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Problem Set 7 Solutions

1. (a) Expand x(ε) as a perturbation series in ε:

$$x(\varepsilon) = x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3) \tag{1}$$

Substitute into our equation and consider the equation at each order:

$$\begin{aligned} O(\varepsilon^0): &\quad x_0^3 - 1 = 0 \\ O(\varepsilon^1): &\quad 3x_0^2x_1 - x_0 = 0 \Rightarrow x_1 = \frac{1}{3x_0} \\ O(\varepsilon^2): &\quad 3(x_0^2x_2 + x_0x_1^2) - x_1 = 0 \Rightarrow x_2 = 0 \end{aligned}$$

The solutions to for x_0 are x_0=1,$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Let's consider only the real root for now. We have found:

$$x(\varepsilon) = 1 + \frac{1}{3}\varepsilon + O(\varepsilon^3) \tag{2}$$

For ε=0.001, MATLAB gives x(ε)=1.00033333332099 while our approximation gives x(ε)≈1.00033333333333. These solutions differ by approximately 1.23×10^{-11}, confirming that our solution is valid at least to O(ε^3).

(b) Expand x(ε) as a perturbation series in ε:

$$x(\varepsilon) = x_0 + x_1\varepsilon + x_2\varepsilon^2 + O(\varepsilon^3) \tag{3}$$

Substitute into our equation and consider the equation at each order:

$$\begin{aligned} O(\varepsilon^0): &\quad x_0^3 - x_0 = 0 \\ O(\varepsilon^1): &\quad 3x_0^2x_1 + x_0^2 - x_1 = 0 \Rightarrow x_1 = \frac{x_0^2}{1-3x_0^2} \\ O(\varepsilon^2): &\quad 3(x_0^2x_2 + x_0x_1^2) + 2x_0x_1 - x_2 = 0 \Rightarrow x_2 = \frac{x_0x_1(2+3x_1)}{1-3x_0^2} \end{aligned}$$

The solutions for x_0 are x_0=-1,0,1. So near the roots we have:

$$\begin{aligned} x(\varepsilon) &= -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3) && (4) \\ x(\varepsilon) &= 0 + O(\varepsilon^3) && (5) \\ x(\varepsilon) &= 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + O(\varepsilon^3) && (6) \end{aligned}$$

Here is a table showing x(ε) as calculated with MATLAB compared to our expansion.

x_0	$x_0 + \varepsilon x_1$	$x_0 + \varepsilon x_1 + \varepsilon^2 x_2$	x(ε)
-1	-1.00050000000000	-1.00050012500000	-1.00050012499999
0	0	0	0
1	0.99950000000000	0.99950012500000	0.99950012499999

2. Write the system as a 2D dynamical system:

$$\dot{x} = v \tag{7}$$
$$\dot{v} = a - x + \mu(1-x^2)v \tag{8}$$

The fixed point of this system is (x,v)=(a,0). The Jacobian is:

$$J = \begin{pmatrix} 0 & 1 \\ -1-2\mu xv & \mu(1-x^2) \end{pmatrix}$$

Evaluated at the fixed point:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \mu(1-a^2) \end{pmatrix}$$

So we have:

$$\lambda_{\pm} = \frac{\mu(1-a^2)}{2} \pm \sqrt{\frac{\mu^2(1-a^2)^2}{4} - 1} \tag{9}$$

To have a Hopf bifurcation we need $\mu(1-a^2)$=0 and $\frac{\mu^2(1-a^2)^2}{4} - 1$<0. Notice that the first condition implies the second condition. So we only need $\mu(1-a^2)$=0. This means that Hopf bifurcations occur on the curves a=$\pm$1 and μ=0. I checked this numerically and found it to be true.

3. The Jacobian is:

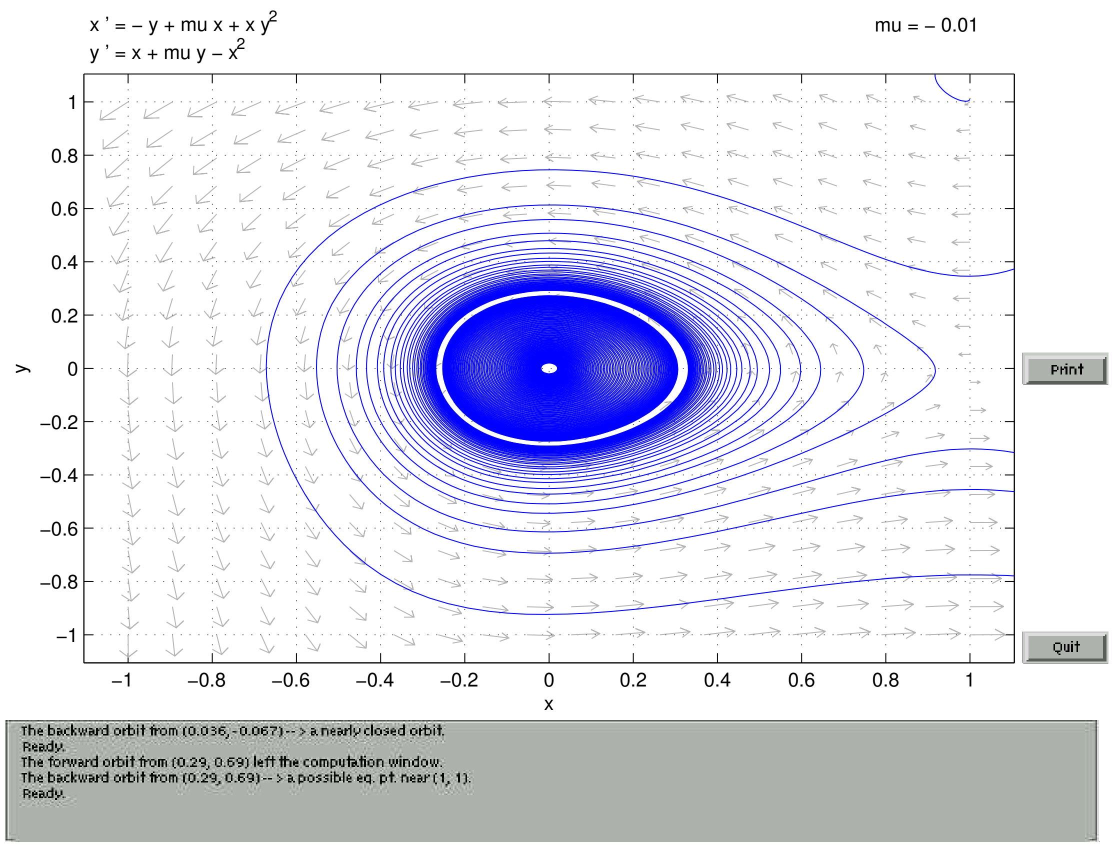
$$J = \begin{pmatrix} \mu + y^2 & 2xy - 1 \\ 1-2x & \mu \end{pmatrix}$$

Evaluate at the origin:

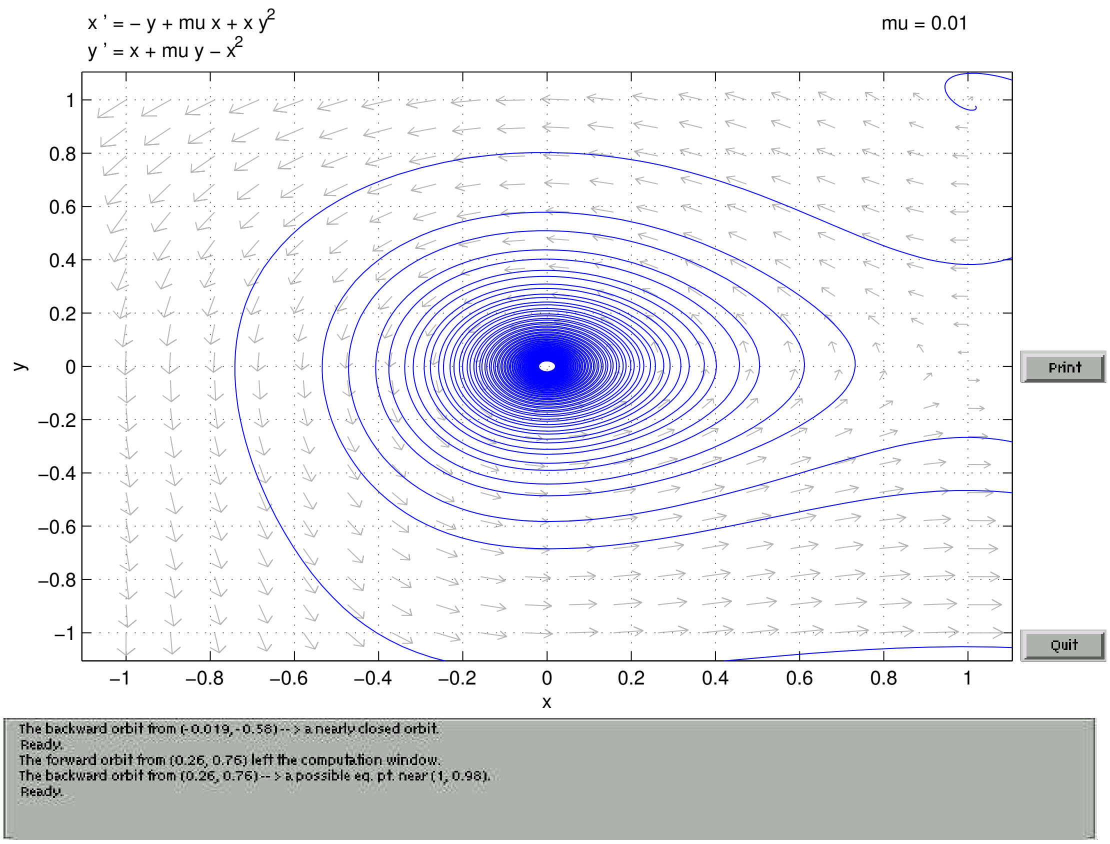
$$J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

So the eigenvalues are $\lambda_{\pm} = \mu \pm i$ and are pure imaginary when μ=0.

4. There is an unstable limit cycle around the origin and a stable spiral at the origin for μ<0:


x ' = − y + mu x + x y2
y ' = x + mu y − x2
mu = − 0.01
y
x
Print
Quit
The backward orbit from (0.036, -0.067) -- > a nearly closed orbit.
Ready.
The forward orbit from (0.29, 0.69) left the computation window.
The backward orbit from (0.29, 0.69) -- > a possible eq. pt. near (1, 1).
Ready.

The origin is a unstable spiral and no limit cycle for μ>0. So a subcritical Hopf bifurcation occurs at μ=0.



5. (a) Use x=r cos(θ) and y=r sin(θ). The system can be rewritten:

$$\dot{r} = \mu r + r^2 cos^2(\theta) sin(\theta)[r sin(\theta) - 1] \quad (10)$$

$$\dot{\theta} = 1 - r cos(\theta)[cos^2(\theta) + r^2 sin^3(\theta)] \quad (11)$$

(b) Consider the average of these equations over θ. This will give us a qualitative idea of the behavior. We expect this method to capture the behavior best for r≪1 when θ is less important for the dynamics.

$$\dot{r} = \mu r + \frac{1}{8} r^3 \quad (12)$$

$$\dot{\theta} = 1 \quad (13)$$

(c) These equations suggest that an unstable limit cycle exists for negative μ with a radius of approximately $\sqrt{-8\mu}$ (consider the 1D dynamics in r) and the origin is unstable for positive μ. There is a subcritical Hopf bifurcation. Although not rigorous, this justifies the numerical results given above.