APM 203 Homework Assignment #1

$\boxed{\text{SOLUTIONS}}$

(A) 1. $\dot{x} = 1 - e^{-x^2} \equiv f(x)$

fixed points : $f(x^*) = 0 \Rightarrow 1 - e^{-x^{*2}} = 0$

or $x^{*2} = 0$ i.e. $\boxed{x^* = 0.}$

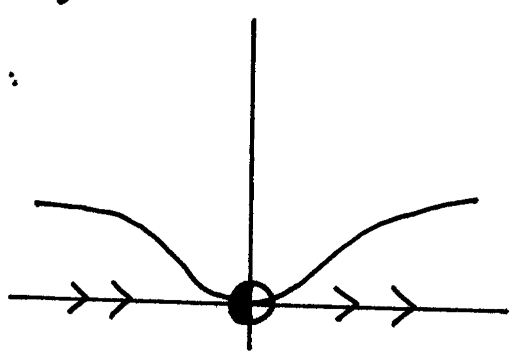
stability : $x = x^* + \delta x = \delta x$

$f(x) = 1 - e^{-\delta x^2} \approx 1 - (1 - \delta x^2) = \delta x^2$

$= \mathcal{O}(\delta x^2)$

$\Rightarrow$ linearization fails.

Graphical approach :



thus $x^* = 0$ is semi-stable (stable on the left of zero.)

2. $\dot{x} = ax - x^3$

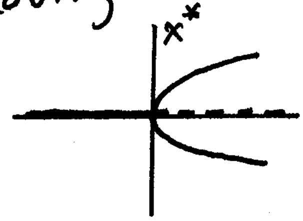
<u>fixed points</u> :

$a < 0 \rightarrow x^* = 0$

$a = 0 \rightarrow x^* = 0$

$a > 0 \rightarrow x^* = 0, \; x^* = \pm\sqrt{a}$

we may recognize this system as having a <u>pitch fork bifurcation</u> at $a = 0$:



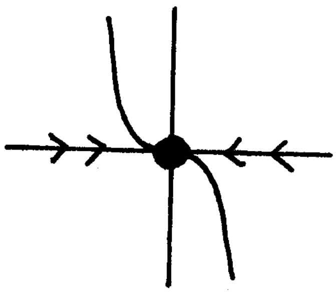
<u>stability</u> : $a < 0$, $x = x^* + \delta x = \delta x$

$\dot{x} \approx a\,\delta x \Rightarrow$ <u>stable</u> $(a < 0!)$

next, $a = 0$, $x = x^* + \delta x = \delta x$

$\dot{x} = -\delta x^3 = \mathcal{O}(\delta x)^3 \Rightarrow$ linearization fails.

Graph:



$\Rightarrow$ stable

next, $a > 0$. Now we have 3 f.p.'s

$x^* = 0$: $\dot{x} = x(a - x^2) \cong a\,\delta x \Rightarrow$ unstable

$x^* = \pm\sqrt{a}$: $\dot{x} \cong (\pm\sqrt{a} + \delta x)(\mp 2\sqrt{a}\,\delta x) \cong -2a\,\delta x \Rightarrow$ stable

3. $\dot{x} = x(1-x)(2-x) = f(x)$

$\underline{\underline{\text{f.p.'s}}}$: $f(x^*) = 0 \Rightarrow x^* = 0, 1$ or 2.

$\underline{\underline{\text{stability}}}$: (i) $x^* = 0$: $x = x^* + \delta x = \delta x$

$f(x^* + \delta x) = \delta x\,(1-\delta x)(2-\delta x) \approx 2\delta x \Rightarrow$ unstable.

(ii) $x^* = 1$: $x = x^* + \delta x = 1 + \delta x$

$f(x^* + \delta x) = (1+\delta x)(-\delta x)(2 - \delta x - 1) \approx -\delta x$

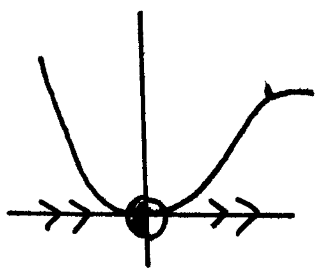
$\Rightarrow$ stable.

(iii) $x^* = 2$: $x = x^* + \delta x = 2 + \delta x$

$f(x^* + \delta x) = (2+\delta x)(-1-\delta x)(-\delta x) = 2\delta x \Rightarrow$ unstable.

4. $\dot{x} = x^2(6-x)$ $\quad$ $\underline{\text{f.p.'s}}$: $x^* = 0$ & $x^* = 6$.

linearization $\underline{\underline{\text{fails}}}$ for $x^* = 0$. Graph:



$\Rightarrow$ semi-stable.

next, $x^* = 6$: $x = 6 + \delta x \Rightarrow$ $\dot{x} \approx (36 + 12\delta x)(-\delta x)$

$\cong -36\,\delta x$

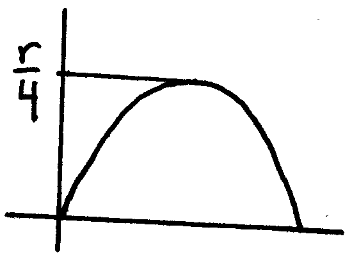
i.e. $\underline{\underline{\text{stable}}}$.

5. $\dot{x} = \ln x$ $\quad$ $\underline{\underline{\text{f.p.}}}$ $\quad f(x^*) = \ln x^* = 0 \Rightarrow x^* = 1$

linearization : $\quad x = 1 + \delta x \Rightarrow \dot{x} = \ln(1+\delta x) \cong \delta x$

i.e. unstable

(B) $\quad x_{n+1} = f(x_n) \qquad f(x) = rx(1-x)$



$x \in [0,1] \ \forall n \Rightarrow r > 0$, otherwise $[0,1] \xrightarrow{f}$ negative x.

furthermore, $f(\frac{1}{2}) = \frac{r}{4} = \max\{f([0,1])\}$

$\Rightarrow r < 4$. Thus $\underline{\underline{0 \le r \le 4}}$.

Now, we find dissipative values of r. The latter requires $|f'(x)| < 1$ everywhere on the interval.

Thus $|r(1-2x)| < 1$. L.H.S. maximizes at $x = 0$ & 1.

so we need $|r(1-2\cdot 0)| = |r(1-2\cdot 1)| < 1$ i.e. $r < 1$, using $r > 0$.

Finally, $\underline{\underline{0 < r < 1}}$.

2. What is the asymptotic behavior for large n? Do this analytically first:

$$x_{n+1} = f(x_n) = r x_n (1 - x_n) \qquad 0 \leq r < 1.$$

since $0 \leq x_n \leq 1$ we therefore have $0 \leq x_{n+1} < x_n$

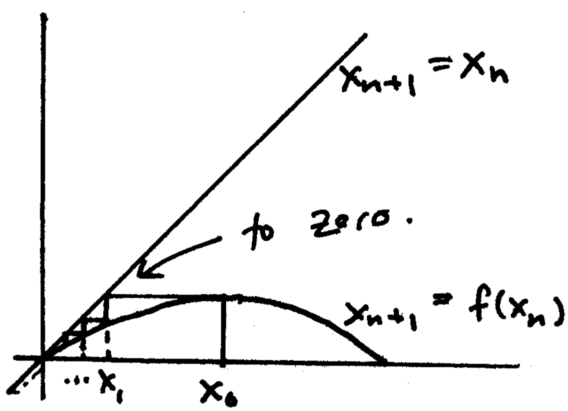
stronger inequality: $0 \leq x_{n+1} < r x_n$

$\Rightarrow \quad 0 \leq x_{n+2} < r x_{n+1} < r^2 x_n \quad \ldots$ ad infinitum

i.e. $0 \leq x_n < r^n x_0$ but $0 \leq r < 1$

thus R.H.S. $\to 0$ and x_n is sent to $\underline{\underline{\text{zero}}}$.

Graphically (use cobwebs!)



3. Fixed points and their stability:

Investigate ranges of r within $0 \leq r \leq 4$.

(i) First, let $0 \leq r < 1$. From part 2, we expect 1 stable f.p. (at $x=0$).

Verify this again: $x^* = f(x^*) = r x^* (1-x^*)$

$\Rightarrow x^* = 0$ (alternate f.p. $1-\frac{1}{r} \notin [0,1]$.)

For stability, linearize: $\delta x_{n+1} = f'(x^*)\,\delta x_n$
$= r\,\delta x_n$

$\Rightarrow$ STABLE

(ii) $r=1$. Stability analysis fails. Now, $x_{n+1} = x_n(1-x_n)$

if $x_0 = 0 \Rightarrow x_1 = 0 \ldots$ etc.

just away from $x_0 = 0$ we have

$x_1 = x_0(1-x_0)$

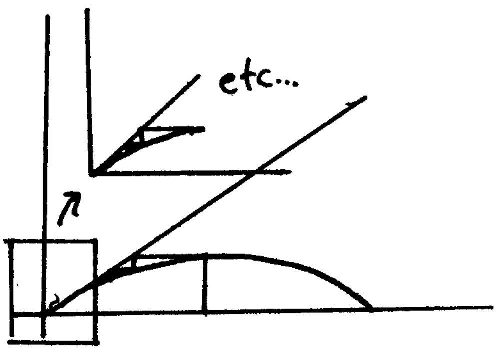
$x_2 = x_0(1-x_0)\left[1 - x_0(1-x_0)\right] = x_0(1-x_0)(1-x_1)$

$\vdots$

$x_n = x_0(1-x_0)(1-x_1)\cdots(1-x_{n-1}) \rightarrow 0$

$\Rightarrow$ stable

also, using cobweb:



(iii) Let $1 < r \leq 4$, fixed points $x^* = 0, 1-\frac{1}{r}$

stability of $x^* = 0$: $\delta x_{n+1} = r\,\delta x_n \Rightarrow$ unstable

of $x^* = 1-\frac{1}{r}$ $\delta x_{n+1} = f'(1-\frac{1}{r})\,\delta x_n$

$$= r(1-2+\frac{2}{r})\delta x_n$$

$$= (2-r)\,\delta x_n$$

Thus, stable for $1 < r < 3$

unstable for $3 < r \leq 4$

and at $r=3$? Investigate behavior of $f(f(x))$.

Then $f(f(\frac{2}{3}+\delta x_n)) = \frac{2}{3}+\delta x_{n+2} = \frac{2}{3}+\delta x_n - 18\delta x_n^3 - 27\delta x_n^4$

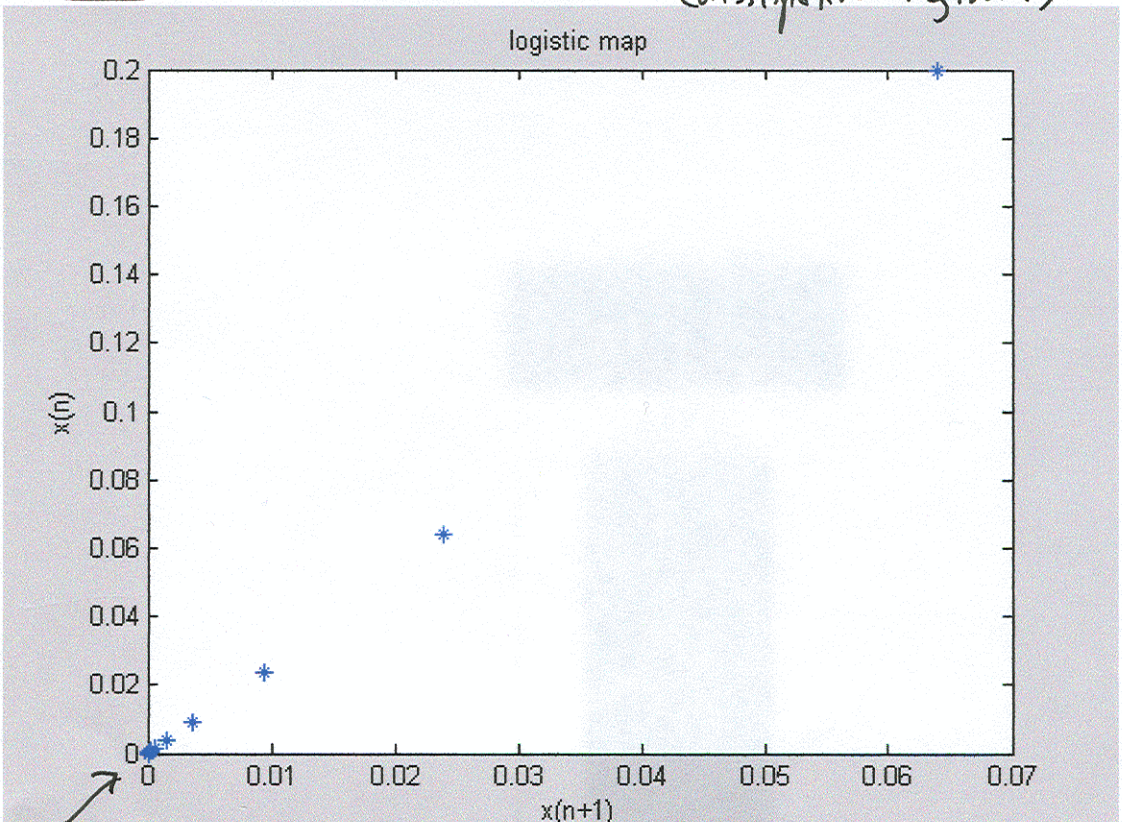
$$\Rightarrow \delta x_{n+2} = \delta x_n \underbrace{(1-18\delta x_n^2 - 27\delta x_n^4)}_{<1}$$

thus $\frac{2}{3}$ is (marginally) stable for $r=3$.

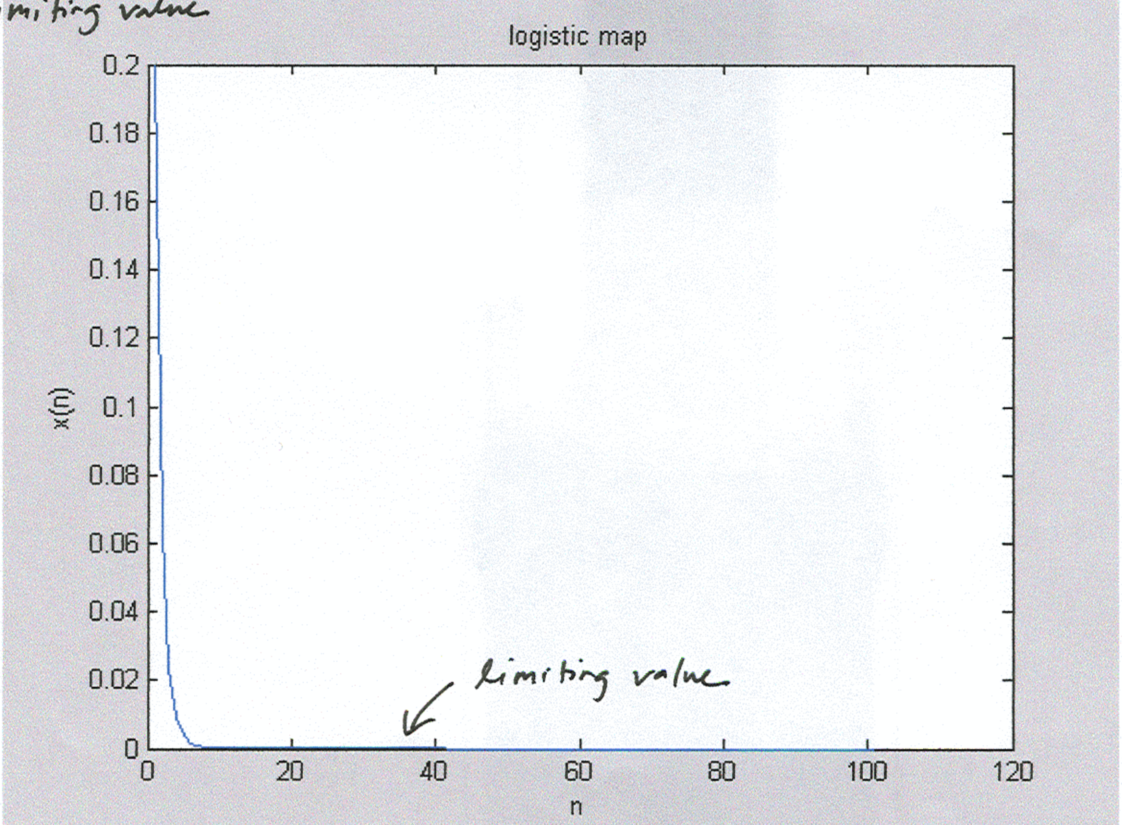
SUMMARY:

r	Fixed pts.		Stability	
$0 \leq r < 1$	0		stable	
$r=1$	0		stable (marginally)	
$1 < r < 3$	0	$1-\frac{1}{r}$	unstable	stable
$r=3$	0	$2/3$	unstable	stable (marginally)
$3 < r \leq 4$	0	$1-\frac{1}{r}$	unstable	unstable

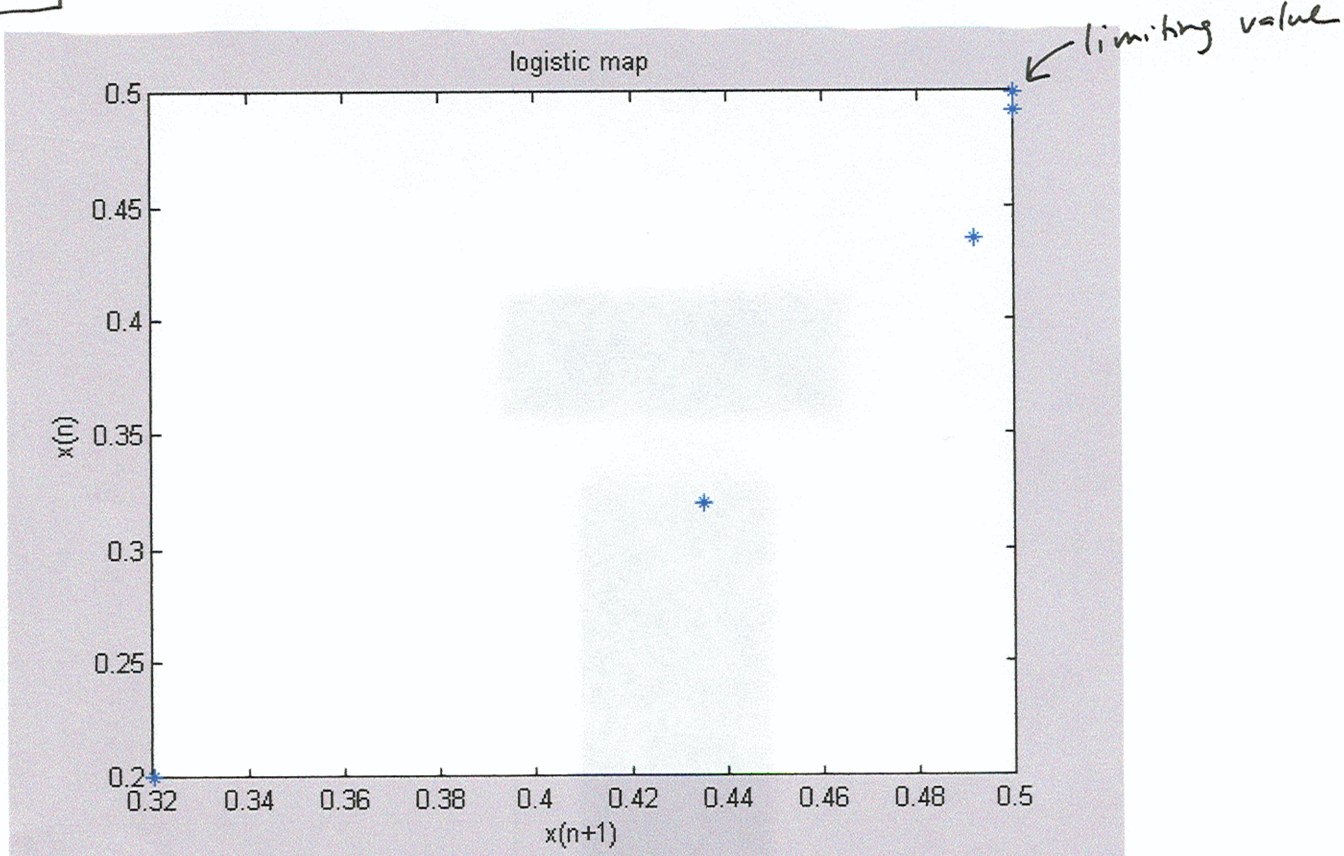
$\boxed{r = 0.4}$ All initial values are mapped to zero as illustrated here (dissipative region?)

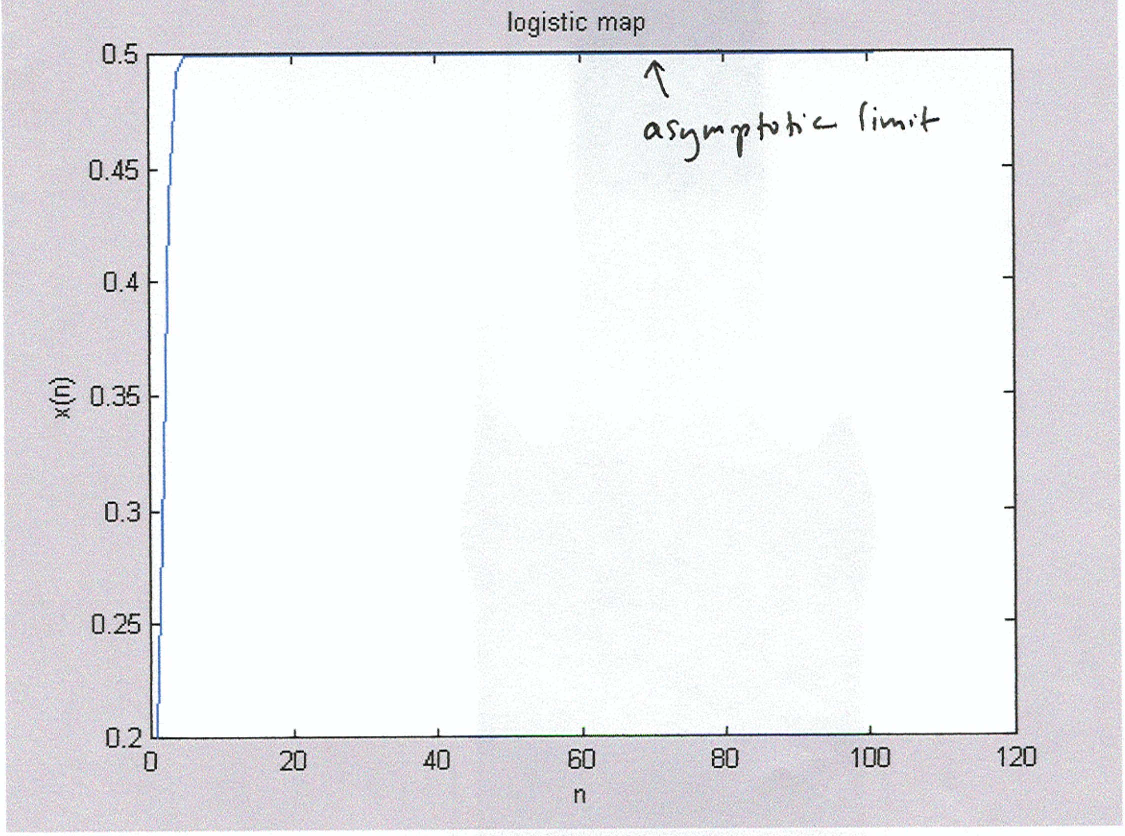


limiting value

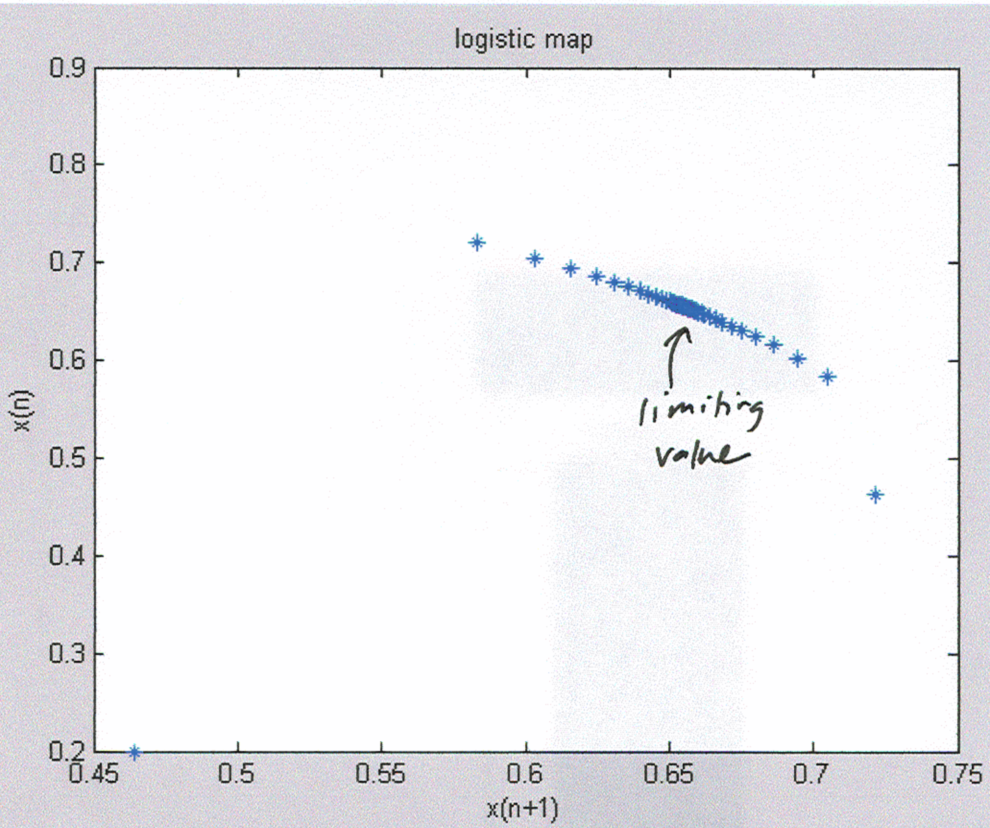


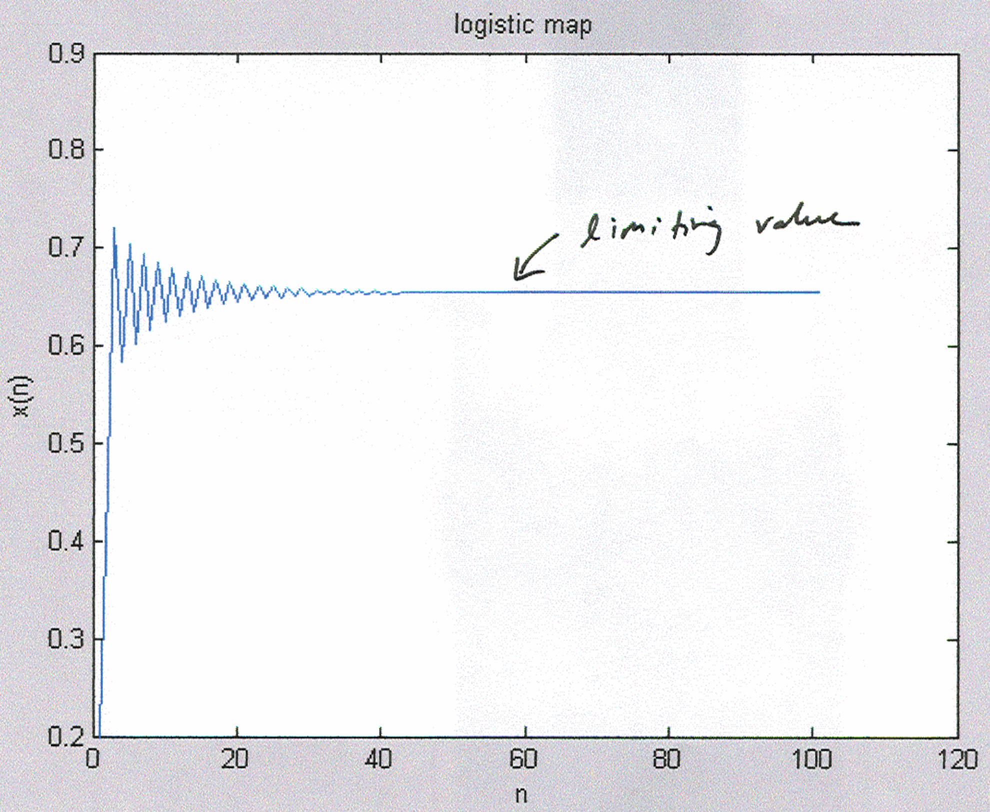
$\boxed{r=2}$ Here, we are in the stable region for f.p. $1-\frac{1}{r}=\frac{1}{2}$.



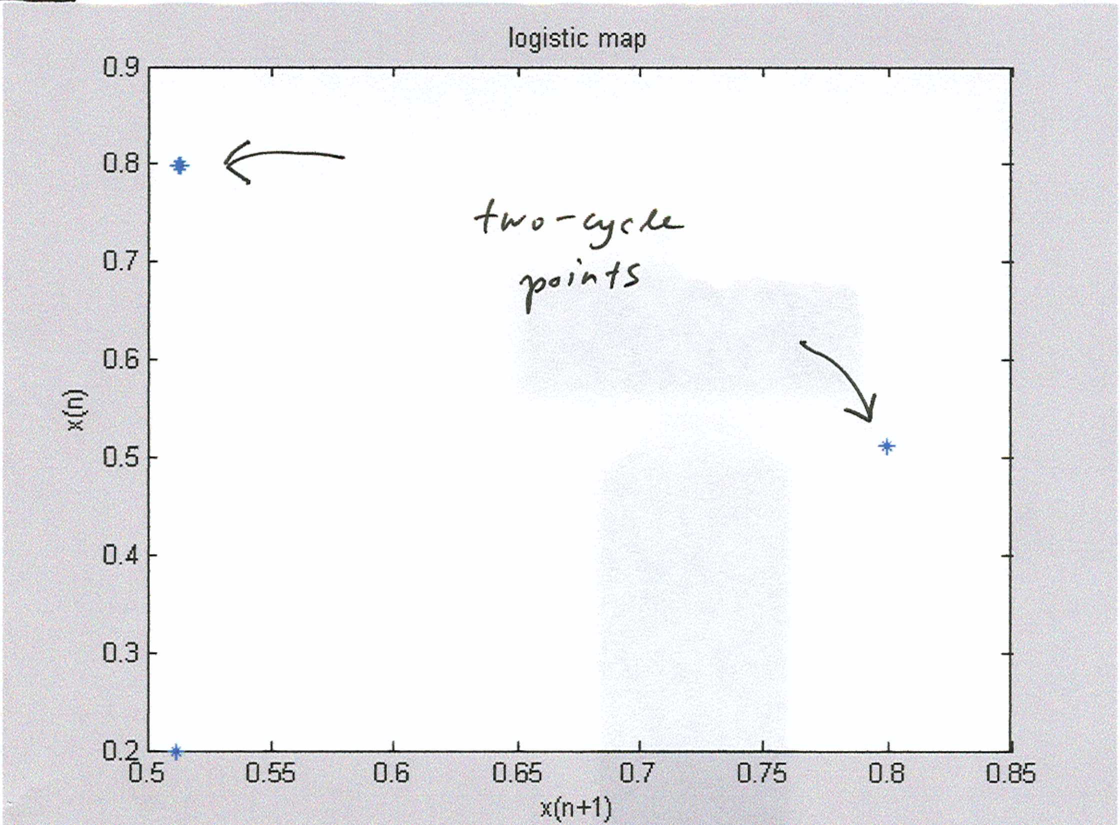


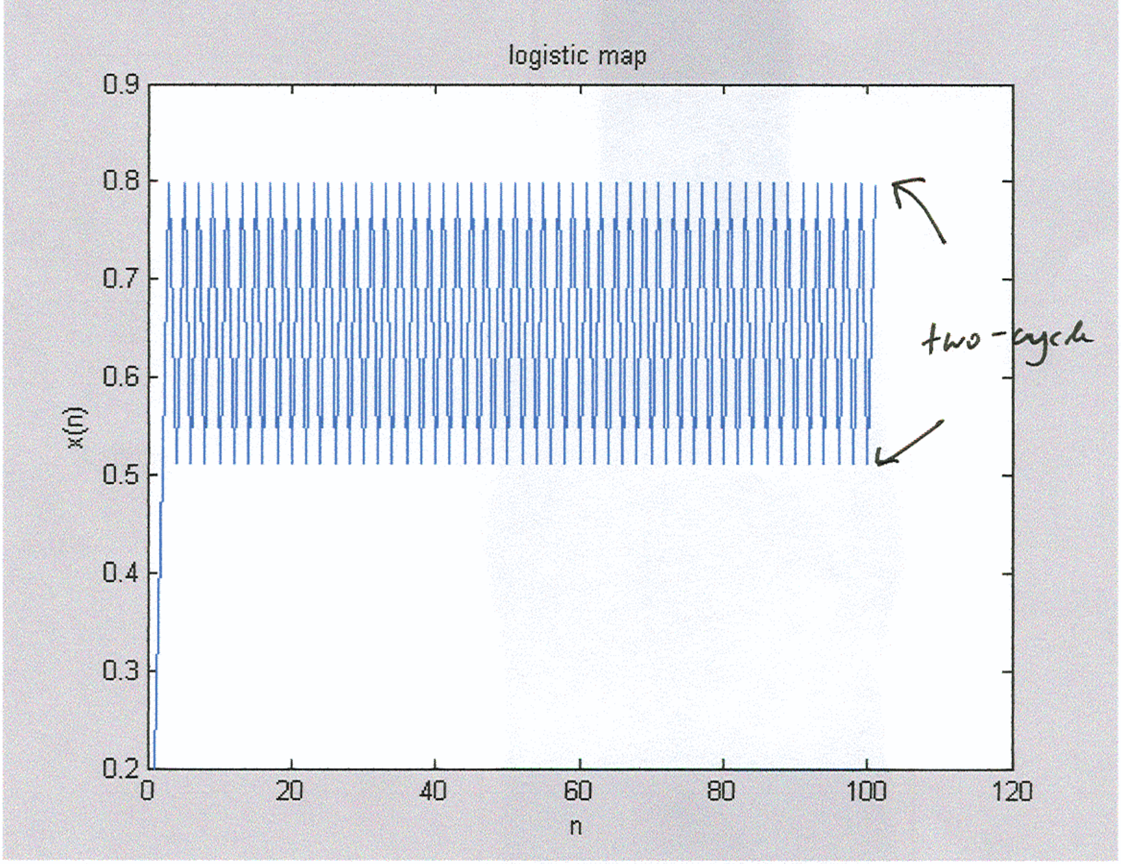
$\boxed{r=2.9}$ Here, we have oscillating motion to the stable fixed point $1-\frac{1}{2.9} \approx .65$





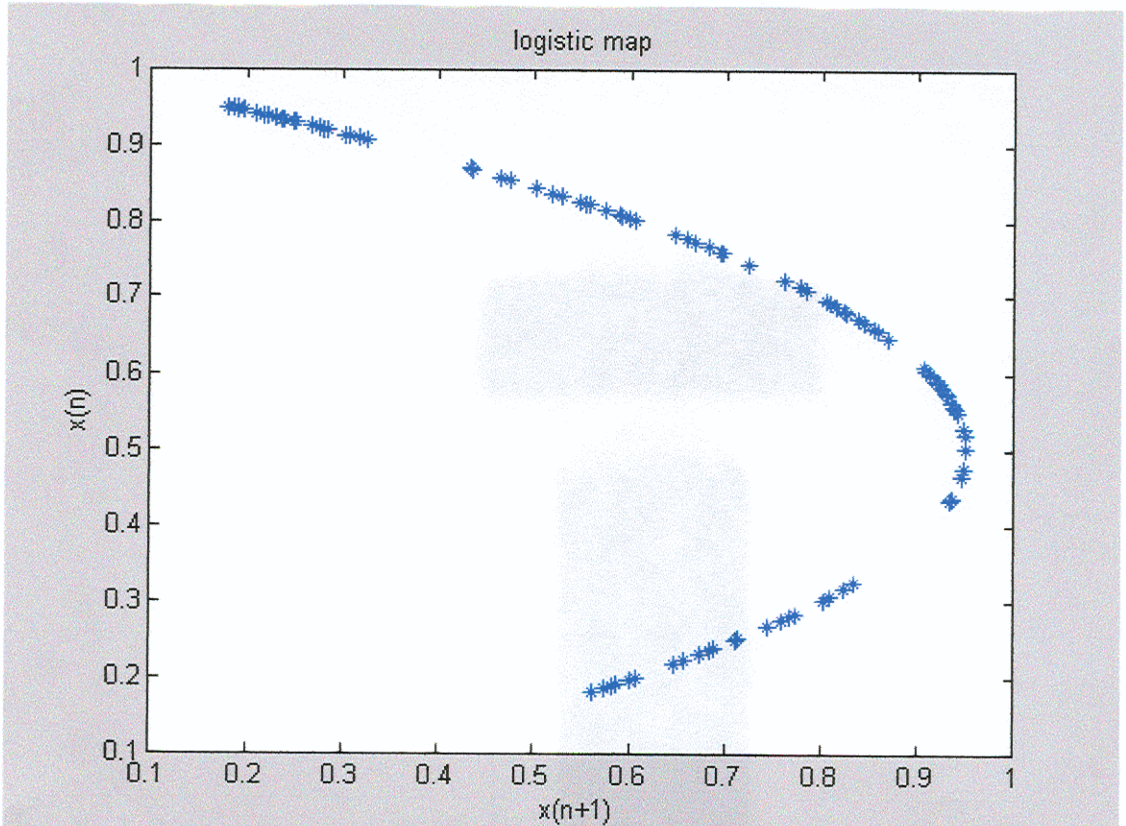
$\boxed{r = 3.2}$ The system enters into a 2-cycle. F.P. $1 - \frac{1}{3.2} \cong 0.69$ is no longer stable.

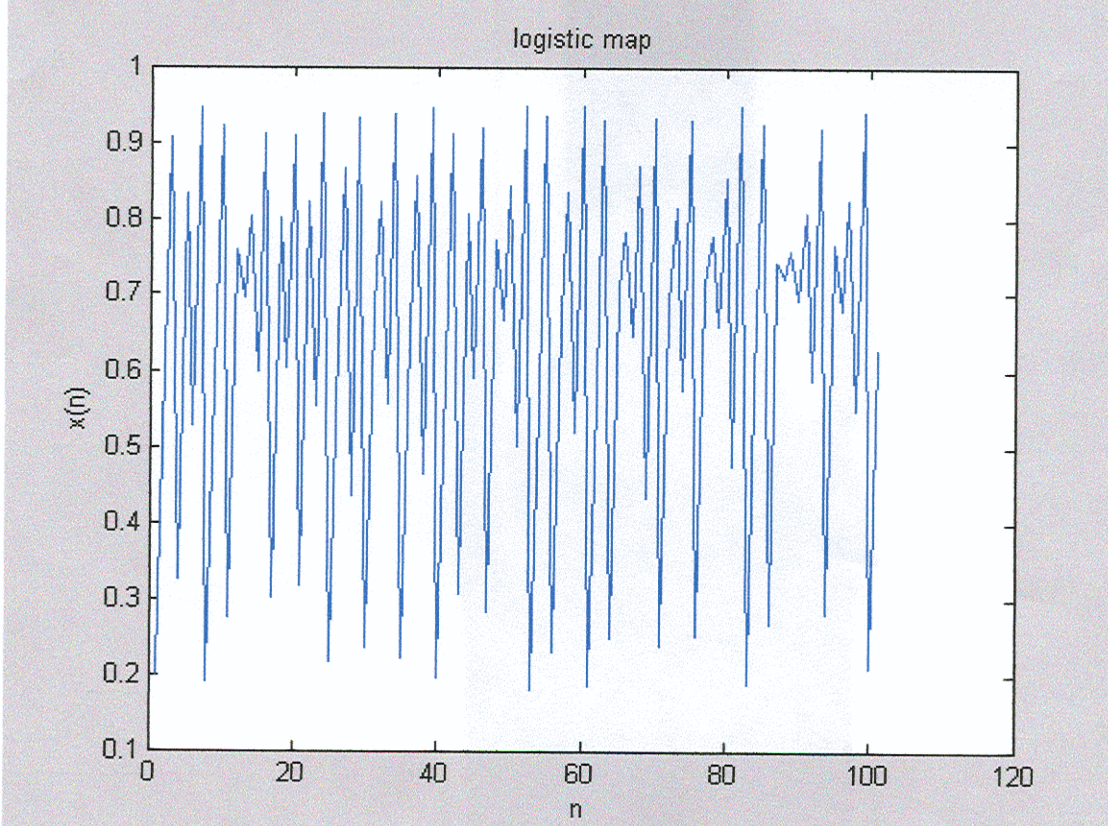




two-cycle points

$\boxed{r = 3.8}$ x_n is filling out a continuous range of values in $[0,1]$.

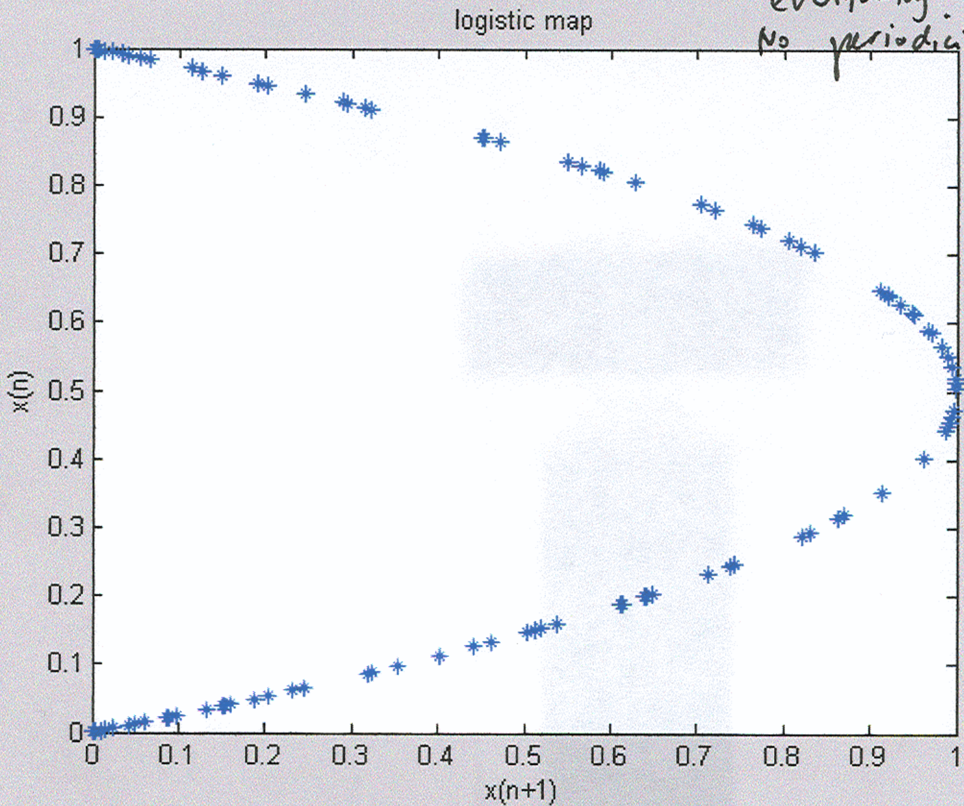


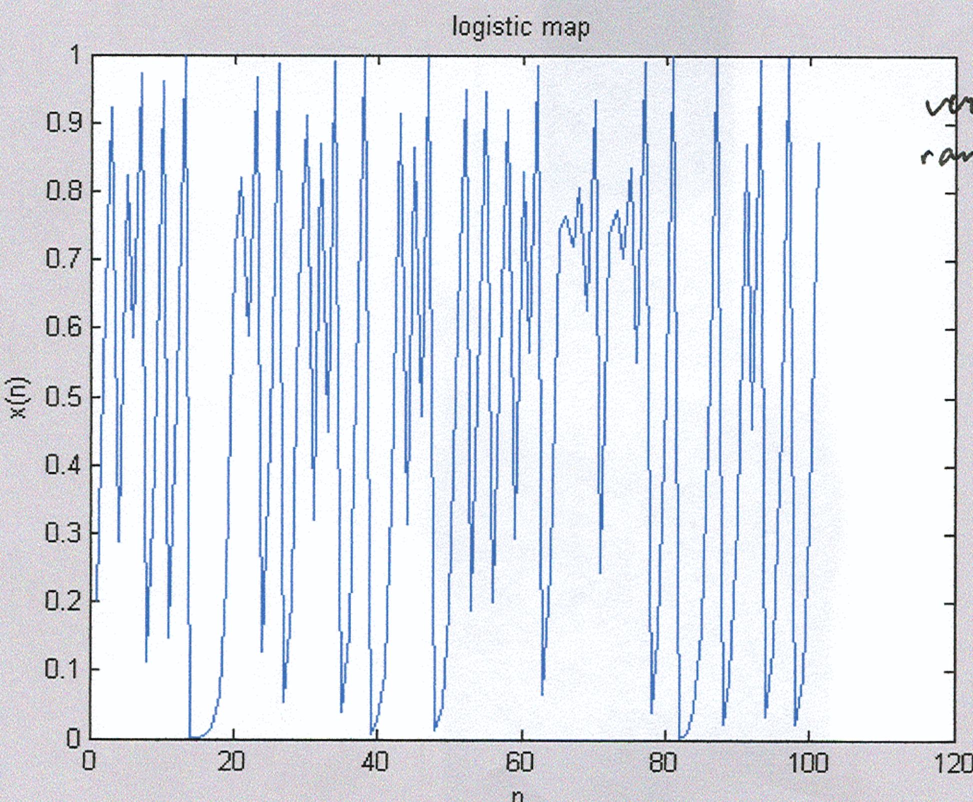


wild, but bounded motion. No periodicity, no asymptotic limit.

$\boxed{r=4}$

Now all values of x in $[0,1]$ are visited, the curve eventually becomes continuous. No periodicity, no limiting value





very erratic and ranges from 0 to 1.

special cases: