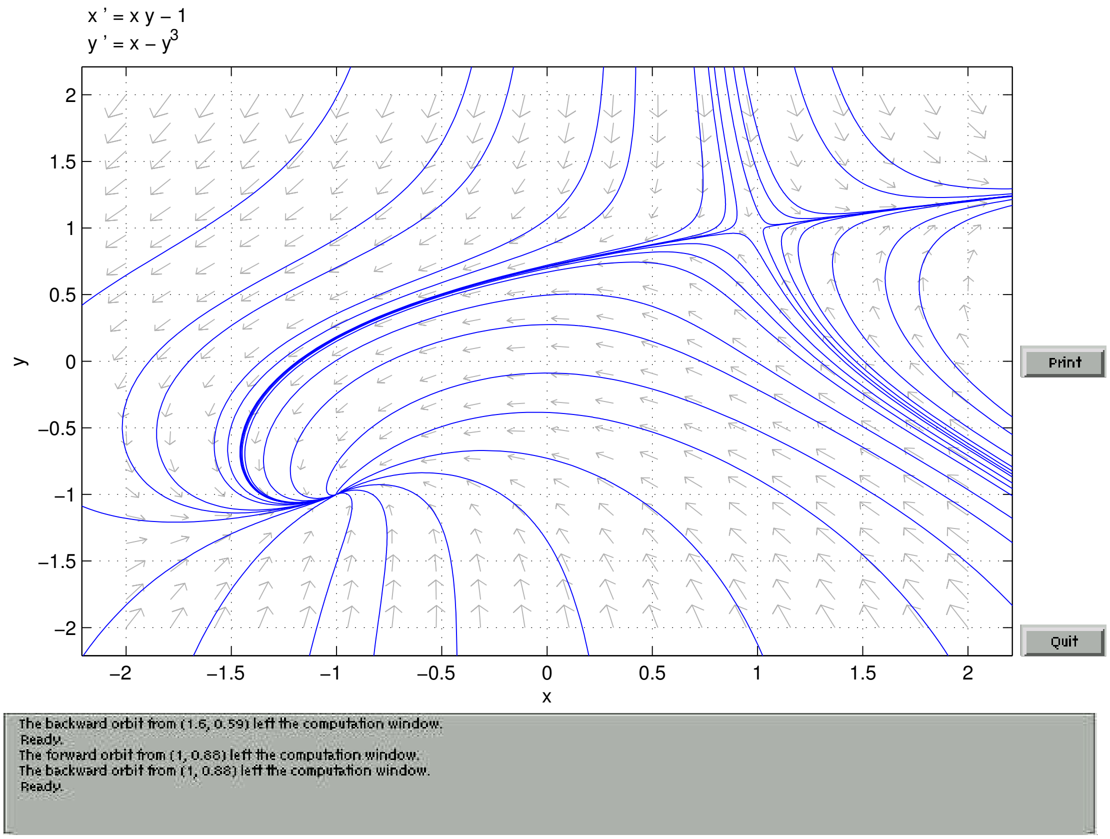
For n+m odd, $\lambda_{1,2} = \pm 1$ and we have saddles. For n+m even, $\lambda_{1,2} = \pm i$ and we have centers. We can have centers in this system, even though it is nonlinear, because it is reversible.



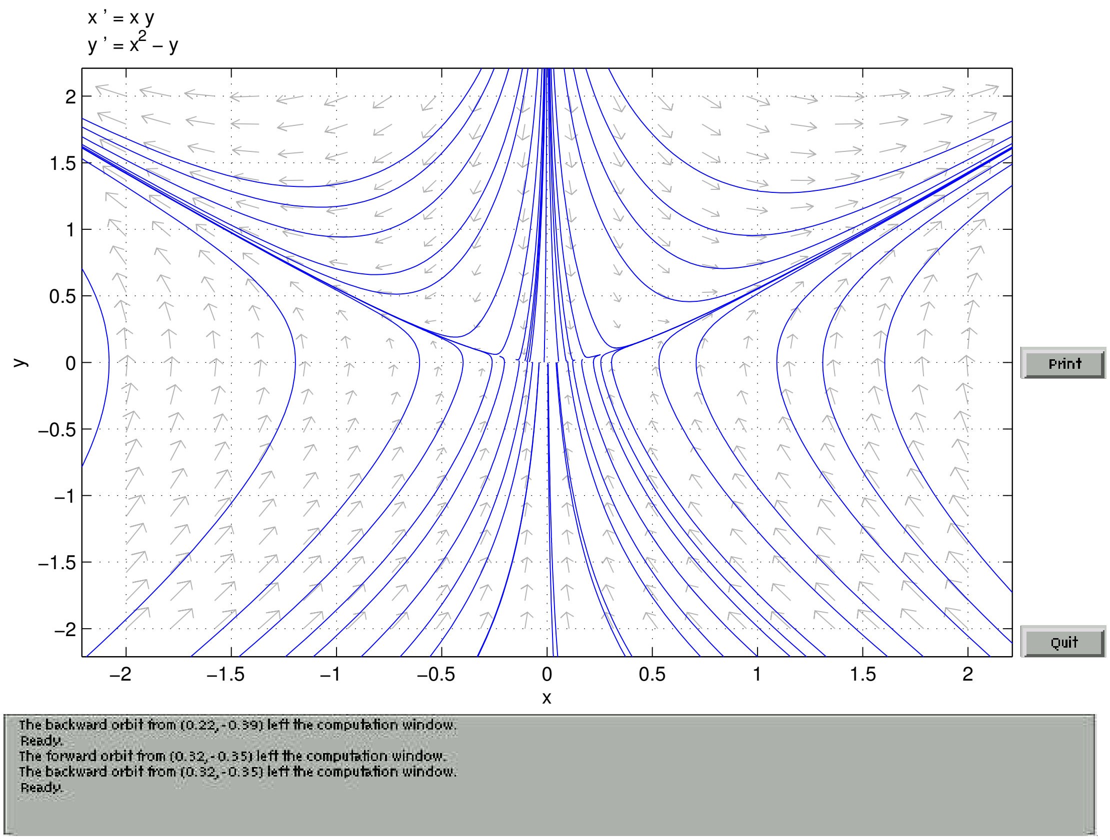
c) The f.p. are at (x,y)=(-1,-1) and (1,1). The Jacobian is J=[y x; 1 -3y^2].

$A_1 = J|_{(-1,-1)} = [-1\ -1; 1\ -3] \Rightarrow\ \tau = -4,\ \Delta = 4\ \Rightarrow\ \lambda_{1,2} = -2.$ In a linear system this could be a star or degenerate node. Since the system is nonlinear, it appears to be a stable node.

$A_2 = J|_{(1,1)} = [1\ 1; 1\ -3] \Rightarrow\ \tau = -2,\ \Delta = -4\ \Rightarrow\ \lambda_{1,2} = -1 \pm \sqrt{5}.$ A saddle point.



d) The f.p. is at (x,y)=(0,0). The Jacobian is J=[y x; 2x -1]. $A = J|_{(0,0)} = [0\ 0; 0\ -1] \Rightarrow \tau = -1,\ \Delta = 0 \Rightarrow \lambda_{1,2} = 0, -1$. In a linear system this would be a non-isolated f.p. The nonlinear plot looks more like a saddle. Although close to the origin it does look similar to a non-isolated f.p., it in fact isn't. This can be seen by noting that the x and y nullclines intersect only at one point.



4) Let's call $\bar{F}$ the circle map with the modulo omitted, that is:

$$\bar{F}(\theta) = \theta + \Omega - \frac{K}{2\pi} sin(2\pi\theta) \tag{1}$$

Assume that for integers p and q, $\bar{F}^q(\theta) = p + \theta$. The winding number is defined as:

$$w = lim_{n\to\infty} \frac{\bar{F}^n(\theta_0) - \theta_0}{n} \tag{2}$$

Let n=jq+l, for integers j and l:

$$w = lim_{j\to\infty} \frac{\bar{F}^{jq+l}(\theta_0) - \theta_0}{jq+l} \tag{3}$$

Use $\bar{F}^{jq+l}(\theta_0) = \bar{F}^l(jp+\theta_0) \approx jp$ in the limit $j \to \infty$ (since $\bar{F}^l$ can only add a maximum of l). With this it is clear that:

$$w = lim_{j\to\infty} \frac{\bar{F}^{jq+l}(\theta_0) - \theta_0}{jq+l} = \frac{p}{q} \tag{4}$$

The important point here is that we should interpret q as the number of times the function is evaluated before it returns to θ_0 and p as the number of circuits it does as it returns.

For p=0, q=1, we have: $\bar{F}(\theta) = \theta \Rightarrow \theta + \Omega - \frac{K}{2\pi} sin(2\pi\theta) = \theta$. So we have: $\Omega(K) = \frac{K}{2\pi} sin(2\pi\theta)$. We can only satisfy this relation for any θ for $|\Omega(K)| \leq \frac{K}{2\pi}$ and since $\Omega \in [0,1]$ the limit of the tongue is given by $\Omega(K) = \frac{K}{2\pi}$.

For p=1, q=1, we have: $\bar{F}(\theta) = 1 + \theta \Rightarrow \theta + \Omega - \frac{K}{2\pi} sin(2\pi\theta) = 1 + \theta$. So we have: $\Omega(K) = 1 + \frac{K}{2\pi} sin(2\pi\theta)$. We can only satisfy this relation for any θ for $\Omega(K) \geq 1 - \frac{K}{2\pi}$ (and again remember $\Omega \in [0,1]$), so the limit of the tongue is given by $\Omega(K) = 1 - \frac{K}{2\pi}$.

5) For p=1, q=2, we have: $\bar{F}^2(\theta) = 1 + \theta$ or:

$$[\theta + \Omega - \frac{K}{2\pi} sin(2\pi\theta)] + \Omega - \frac{K}{2\pi} sin(2\pi[\theta + \Omega - \frac{K}{2\pi} sin(2\pi\theta)]) = 1 + \theta \tag{5}$$

We look near K=0 and take $\Omega = \frac{1}{2} + \varepsilon$ for small ε, so:

$$\begin{aligned} 2\varepsilon - \frac{K}{2\pi} sin(2\pi\theta) - \frac{K}{2\pi} sin[2\pi(\theta + \frac{1}{2}) + 2\pi\varepsilon - K sin(2\pi\theta)] &= 0 \\ 2\varepsilon - \frac{K}{2\pi} sin(2\pi\theta) + \frac{K}{2\pi} sin[2\pi\theta + 2\pi\varepsilon - K sin(2\pi\theta)] &= 0 \\ 2\varepsilon - \frac{K}{2\pi} sin(2\pi\theta) + \frac{K}{2\pi} sin(2\pi\theta) + \frac{K}{2\pi} cos(2\pi\theta)(2\pi\varepsilon - K sin(2\pi\theta)) &\approx 0 \end{aligned} \tag{6}$$

A Taylor expansion for small K,ε has been used. It is clear from the above equation that $\varepsilon = O(K^2)$, so neglecting ε with respect to K:

$$\begin{aligned} \varepsilon &\approx \frac{K^2}{4\pi} cos(2\pi\theta) sin(2\pi\theta) \\ &= \frac{K^2}{8\pi} sin(4\pi\theta) \end{aligned} \tag{7}$$

(8)

So the limits of the tongue are $\varepsilon = \pm\frac{K^2}{8\pi}$ or $\Omega = \frac{1}{2} \pm \frac{K^2}{8\pi}$.