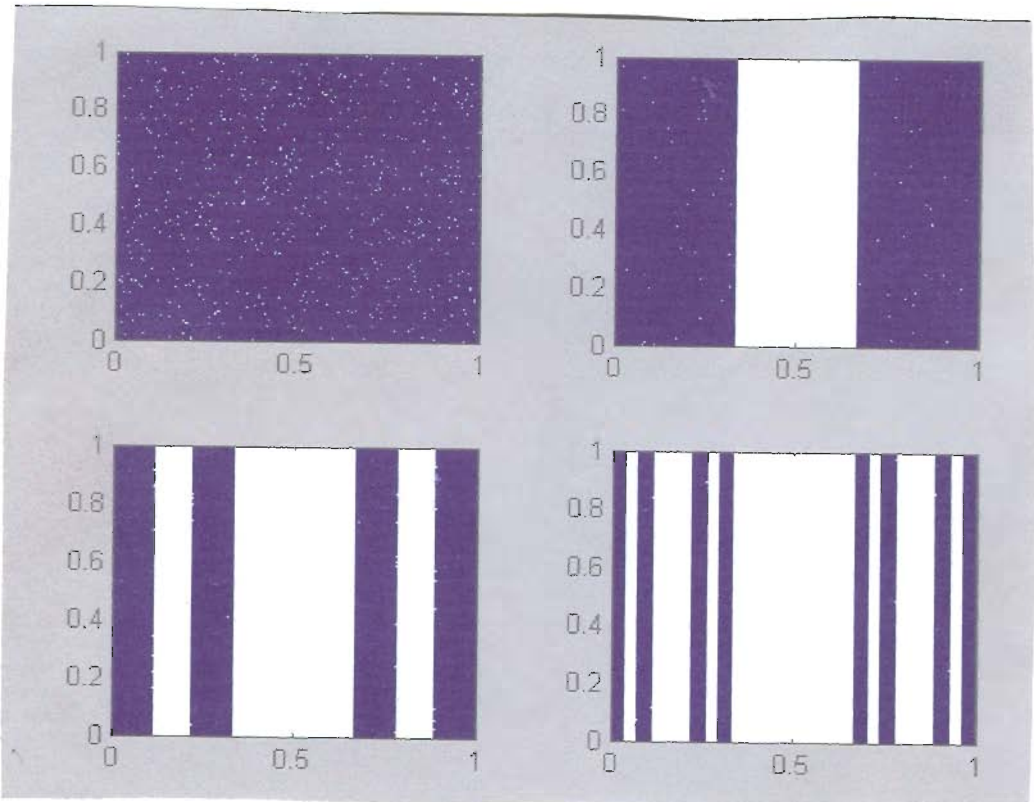
APM203 Homework #8 Solutions

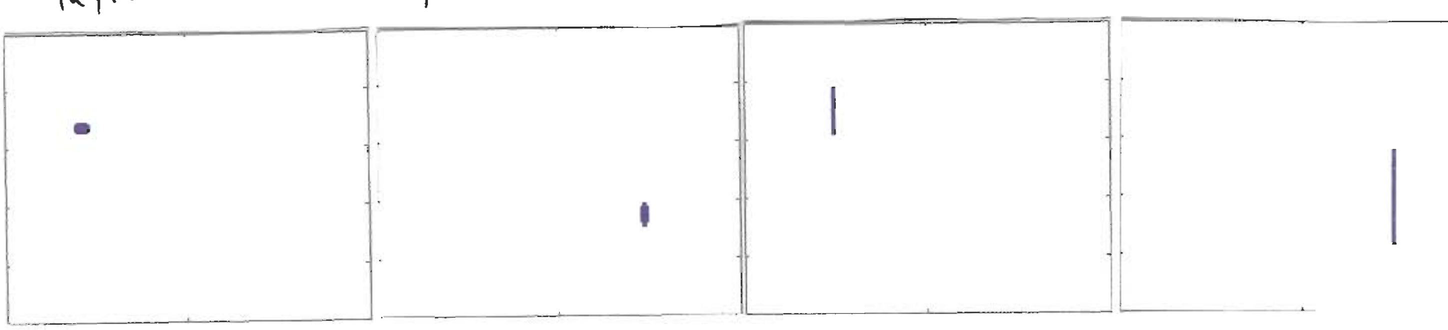
Problem #1

The baker map:

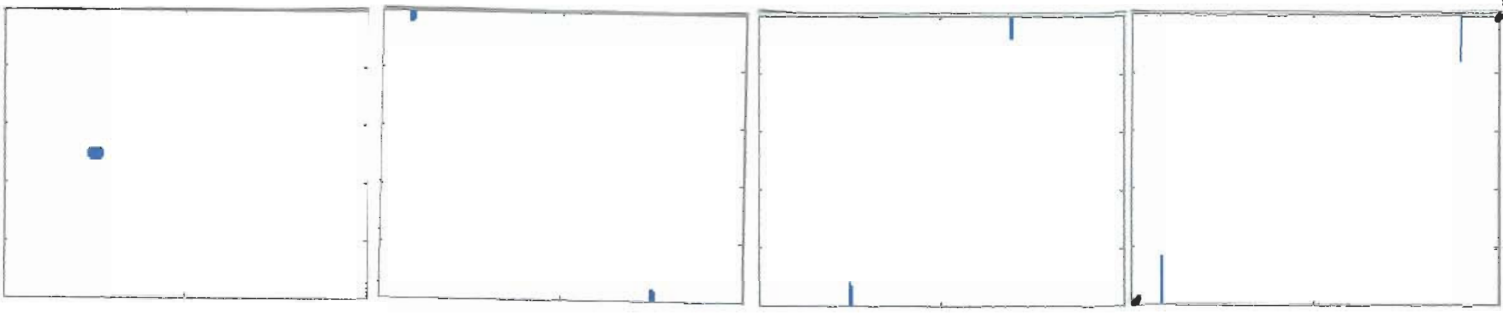
$$B(x,y) = \begin{cases} \left(\frac{1}{3}x,\ 2y\right) & 0 \le y \le \frac{1}{2} \\ \left(\frac{1}{3}x + \frac{1}{3},\ 2y-1\right) & \frac{1}{2} \le y \le 1 \end{cases}$$



Above, I've provided illustrations of the first 3 mappings of the unit square by B. The pattern becomes a Cantor set (middle third) as we continue ad infinitum. Map of a small circle:



If the circle is ever on the $y=\frac{1}{2}$ line, it will ultimately get mapped to the edges. The portion on the $y=\frac{1}{2}$ line gets mapped to the corners (dimension zero.)



(b) Lyapunov Exponent:

$$h = \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{|y_n|}{|y_0|}\right) = \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{|DB^n y_0|}{|y_0|}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2}\cdot\frac{1}{n}\cdot \log\left(y_0^{+} DB^n(y_0)^{+} DB^n(y_0) y_0\right)$$

$$DB = \begin{pmatrix} 1/3 & 0 \\ 0 & 2 \end{pmatrix}$$ independent of y_0. Thus

$$DB^n = \begin{pmatrix} 1/3^n & 0 \\ 0 & 2^n \end{pmatrix}$$ and letting $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$,

$$\Rightarrow \quad DB\,\hat{e}_1 = \frac{1}{3}\hat{e}_1$$

$$DB\,\hat{e}_2 = 2\,\hat{e}_2$$

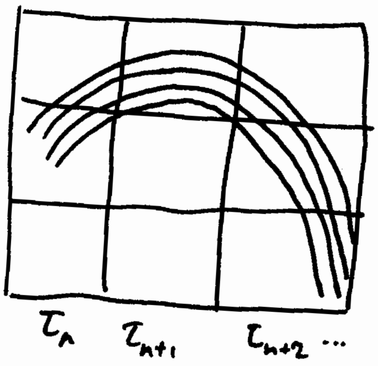
and $h_1 = \lim_{n\to\infty} \frac{1}{2n} \log\left(\frac{1}{3^n}\right)^2 = -\log 3 < 0$

$h_2 = \log 2 > 0$

circles (of the $y=\frac{1}{2}$ line) are stretched and compressed into ellipses with semimajor axis along $\hat{y}$ direction.

$\boxed{\text{Problem \# 2}}$ $K = \lim_{l \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} (K_{n+1} - K_n)$

a. Periodic system: ($\Rightarrow$ deterministic)



$$K_n = -\sum_{i_0 \cdots i_n} P_{i_0 \cdots i_n} \log P_{i_0 \cdots i_n}$$

where $P_{i_0 \cdots i_n}$ is the joint probability that given $x(n)$ is in i_0 at $\tau = 0$, i_1 at $\tau = 1, \ldots, i_{n-1}$ at $\tau = n-1$ then $x(n)$ is @ i_n at $\tau = n$.

But, for any n, $P_{i_0 \cdots i_n} = P_{i_0} \delta_{i_n i_n'} \delta_{i_{n-1} i_{n-1}'} \cdots \delta_{i_1 i_1'}$

thus, $K_n = -\sum_{i_0} P_{i_0} \log P_{i_0} = K_0$.

Now, $K = \lim_{l \to 0} \lim_{N \to \infty} \frac{1}{N} (K_N - K_0) = 0$

b. Chaotic system with one negative Lyapunov exponent and ~~two~~ positive exponents. Now, points spread into a factor of $\bar{N} = e^{\lambda_1 + \lambda_2}$ more squares at each time step. thus

$$P_{i_0} \cdots P_{i_n} = P_{i_0} \cdot \left(\frac{1}{\bar{N}}\right)^n$$

$$K_N = -\sum_{i_0 \cdots i_n} P_{i_0 \cdots i_n} \ln P_{i_0 \cdots i_n} = -\sum_{i_0 \cdots i_n} P_{i_0} \cdot \frac{1}{\bar{N}^n} \log P_{i_0} \cdot \frac{1}{\bar{N}^n}$$

$$= -\sum_{i_0} P_{i_0} \bar{N}^n \frac{1}{\bar{N}^n} \log P_{i_0} \cdot \frac{1}{\bar{N}^n}$$

$= + K_0 + N \log \bar{N}$. Thus $K = \lim_{n \to \infty} \frac{1}{N} (K_N - K_0)$

$= \lambda_1 + \lambda_2$.

c. Stochastic system. All points are evenly mixed in the next time step. Similar to in part

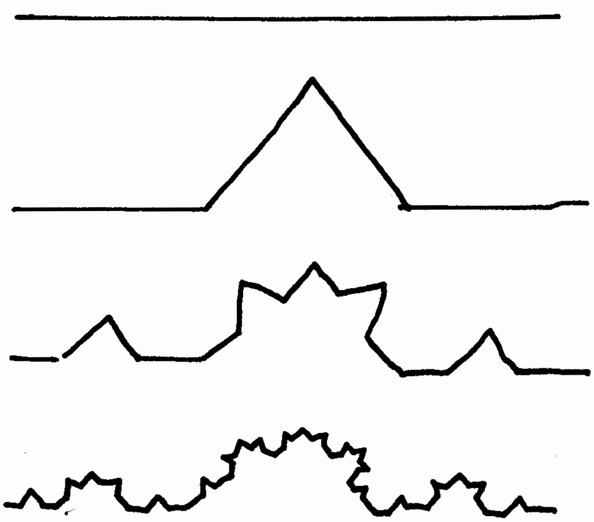
c, $P_{i_0 \cdots i_n} = P_{i_0} \cdot \left(\frac{1}{\bar{N}}\right)^n$ but now, $\bar{N} \to \infty$.

thus $K = \log \bar{N} = \infty$.

Problem #3

construction of Koch curve:

(i)



- segment length 1
- segment no. 1

- segment length $\frac{1}{3}$
- segment no. 4

- segment length $(\frac{1}{3})^2$
- segment no. 4^2

- segment length $(\frac{1}{3})^3$
- segment no. 4^3

$$\Rightarrow \quad d = \lim_{n\to\infty} \frac{\log[N(\varepsilon)]}{\log(1/\varepsilon)} = \lim_{n\to\infty} \frac{\log 4^n}{\log 3^n}$$

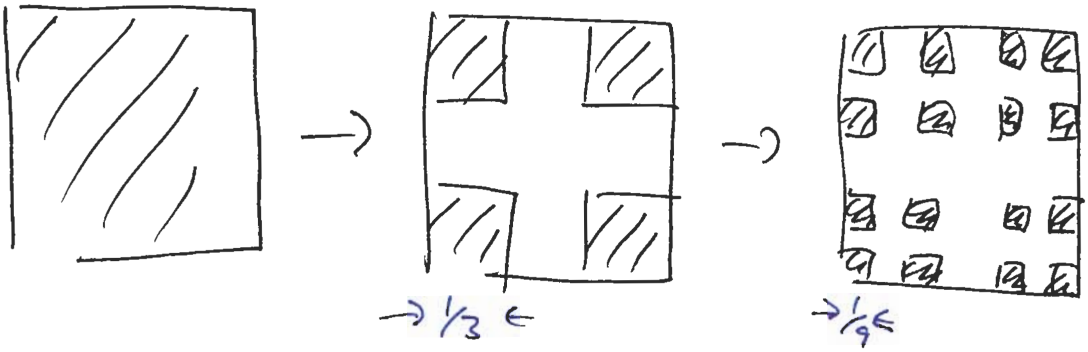
$$= \log(4)/\log(3) = 1.26\ldots$$

Notice the length of this curve is

$\lim_{n\to\infty} \frac{4}{3}\cdot\frac{4}{3}\cdot\frac{4}{3}\cdots \to \infty$! But occupies only a finite area.

That fact is quantified of course by $1 < d < 2$.

(ii)



$$\left.\begin{array}{l} N(\varepsilon) = 4^n \\ \varepsilon = 1/3^n \end{array}\right\} \quad d = \lim_{n\to\infty} \frac{\log 4^n}{\log 3^n} = \frac{\log 4}{\log 3} = 1.26\ldots$$

Same dimension as Koch curve!

$\boxed{\text{part 5.}}$ There are 3^N hypercubes in a single unit and of these, $2N+1$ are removed (accounting for $2N$ faces and 1 center cube.)

Thus, of every cube we cut up $3^N - 2N - 1$ after another iteration.

Hence, at the nth iteration, we're left with $(3^N - 2N - 1)^n$ cubes of length $1/3^n$.

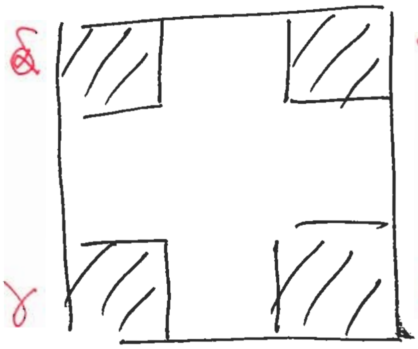
i.e. $$d_N = \lim_{n\to\infty} \frac{\log (3^N - 2N - 1)^n}{\log 3^n}$$

$$= \frac{\log (3^N - 2N - 1)}{\log 3}.$$

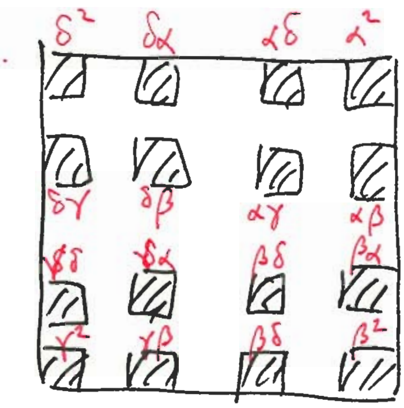
Note, interestingly $d_N \simeq N$ for large N.

Problem # 4

Dimension spectrum and multifractals



$\rightarrow$



$\rightarrow \dots$

$\alpha+\beta+\gamma+\delta=1$
$\alpha,\beta,\gamma,\delta>0.$

$$D_q = \frac{-1}{q-1} \lim_{\epsilon\to 0} \frac{\log I(q,\varepsilon)}{\log 1/\varepsilon} \qquad \varepsilon_n = \frac{1}{3^n}$$

$$I(q,\varepsilon) = \sum_{i=1}^{N(\varepsilon)} \mu_i^q = (\alpha^q+\beta^q+\gamma^q+\delta^q)^n$$

thus,
$$D_q = \frac{-1}{q-1} \lim_{n\to\infty} \frac{\log (\alpha^q+\beta^q+\gamma^q+\delta^q)^n}{\log 3^n}$$

$$= \frac{-1}{q-1} \frac{\log(\alpha^q+\beta^q+\gamma^q+\delta^q)}{\log 3}$$

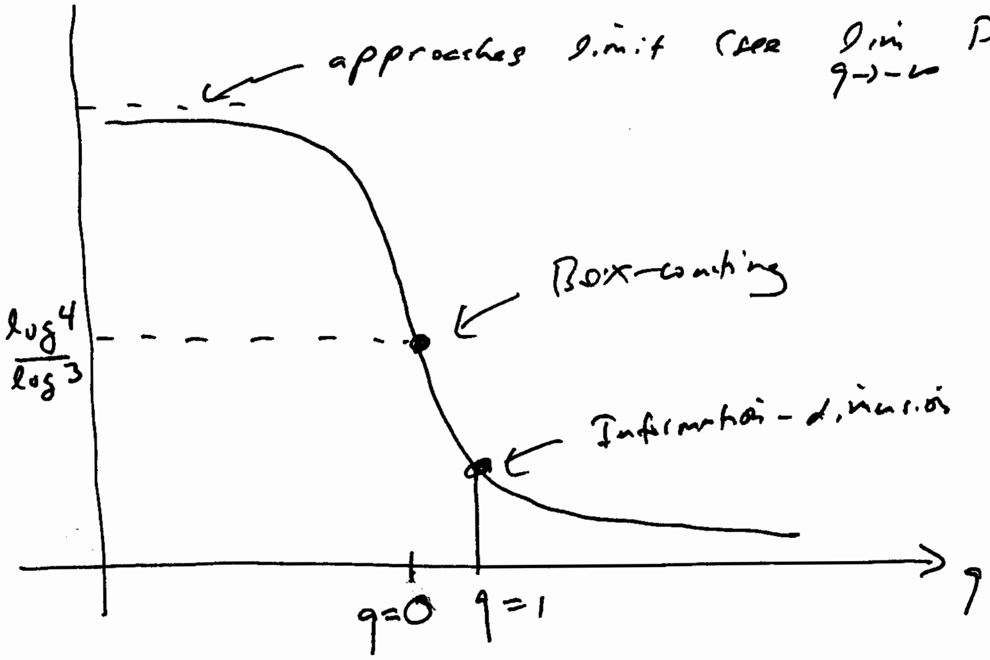
note $\alpha=\beta=\gamma=\delta=1/4 \Rightarrow D_q = \frac{-1}{q-1} \frac{\log 4\cdot(\frac{1}{4})^q}{\log 3} = \frac{\log 4}{\log 3}$

i.e. the Box-counting dimension $= D_0$.

$$D_1 = \frac{\alpha\log\alpha+\beta\log\beta+\gamma\log\gamma+\delta\log\delta}{\log(1/3)} = \text{information dimension}$$
(use L'Hôpital's Rule)

Also:
$$\lim_{q\to\infty} D_q = \frac{-\log \max(\alpha,\beta,\gamma,\delta)}{\log 3}$$

$$\lim_{q\to-\infty} D_q = \frac{-\log \min(\alpha,\beta,\gamma,\delta)}{\log 3}$$



$\boxed{\text{part b.}}$

$$f(\alpha(q)) = -(q-1)\, D_q + q\,\alpha(q)$$

$$\alpha(q) = \frac{d}{dq}\left[(q-1)\, D_q\right] =$$

$$= -\frac{\alpha^q \log \alpha + \beta^q \log \beta + \gamma^q \log \gamma + \delta^q \log \delta}{(\alpha^q + \beta^q + \gamma^q + \delta^q)\, \log 3}$$

Parametric Plot :