Homework #6

Nonlinear dynamics and chaos

1. Solve the Van der pol equation numerically, using the Matlab program from the course home page `van_der_pol.m`. Starting from this program,

   (a) Plot the solution at the weakly nonlinear regime and the strongly nonlinear regime and discuss how they are compared with the theoretical derivations shown in class for both regimes.

   (b) In the strongly nonlinear regime, try different values of μ and show how the time scale of the fast and slow phases of the oscillation depend on μ.

   (c) Plot x vs $\dot{x}$ in the strongly nonlinear regime; super impose on that a plot of the vector field in the phase space of $(x, \dot{x})$ (use the Matlab quiver command); analyze the oscillation following the analysis done in class.

2. (Strogatz p 235) Weakly nonlinear oscillations: For each of the following systems $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$, with $0 < \varepsilon \ll 1$, calculate the averaged equations, and analyze the long term behavior of the system. Find the amplitude and frequency of any limit cycle for the original system. If possible solve the averaged equations explicitly for $x(t, \varepsilon)$, given the initial conditions $x(0) = a$, $\dot{x}(0) = 0$

$$h(x,\dot{x}) = x \qquad (1)$$

$$h(x,\dot{x}) = x\dot{x} \qquad (2)$$

$$h(x,\dot{x}) = (x^2 - 1)\dot{x}^3 \qquad (3)$$

3. (Strogatz 6.5.14) A glider: Let $v =$ speed of glider and $u =$ the angle which the flight path makes with the horizontal direction. The dimensionless equations of motion are:

$$dv/dt = -\sin u - Dv^2 \qquad (4)$$

$$v du/dt = -\cos u + v^2 \qquad (5)$$

   where the trigonometric terms represent the effects of gravity and the v^2 terms represent the effects of drag and lift, correspondingly.

   (a) If friction vanishes ($D = 0$), show that there is a conserved quantity $v^3 - 3v\cos\theta$ and use it to solve for $v(u)$ obtain an exact expression for the trajectories in the (u, v) phase space.

   (b) (for $D = 0$) Using your result above, obtain an exact expression for the separatrix in this system.

(c) (for $D = 0$) What does the flight path of the glider look like (in both real space and phase space of u, v) for motions inside the separatrix versus motions outside the separatrix? Sketch the glider's flight path in both cases.

(d) Analyze and describe what happens if the drag force does not vanish

(e) Using numerical integration, try to sketch the trajectories in the vertical plane with and without friction, and inside and outside of the separatrix (can start from the Matlab program `glider.m` from the course home page).

4. **An optional challenge question:** Consider the system

$$\begin{aligned} \ddot{u} + \omega^2 u &= (\varepsilon - \alpha z)\dot{u} \qquad (6) \\ \dot{z} + \tau z &= u^2 \qquad (7) \end{aligned}$$

ω and α are constants, τ is a positive, O(1), constant and ε is a small positive parameter. We will use the method of multiple scales to analyze this problem.

(a) Rewrite the equations in the form of a 3D dynamical system and find the fixed point around which we expect a limit cycle (since we expect periodic behavior in time, we expect a limit cycle).

(b) Expand each variable as a series in δ, a small parameter. Argue that there should be no zeroth order terms. Use three time scales, $T_0 = t$, $T_1 = \delta t$, and $T_2 = \delta^2 t$. Argue that the $\varepsilon = O(\delta^2)$.

(c) Show that, to the first approximation,

$$u \approx a\cos(\omega t + \beta)$$

where a=O($\sqrt{\varepsilon}$) and

$$\begin{aligned} \dot{a} &= \frac{1}{2}\varepsilon a - \frac{\alpha(\tau^2 + 8\omega^2)}{8\tau(\tau^2 + 4\omega^2)}a^3 \qquad (8) \\ \dot{\beta} &= -\frac{\alpha\omega}{4(\tau^2 + 4\omega^2)}a^2. \qquad (9) \end{aligned}$$

Which bifurcation do these equations describe? As function of which parameter? Why does the multiple scale approximation make sense to use here?