# Solving the diffusion-advection equation using finite differences

Ian, 4/27/04

We want to numerically find how a chemical concentration (or temperature) evolves with time in a 1-D pipe filled with fluid flowing at velocity $u$, i.e., to compute C(x,t) given C(x,0). C(x,t) evolves according to the diffusion-advection equation,

$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} = \kappa\frac{\partial^2 C(x,t)}{\partial x^2} \tag{1}$$

Approximate the derivatives using the centered difference scheme:

$$\frac{\partial C_{i,j}}{\partial t} \approx \frac{C_{i,j+1} - C_{i,j-1}}{2\Delta t} \tag{2}$$

$$\frac{\partial C_{i,j}}{\partial x} \approx \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \tag{3}$$

$$\frac{\partial^2 C_{i,j}}{\partial x^2} \approx \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} \tag{4}$$

Inserting (2)-(4) into (1) gives

$$\frac{C_{i,j+1} - C_{i,j-1}}{2\Delta t} + u\frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \approx \kappa\frac{C_{i+1,j-1} - 2C_{i,j-1} + C_{i-1,j-1}}{\Delta x^2} \tag{5}$$

(Note that we evaluate the diffusion term at time $j-1$ instead of time $j$ to avoid numerical instability.) Rephrasing the goal above, we want to compute $C_{i,j+1}$ (concentration at next time step) given $C_{i,j}$ and $C_{i,j-1}$ (concentration at current and previous time steps). Solve (6) for $C_{i,j+1}$:

$$C_{i,j+1} \approx C_{i,j-1} - u\frac{\Delta t}{\Delta x}(C_{i+1,j} - C_{i-1,j}) + \kappa\frac{2\Delta t}{\Delta x^2}(C_{i+1,j-1} - 2C_{i,j-1} + C_{i-1,j-1}) \tag{6}$$

We have solved for the diffusion-advection equation for the time evolution using centered finite difference schemes in time and space. This method, represented by (6), is called the *leapfrog scheme*. It is implemented in Matlab in `pipe_1d_tracer.m`.