Homework #8
Nonlinear dynamics and chaos

1. **Global bifurcation of limit cycles:** (Strogatz 8.4.2) Analyze the bifurcations as function of μ of the system

$$\begin{aligned}\dot{r} &= r(\mu - \sin r)\\ \dot{\theta} &= 1\end{aligned}$$

2. **Hopf bifurcation in Lorenz equations:** Find the critical r_H at which a Hopf bifurcation of the C^+, C^- points occurs in the Lorenz system.

3. **Pitchfork bifurcation in the Lorenz equations:** Plot the fixed points for x, y and z as function of r for the Lorenz system. Plot also $x^2+y^2+z^2$ for these fixed points as function of r. Explain each of your plots. Note that this is a 3d system that undergoes the pitchfork bifurcation whose normal form is 1d.

4. **Hysteresis for the driven pendulum:** (Numerical, use `driven_pendulum.m` from the course home page).

   (a) Find values of the friction α and forcing I in the equation $\phi'' + \alpha\phi + \sin\phi = I$ for which there are both a stable limit cycle and a stable fixed point. Solve numerically, show and explain how the system approaches these two different solutions for different initial conditions.

   (b) Estimate the period as function of the bifurcation parameter I for $\alpha = 1.5$ as I approaches 1 from above. Plot $period(I)$ together with the expected dependency for this kind of a bifurcation. Discuss the results. Plot the oscillations for $I = 1.001$ or for similar value just above 1. This form of oscillations is typically found in experimental or model systems for infinite period bifurcations.

5. **Numerical integration of the Lorentz system:** Set $b = 8/3$; $\sigma = 10$. Use the solver `lorenz.m` on the course home page to plot the time series of the Lorenz system in the regimes (a) $r < 1$; (b) $1 < r < r_H$; (c) $r = r_h + \varepsilon$ for some small ε, (in this case, start with initial conditions very close to the location of one of the C^+/C^- fixed point; (d) $r = 28$. For each of these values of r, plot a time series of $y(t)$ as well as a phase trajectory in the (x, z) plane, and explain what you see in terms of the bifurcation behavior of r analyzed in class.