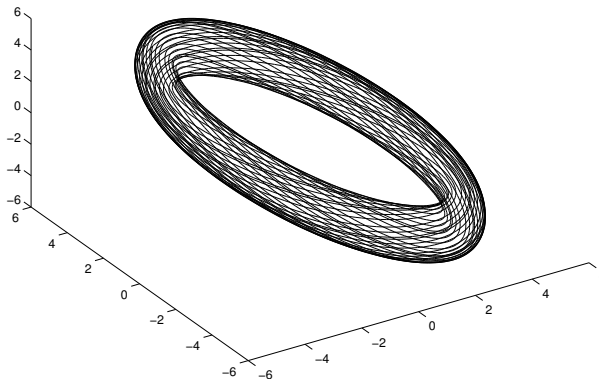


Homework #5
Nonlinear dynamics and chaos

1. **Hopf bifurcation in Lorenz equations:** Find the critical r_H at which a Hopf bifurcation of the C^+ , C^- points occurs in the Lorenz system. (Note that the optional question below may be done instead of this one).
2. **Reconstructed phase space and quasi-periodicity:** Often in experimental systems, the equations are not known, only a time series of a certain measurement is available, but we still would like to plot things in a “reconstructed” phase space based on the available time series. There are several standard ways to do this, two of which are demonstrated here. Consider a time series of a “quasi-periodic” process $x(t) = a \sin(\omega_1 t) + b \sin(\omega_2 t)$ in a reconstructed 3d phase space and show that it produces a torus. (A quasi-periodic process is one with two frequencies whose ratio does not form a rational number). Do this in the two following ways to see that they produce topologically equivalent (i.e. two pictures that look similar) phase space portraits.
 - (a) Use “delay coordinates” reconstructed phase space, such that each point in phase space is given by $\mathbf{x}(t) = (x(t), x(t - \tau), x(t - 2\tau))$ for some delay τ .
 - (b) Use a phase space based on $\mathbf{x}(t) = (x(t), \frac{d}{dt}x(t), \frac{d^2}{dt^2}x(t))$

Try various choices for the parameters $a, b, \omega_1, \omega_2, \tau$ to get a nice looking torus as in the figure below. Explore cases in which $\omega_1/\omega_2 = p/q$ with p, q integers, for both $p > q$ and $p < q$, as well as cases in which ω_1/ω_2 is irrational.

Explain why is it that while the time series in a case where ω_1/ω_2 is irrational never repeats itself, it is still not chaotic.



3. **Dangers of numerics:**

- (a) Using the Matlab program on the course home page (or similar), integrate the Lorenz equation in time in the chaotic regime $\sigma = 10, b = 8/3, r = 28$. Plot the butterfly attractor (z vs x). Start plotting the attractor only after some initial integration time, to allow transients to disappear.
- (b) Integrate the Lorenz system using the leap-frog scheme, as well as the modified Euler scheme. The Matlab program on the course home page uses the latter, and changing it

to a leap frog scheme requires only a very simple modification. Find the smallest Robert Filter coefficient that is needed to stabilize the leap frog scheme. Start with exactly the same initial conditions for both schemes, and plot the two time series using the two schemes, as well as the separation between the two time series. Repeat these plots in the chaotic regime and in the regime with a stable spiral (for $r < r_H$). Show the plots and explain them.

(c) Using the leapfrog scheme, plot the results (for a few time steps) without the Robert Filter. Apply the Robert Filter, and calculate numerically the value of r_H as function of the Robert filter coefficient (using a few values of the latter only). Plot the results and discuss their implications to the numerical study of nonlinear systems.

4. **Optional challenge question/ extra credit:** This question may be solved in addition to or instead of the first one above. Consider the system

$$\ddot{u} + \omega^2 u = (\varepsilon - \alpha z)\dot{u} \quad (1)$$

$$\dot{z} + \tau z = u^2 \quad (2)$$

ω and α are constants, τ is a positive constant that is away from zero, and ε is a small positive parameter. Use the method of multiple scales and show that, to the first approximation,

$$u \approx a \cos(\omega t + \beta)$$

where

$$\dot{a} = \frac{1}{2}\varepsilon a - \frac{\alpha(\tau^2 + 8\omega^2)}{8\tau(\tau^2 + 4\omega^2)}a^3 \quad (3)$$

$$\dot{\beta} = -\frac{\alpha\omega}{4(\tau^2 + 4\omega^2)}a^2. \quad (4)$$

Which bifurcation do these equations describe? as function of which parameter? why does the multiple scales approximation make sense to use here?