Homework #11
Nonlinear dynamics and chaos

1. Find the fixed points of the standard map,

$$\theta_{n+1} = \theta_n + p_n \mod 2\pi; \qquad p_{n+1} = p_n + K \sin\theta_{n+1}$$

in the region $-\pi < p < \pi$. Determine their stability as function of K. In what range of $K \geq 0$ is there an elliptic fixed point?

2. *Canonical perturbation theory for a simple pendulum:* Consider a simple pendulum under gravity,

$$H = \frac{1}{2}Gp^2 - F\cos(\phi)$$

   (a) Expand Hamiltonian in a Taylor series,

$$H = C_1 p^2 + C_2\phi^2 + \varepsilon C_4 \phi^4 + \varepsilon^2 C_6 \phi^6 + \ldots,$$

   find the constants in the expansion, and treat the deviations from harmonic oscillator as a perturbation, retaining only lowest order perturbations, $O(\phi^4)$. The ε has been inserted above to conveniently identify the perturbation terms.

   (b) Derive the action-angle variables (J,θ) for the unperturbed harmonic oscillator, following the outline done in class.

   (c) Write the perturbed Hamiltonian in terms of the AA variables of the unperturbed one. The resulting expression has $\sin^4\theta$ and $\sin^6\theta$ etc. Expand this in Fourier series and show that

$$H_0 = \omega_0 J; \qquad H_1 = C(3 - 4\cos 2\theta + \cos 4\theta)$$

   and find ω_0 and the constant C.

   (d) Deduce the order epsilon correction to the frequency, showing that

$$\omega = \frac{\partial \bar{H}}{\partial \bar{J}} = \omega_0 - \frac{\varepsilon}{8} G\bar{J},$$

   and interpret the results physically.

   (e) Use the relation derived in class

$$\omega \frac{\partial S_1}{\partial \bar{\theta}} = H_1 - \langle H_1 \rangle,$$

   integrate it and show also that the $O(\varepsilon)$ correction to the generating function is given by

$$S_1 = -\frac{GJ^2}{192\omega_0}(8\sin 2\theta - \sin 4\theta).$$

   (f) Find the new AA variables to order ε using the expression for the generating function and

$$\theta = \bar{\theta} - \varepsilon \frac{\partial S_1(\bar{J},\bar{\theta})}{\partial \bar{J}}$$

3. *Chaotic mixing in 2d flows:* Consider the two-dimensional flow given by the stream function

$$\psi(x,y,t) = \sin kx \sin \pi y + \varepsilon \cos \omega t \cos kx \cos \pi y.$$

   (a) Choose k, ω and plot the flow (both the stream function and the velocity field using something like "quiver" in Matlab) for $\varepsilon = 0$ and also for small ε at times $0, \pi/\omega, 3\pi/2\omega, 2\pi/\omega$.

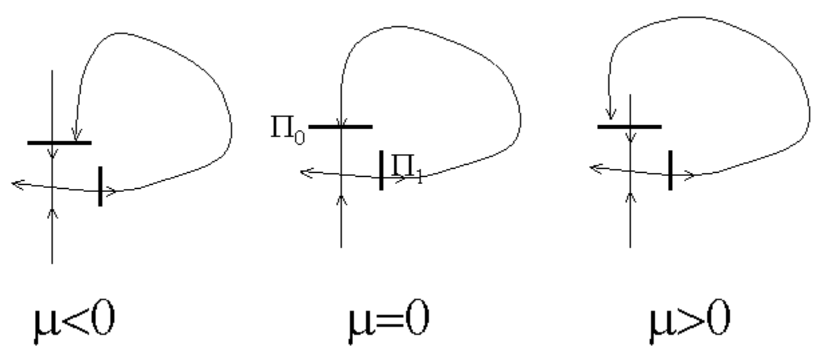
   (b) Solve for fluid trajectories $x(t), y(t)$ in case $\varepsilon = 0$, showing therefore that the system is integrable then. What do the trajectories look like?

   (c) Solve and plot with $\varepsilon > 0$, for the trajectories of a few fluid particles that are initially nearby. This needs to be done numerically, by integrating the appropriate two coupled ODEs for $x(t), y(t)$. Can you see sensitivity to initial conditions of the trajectories? Can you see how the an initial patch of initial conditions undergoes stretching and folding by the flow?

4. *Using the methods used in class for studying the Shilnikov phenomenon:* (Vered Rom-Kedar) Consider a two dimensional autonomous system depending on one parameter μ:

$$\frac{dx}{dt} = f(x,y;\mu), \frac{dy}{dt} = g(x,y;\mu).$$

Assume the system has a saddle fixed point at the origin, with eigenvalues $\lambda > 0 > \gamma$ so that for small μ the following global homoclinic bifurcation occurs:



Follow the following steps to find when a limit cycle is created, i.e. for a given λ, γ is it created for $\mu > 0$ or for $\mu < 0$?

   (a) Construct a local map from a small cros-section Π_0 near the fixed to the cross-section Π_1 using the linearized dynamics.

   (b) Explain why the following form may be used for the return map from Π_1 to Π_0:

$$\begin{pmatrix} x' \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ y \end{pmatrix} + \begin{pmatrix} -\mu \\ \varepsilon \end{pmatrix} + O(2)$$

   where $a > 0$.

   (c) Write the composed map from Π_0 to itself (consider only the equation for x).

(d) Find the fixed point of this map and its stability (consider the two cases $\frac{\lambda}{|\gamma|} < 1$, $\frac{\lambda}{|\gamma|} > 1$ separately; use the fact that x is small where the map is evaluated). What does the existence and stability of the fixed points of the map tell us about the original dynamical system? Can anything be said for the case $\frac{\lambda}{|\gamma|} = 1$? Why?

5. **Challenge/ extra credit:** show that the second order correction to the Hamiltonian of question (2) is

$$H_2 = \frac{G^2 J^3}{2880\omega_0}(10 - 15\cos 2\theta + 5\cos 4\theta - \cos 6\theta),$$

find and plot the first and second order corrections to the frequency as function of the energy.

6. **Optional:** Read some interesting facts about the proof of the KAM theorem and its history in "Math-World", see link in the "Misc" section at the bottom of the course home page. Browse the proof of the KAM theorem in the lecture by James Colliander (see link in course home page).