Homework #8
Nonlinear dynamics and chaos

1. **Lyapunov exponents:** (a) Plot the regions to which the two halves ($y > 1/2$ and $y < 1/2$) of the unit square are mapped by the baker map:

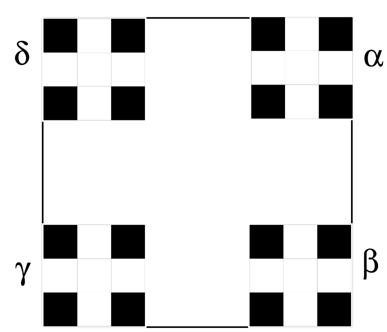
$$B(x,y) = \begin{cases} (\frac{1}{3}x, 2y) & \text{if } 0 \le y \le \frac{1}{2} \\ (\frac{1}{3}x + \frac{2}{3}, 2y-1) & \text{if } \frac{1}{2} < y \le 1 \end{cases} \tag{1}$$

   Plot (using numerical or analytical calculation) the mapping of a small circle of initial conditions by the baker map after one and after two iterations. Consider also the case in which a part of the circle is mapped to the $y = 1/2$ line. (b) Calculate (analytically) the Lyapunov exponents for the baker map. Use your results for the Lyapunov exponents to explain the mapping of a small circle by the baker map (for the case in which it is not mapped to $y = 1/2$).

2. **Kolmogorov entropy:** Follow the heuristic arguments in Schuster (given in the table whose schematic drawings were also made in class; Table 7 page 100 in his 1st edition from 1984), and calculate the Kolmogorov entropy for a periodic system, chaotic system with **two positive and one negative** Lyapunov exponents, and a purely stochastic system. Show explicitly all the steps leading to the final answer, using the definition of the Kolmogorov entropy. You may assume that the expansion rates are the same at all iterations/ steps of the dynamics system.

3. **Fractal dimensions:**

   (a) (i) Calculate the dimension of the Koch curve. (ii) Calculate the box dimension of the following object: take a square, divide it into 9 equal size squares, and remove the middle and side ones as shown in the plot. Repeat this for each of the smaller remaining squares (as shown in figure). Iterate an infinite number of times.



   (b) **Challenge/ extra credit:** (Strogatz 11.3.9, p. 419) Consider a 3d cube, divide it into 27 equal-size cubes, remove the center cube as well as the center cube from each of the faces. This is equivalent to drilling three square holes through the centers of the three faces of the cube. Iterate an infinite number of times. Now the question: find the fractal dimension of the final object in the case where a similar process is done for an arbitrary N-dimensional cube.

4. **Dimension spectrum and multifractals:**

(a) (Ott problem 3.7, page 103) Consider the fractal of question 3a(ii). Remember that the dimension spectrum D_q makes sense only when there is a non-uniform measure on the object being considered. Put a measure on this fractal as follows. Let $\alpha, \beta, \gamma, \delta$ be positive numbers such that $\alpha + \beta + \gamma + \delta = 1$. For the first stage of constructing the fractal, let the measure on the squares be as marked on the above figure. At the next stage, when each square is again divided into 4 smaller squares, let the total measure on the four smaller squares be equal to what it was in the previous iteration, and let this total be divided as in the previous iteration. That is, the fraction of the weight that is assigned to the upper right smaller box is α of the total assigned to the entire larger box in the previous iteration, etc. Calculate D_q and plot it (plot by assuming some values for $\alpha, \beta, \gamma, \delta$).

(b) Try to calculate and plot $f(\alpha)$ for this fractal.