EPS 131 Problem set 4 SOLUTIONS

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① The relavent terms in the momentum equation are

$$\frac{\partial u}{\partial t} - fv = -ru$$

$$\frac{\partial v}{\partial t} + fu = -rv$$

$\frac{\partial}{\partial t}\vec{u} = \begin{pmatrix} -r & f \\ -f & -r \end{pmatrix}\vec{u} \Rightarrow$ guess $\vec{u} = \vec{u}_0 e^{at}$, which gives

$a\vec{u} = \begin{pmatrix} -r & f \\ -f & -r \end{pmatrix}\vec{u}$, or

$0 = \begin{pmatrix} -r-a & f \\ -f & -r-a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, so the determinant must be zero

$0 = (-r-a)^2 + f^2 \Rightarrow (r+a)^2 = -f^2 \Rightarrow r + a = \pm if \Rightarrow a = \pm if - r$

$\vec{u} = \mathrm{Re}[\vec{u}_0 e^{(\pm if - r)t}] = \vec{u}_0 e^{-rt} \cos ft$ as in class

u_0 and v_0 are related. Plug $\vec{u} = \vec{u}_0 e^{at}$ into first equation:

$a u_0 e^{at} - f v_0 e^{at} = -r u_0 e^{at}$

$\boxed{v_0 = \left(\frac{a+r}{f}\right) u_0 = \pm i u_0}$.

$v = \mathrm{Re}[\pm i u_0 e^{\pm ift} e^{-r}] = -u_0 e^{-r} \sin ft$. So we have the velocities,

$$\boxed{\begin{aligned} u &= u_0 e^{-rt} \cos ft \\ v &= -u_0 e^{-rt} \sin ft \end{aligned}}$$

Only u_0, the radius, is unspecified.

Reasonable assumption: u_0 is constant (doesn't vary with (x,y)).

$\frac{d\vec{x}}{dt} = \vec{u} = \vec{u}_0 e^{(if-r)t}$ [real part implied, $+if$ solution chosen]

$\vec{x} = \frac{\vec{u}_0}{(if-r)} e^{(if-r)t} + \vec{c}_1$ [integrated to get particle trajectory; $\vec{c}_1$ is constant]

Taking the real part gives

$$\boxed{\begin{aligned} x &= \frac{u_0}{f^2+r^2} e^{-rt} (-r\cos ft + f \sin ft) \\ y &= \frac{u_0}{f^2+r^2} e^{-rt} (f \cos ft + r \sin ft) \end{aligned}}$$

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x = u0/(f^2 + r^2) e^(-r t) (-r Cos[f t] + f Sin[f t]); y = u0/(f^2 + r^2) e^(-r t) (f Cos[f t] + r Sin[f t]);
u0 = 1; f = 1; r = 0.1; ParametricPlot[{x, y}, {t, 0, 20}, Axes → False, Frame → True]
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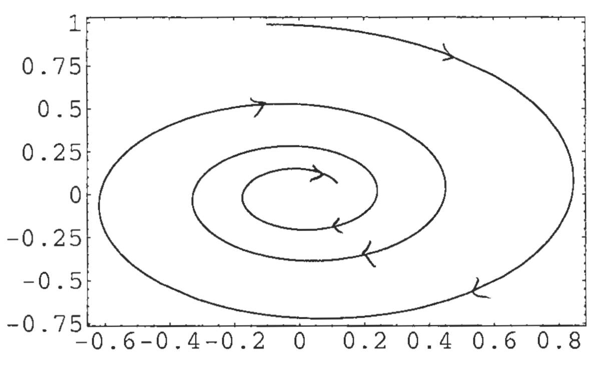
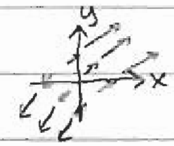
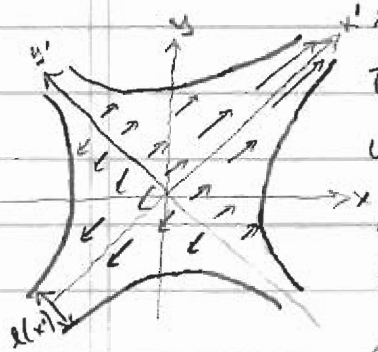


using Mathematica to generate plot.

(2) The idea is that deep water is both cold and nutrient rich, so regions with upwelling should have cold sea surface temperature and green color (from phytoplankton thriving on the nutrients). This idea has decent, but not terrific, agreement with data (better in some upwelling regions than others)

(3) 

The flow is always in the ⤢ direction, suggesting a transformation $y' \nwarrow \nearrow x'$



$x' = x+y, \quad y' = y-x \Rightarrow u' = u+v = 2ax', \ v' = u-v = 0$

Fluid is conserved along the channel, so $u'(x') \cdot l(x')$ should be constant in x', the long-channel distance, where l is the width.

So $l(x')\,u'(x') = c_1 \Rightarrow l = \frac{c_1}{u'} = \frac{c_1}{2ax'} = \frac{c_2}{x'}$;

c_1 and c_2 are constants, i.e.,

$\boxed{l = \frac{c_2}{x'} = \frac{c_2}{x+y}}$ is the channel width

Eulerian: $a = \frac{du'}{dt} = u'\frac{\partial u'}{\partial x'} = (2ax')(2a) = \boxed{4a^2x'} = 4a^2(x+y)$

Lagrangian: $\frac{dx'}{dt} = u' = 2ax' \Rightarrow x' = x_0 e^{2at} \Rightarrow \frac{d^2x'}{dt^2} = a = \boxed{4a^2 x_0' e^{2at}}$

(4) $f(-v,u) = -\frac{1}{\rho_0}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) p$

$(u,v) = \frac{1}{f\rho_0}\left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right) p$

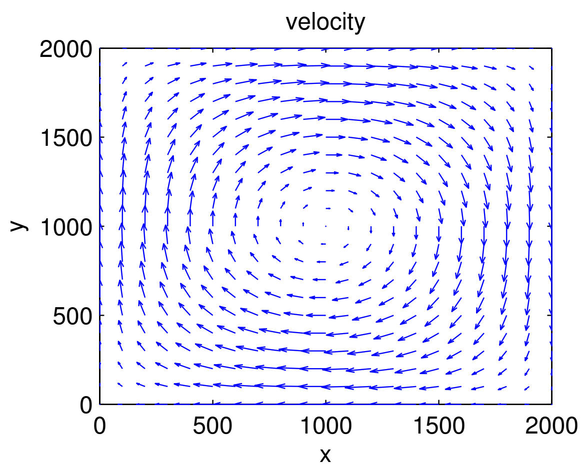
$$\boxed{\begin{aligned} u &= -\frac{1}{f\rho_0}\,\frac{\pi}{a}\,\sin\frac{\pi x}{a}\,\cos\frac{\pi y}{a}\,Hg\rho_0 \\ v &= \frac{1}{f\rho_0}\,\frac{\pi}{a}\,\cos\frac{\pi x}{a}\,\sin\frac{\pi y}{a}\,Hg\rho_0 \end{aligned}}$$

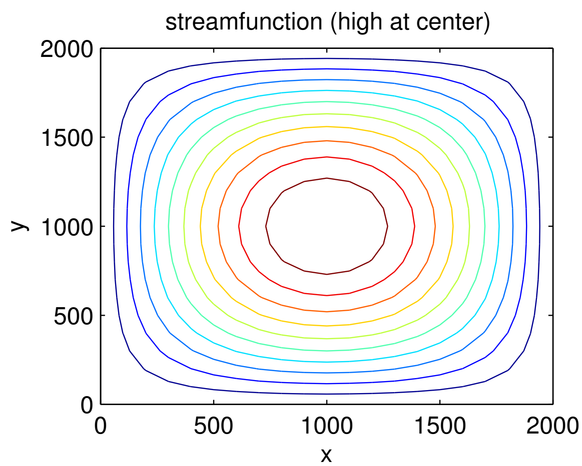
The stream function Ψ is defined such that

$(u,v) = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)\Psi$, so $\boxed{\Psi = \frac{1}{f\rho_0}p}$, with units of $\boxed{\frac{m^2}{s}}$.

See attached Matlab code and figures

4. cont'd



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%matlab code to generate velocity and stream function plots (above)
H=1;g=9.8;p0=1024;a=2000;f=2*(2*pi/(360*24*3600))*sin(40*pi/180);
[x,y]=meshgrid(0:100:2000,0:100:2000);
p=sin(pi*x/a).*sin(pi*y/a)*H*g*p0;
u=-1/(f*p0)*pi/a*sin(pi*x/a).*cos(pi*y/a)*H*g*p0;
v=1/(f*p0)*pi/a*cos(pi*x/a).*sin(pi*y/a)*H*g*p0;
subplot(2,2,1)
quiver(x,y,u,v)
set(gca,'XLim',[0 2000],'YLim',[0 2000])
xlabel('x'),ylabel('y'),title('velocity')
subplot(2,2,2)
contour(x,y,p/(f*p0),10)
xlabel('x'),ylabel('y'),title('streamfunction (high at center)')
```