APM203 Homework #9 solutions

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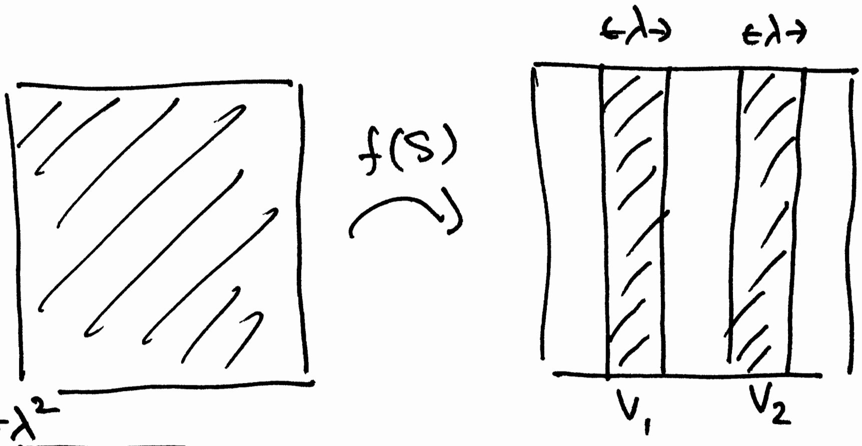
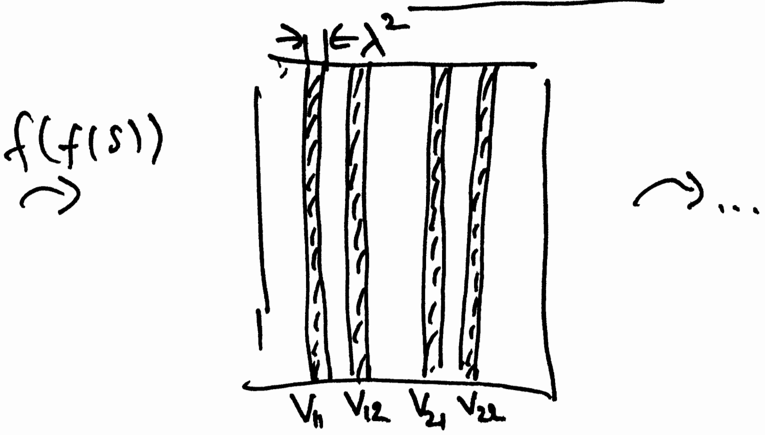
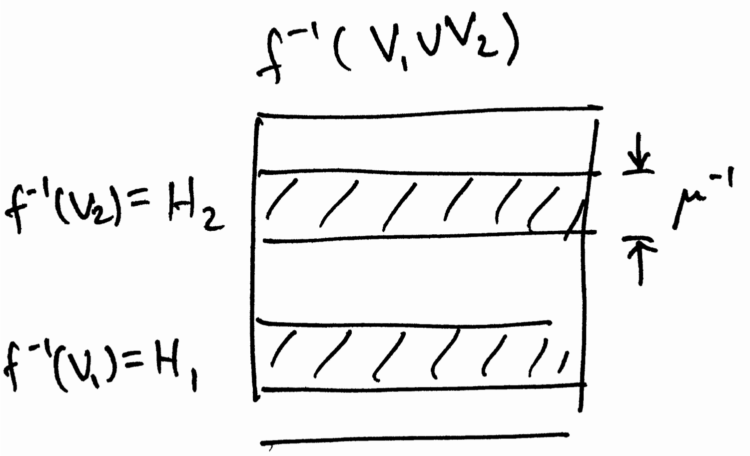
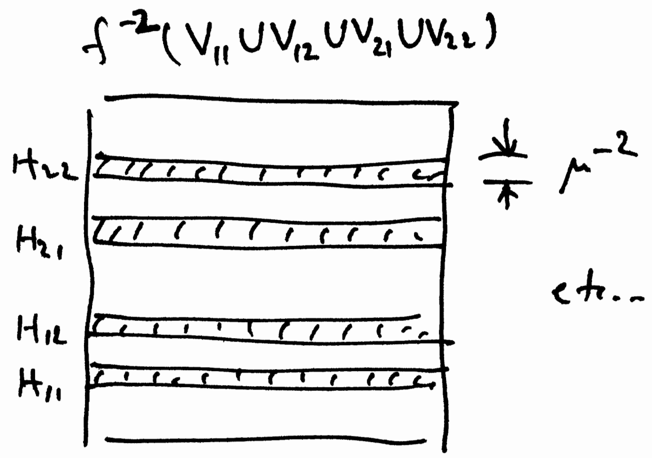
Problem #1









Invariant set is contained in $(H_1 \cup H_2) \cap (V_1 \cup V_2)$, $(H_{11} \cup H_{12} \cup H_{21} \cup H_{22}) \cap (V_{11} \cup V_{12} \cup V_{21} \cup V_{22})$, etc.

Thus, invariant set is fractal with Lebesgue measure zero. Each dimension can be calculated separately and then summed to get full dimension of invariant set. viz.

$$D = D_1 + D_2 = \lim_{n\to\infty} \frac{\log 2^n}{\log 1/\lambda^n} + \lim_{n\to\infty} \frac{\log 2^n}{\log \mu^n}$$

$$= \frac{\log 2}{\log 1/\lambda} + \frac{\log 2}{\log \mu} .$$

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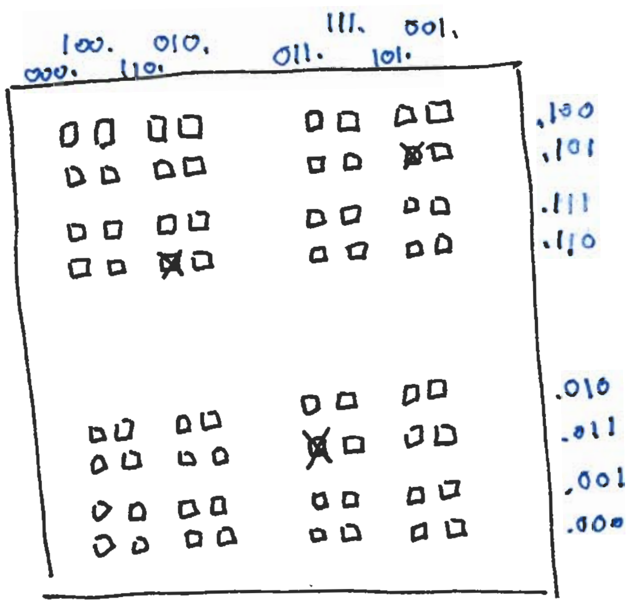
part b.

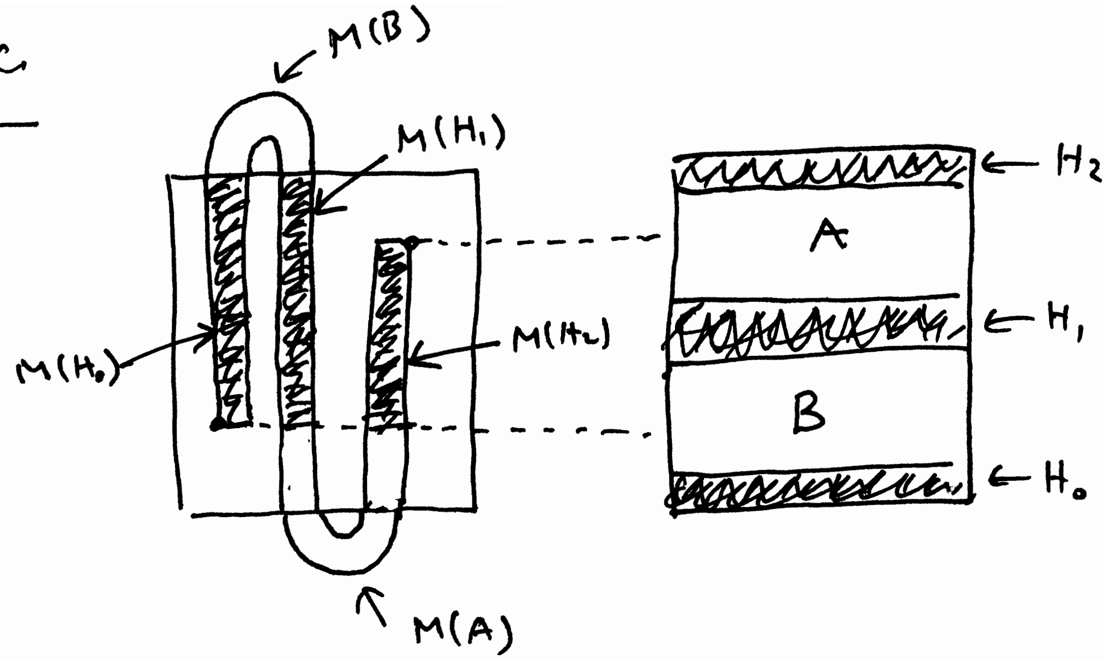


3-CYCLE : $.\overline{011}$ = ...011011.011011...

We can locate the orbit to an accuracy of $\frac{1}{27}$ since these are the cube sizes. The orbit occupies the X'ed squares above :

011.011 → 101.101 → 110.110

part c.



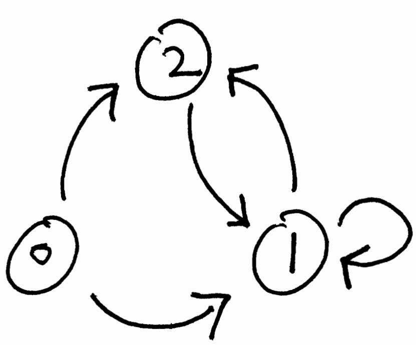
Thus, from picture above, $H_1 \to H_1$ or H_2 in invariant set, $H_0 \to H_1$ or H_2 and $H_2 \to H_1$ only.

This means period-2 orbits may only be:

$.\overline{12}$ (which is also period-4)

period-3 : $.\overline{112}$

period 4: $.\overline{1112}$



period-1 : only $.\overline{1}$. (also period-2 and period-4)

Note - clearly if we leave 0, we don't return to it, so 0 will never appear in orbits that are periodic.

Problem #2

Hénon Map:
$$x_{n+1} = a + b y_n - x_n^2$$
$$y_{n+1} = x_n$$

This map is the same, it's straightforward to see, as the map
$$\bar{x}_{n+1} = 1 + \bar{y}_n - a\bar{x}_n^2$$
$$\bar{y}_{n+1} = b\bar{x}_n$$

if: $a\bar{x}_n \equiv x_n$ and $\frac{a}{b}\bar{y}_n \equiv y_n$.

From the second set of Eqns., f.p.'s satisfy
$$a\bar{x}_0^2 + (1-b)\bar{x}_0 - 1 = 0$$

sol'ns are: $\bar{x}_0 = \frac{1}{2a}\left\{ (b-1) \pm \sqrt{(1-b)^2 + 4a} \right\}$

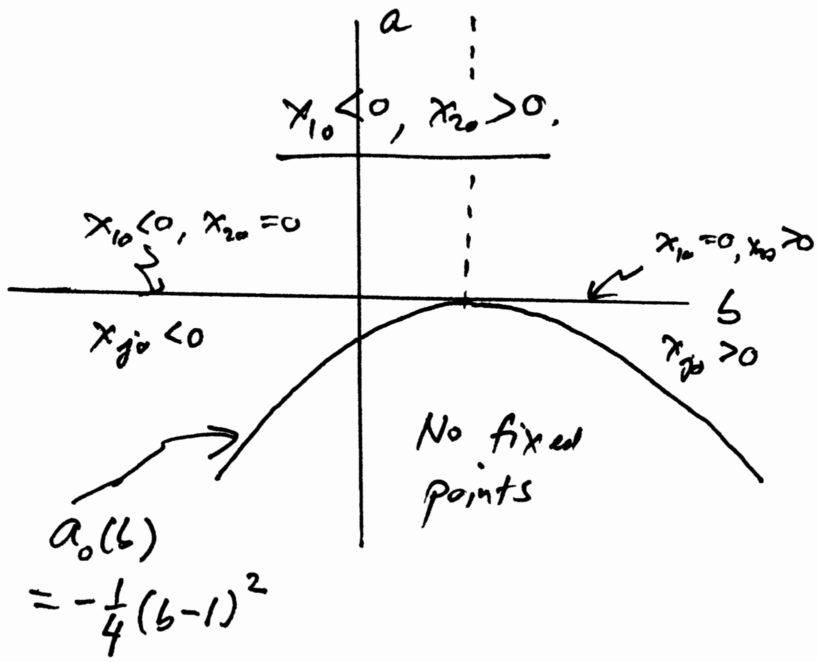
or, in original units,
$$x_0 = \frac{1}{2}\left\{ (b-1) \mp \sqrt{(1-b)^2 + 4a} \right\}.$$

when $a > -\frac{1}{4}(1-b)^2$ there are 2 real sol'ns to quadratic eqn.

corresponding f.p.'s are:
$$(x_{10}, y_{10}) = \left(\frac{1}{2}\left\{ (b-1) - \sqrt{(1-b)^2 + 4a} \right\},\ x_{10} \right)$$
$$(x_{20}, y_{20}) = \left(\frac{1}{2}\left\{ (b-1) + \sqrt{(1-b)^2 + 4a} \right\},\ x_{20} \right)$$

Thus, if $a>0$, there are always two fixed points. One is always negative (e.g. in the 3rd quadrant) the other positive (e.g. in the 1st quadrant).
If $a<0$, Fixed points exist only if $a > -\frac{1}{4}(b-1)^2$ as we've seen above. This means that for a given a, a saddle node bifurcation occurs at $b_<$ and $b_>$ where $b_< = 1-\sqrt{-4a}$ and $b_> = 1+\sqrt{-4a}$. At these bifurcating the newborn fixed points are sitting on top of each other at $(x,y) = \pm\left(\frac{1}{2}\sqrt{-4a}, \frac{1}{2}\sqrt{-4a}\right) = \pm(\sqrt{-a}, \sqrt{-a})$.
Furthermore, for $b<b_<$, $x_{j0}, y_{j0} < 0$ and for $b>b_>$, $x_{j0}, y_{j0} > 0$. Summarize:

a	b	x_{10}	x_{20}
<0	$<b_<$	<0	<0
<0	$>b_>$	>0	>0
<0	$b_< < b < b_>$	D.N.E.	
>0	$-\infty < b < \infty$	<0	>0
$=0$	$<b_<=1$	<0	$=0$
$=0$	$>b_>=1$	$=0$	>0



In the following, we restrict our analysis to $|b| \leq 1$ so that the system is dissipative ($|b|<1$) or area-preserving ($|b|=1$).

Jacobian of Map:

$$D_x F = \begin{pmatrix} -2x_{j0} & b \\ 1 & 0 \end{pmatrix} \Rightarrow \boxed{\lambda_\pm = -x_{j0} \pm \sqrt{x_{j0}^2 + b}}$$

- On $a_0(b)$, we've seen a single f.p. exists (which splits into two for slightly larger a). From $\lambda_\pm$ above, its eigenvalues are $\lambda = \{1, -b\}$ and is thus MARGINALLY STABLE. Note — 2 f.p.'s are developing in a saddle-node bifurcation at 1, the bndy. of the unit circle. So we expect one stable, one unstable f.p. for $a \gtrsim a_0(b)$.

- Statement: $\boxed{\text{f.p. at } x_{10} \text{ is always unstable.}}$

  Proof: we have $x_{10} = \frac{1}{2}\{(b-1) - \sqrt{(b-1)^2 + 4a}\}$

  $= -c - \sqrt{c^2 + a}$ where $c \equiv \frac{1}{2}(1-b)$. $0 \leq c \leq 1$.

  Thus, $\lambda_+ = -x_{10} + \sqrt{x_{10}^2 + b} = c + \sqrt{c^2+a} + \sqrt{(c+\sqrt{c^2+a})^2 + \underset{1-2c}{\underset{\uparrow}{b}}}$

  now $a \geq -c^2 \Rightarrow \lambda_+ \geq c + \sqrt{c^2 - 2c + 1} = 1$.

  Thus x_{10} is unstable. (any $a > a_0(b)$.)

- statement: $\boxed{\text{f.p. at } x_2 \text{ is stable for a slightly larger than } a_0(b).}$ i.e. $a \gtrsim -c^2$. More precisely, let $a = -c^2 + \varepsilon^2$ (where ε^2 is positive and small.

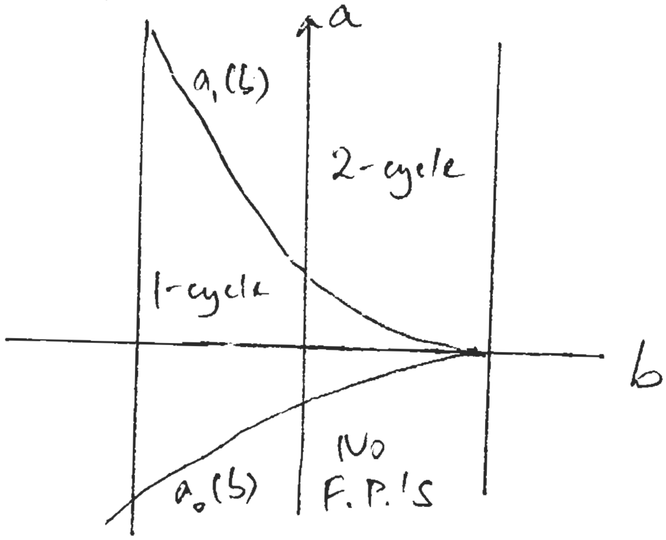
Then $\lambda_\pm = c - \varepsilon \pm (1-c)\left(1 - \frac{c\varepsilon}{(1-c)^2}\right) =$

$$= \begin{cases} 1 - \frac{\varepsilon}{1-c} \\ (2c-1)\left(1 + \frac{\varepsilon}{1-c}\right) \end{cases}$$

which are both smaller than 1 in magnitude if b is $o(\varepsilon)$ away from 1 and -1, which we'll take for granted.

- x_{2*} destabilizes (one eigenvalue crosses over -1) at $a = \frac{3}{4}(b-1)^2$: let $\lambda = -1 \Rightarrow x = -\left(\frac{b-1}{2}\right)$ which, we've seen, is only possible for x_{2*} since $x_{1*} < 0$. plugging into eqn. for x_{2*} $\Rightarrow a = \frac{3}{4}(b-1)^2$ ✓.

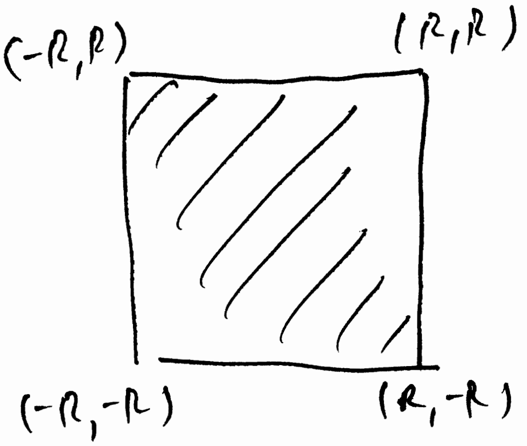
It can be shown that the map has a stable 2-cycle above $a_1(b) = \frac{3}{4}(b-1)^2$. (for $|b| < 1$.)



$\boxed{\text{part d. (and g.)}}$

$$\rho^2 + (|b|+1)\rho - a = 0$$

then:
$$R = \frac{|b|+1}{2} + \frac{1}{2}\sqrt{(|b|+1)^2 + 4a}$$



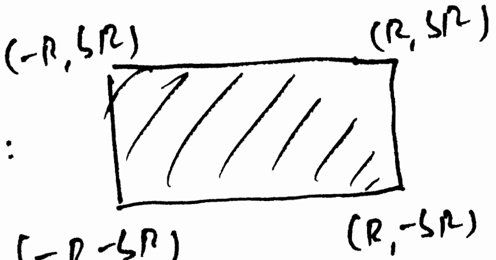
Note first that Hénon map may be written as 3 successive transformations $T'''T''T'$ viz.

$$T': \left.\begin{aligned} x' &= x \\ y' &= by \end{aligned}\right\} \text{ compressive}$$

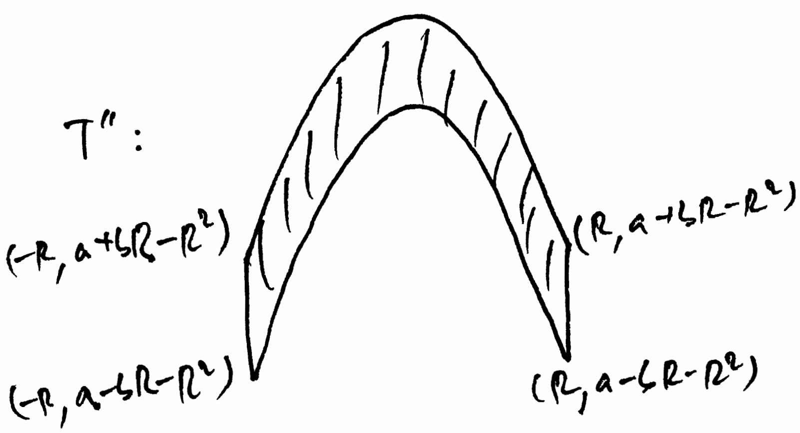
$$T'': \left.\begin{aligned} x' &= x \\ y' &= a + y - x^2 \end{aligned}\right\} \begin{array}{l}\text{area-preserving} \\ \text{and stretches.}\end{array}$$

$$T''': \left.\begin{aligned} x' &= y \\ y' &= x \end{aligned}\right\} \begin{array}{l}\text{area-preserving} \\ \text{(reflects across} \\ \text{y=x line.)}\end{array}$$

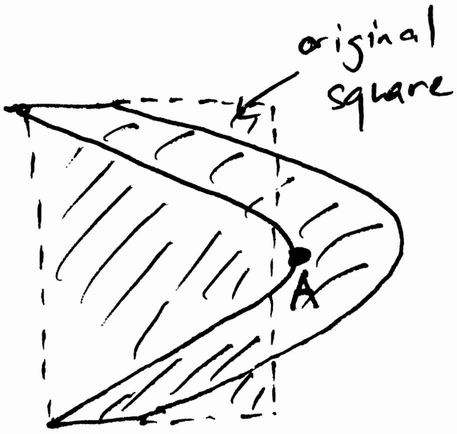
T':



T'':



$\rightarrow T'''$



Thus, we notice that the map has similar squeezing and stretching properties as the horseshoe map. Furthermore, the point labeled A traverses the right side of the square as b changes for a given a. A is on the side when

$x' = a - bR = R$. thus, we get stripes

when $a - bR > R$ or $b < \frac{a-R}{R}$.
