Large-step semantics, continued

Lecture 4 Thursday, February 4, 2016

Last lecture we saw inference rules for a large-step semantics for our arithmetic language.

To see how we use these rules, here is a proof tree that shows that $\langle \mathsf{foo} := 3; \mathsf{foo} \times \mathsf{bar}, \sigma\rangle \Downarrow \langle 21, \sigma'\rangle$ for a store σ such that $\sigma(\mathsf{bar}) = 7$, and $\sigma' = \sigma[\mathsf{foo} \mapsto 3]$.

$$\textsc{Asg}_{\textsc{Lrg}}\ \dfrac{\textsc{Int}_{\textsc{Lrg}}\ \dfrac{}{\langle 3, \sigma\rangle \Downarrow \langle 3, \sigma\rangle} \qquad \textsc{Mul}_{\textsc{Lrg}}\ \dfrac{\textsc{Var}_{\textsc{Lrg}}\ \dfrac{}{\langle \mathsf{foo}, \sigma'\rangle \Downarrow \langle 3, \sigma'\rangle} \qquad \textsc{Var}_{\textsc{Lrg}}\ \dfrac{}{\langle \mathsf{bar}, \sigma'\rangle \Downarrow \langle 7, \sigma'\rangle}}{\langle \mathsf{foo} \times \mathsf{bar}, \sigma'\rangle \Downarrow \langle 21, \sigma'\rangle}}{\langle \mathsf{foo} := 3; \mathsf{foo} \times \mathsf{bar}, \sigma\rangle \Downarrow \langle 21, \sigma'\rangle}$$

A closer look to this structure reveals the relation between small step and large-step evaluation: a depth-first traversal of the large-step proof tree yields the sequence of one-step transitions in small-step evaluation.

1 Equivalence of semantics

So far, we have specified the semantics of our language of arithmetic expressions in two different ways: small-step operational semantics and large-step operational semantics. Are they expressing the same meaning of arithmetic expressions? Can we show that they express the same thing?

Theorem (Equivalence of semantics)**.** *For all expressions e, stores σ and σ', and integers n, we have:*

$$\langle e, \sigma\rangle \Downarrow \langle n, \sigma'\rangle \iff \langle e, \sigma\rangle \longrightarrow^* \langle n, \sigma'\rangle.$$

Proof sketch.

- $\Longrightarrow$. We proceed by structural induction on expressions e. The inductive hypothesis is:

  $$P(e) = \forall \sigma, \sigma' \in \textbf{Store}.\ , \forall n \in \textbf{Int}.\ \langle e, \sigma\rangle \Downarrow \langle n, \sigma'\rangle \Longrightarrow \langle e, \sigma\rangle \longrightarrow^* \langle n, \sigma'\rangle$$

  We have to consider each of the possible axioms and inference rules for constructing an expression.

  - **Case $e \equiv n$.**
    Here, we are consider the case where expression e is equal to some integer n. But then $\langle n, \sigma\rangle \longrightarrow^* \langle n, \sigma\rangle$ holds trivially because of reflexivity of $\longrightarrow^*$.
  - **Case $e \equiv x$.**
    Here, we are considering the case where the expression e is equal to some variable x. Assume that for some σ, σ', and n we have $\langle x, \sigma\rangle \Downarrow \langle n, \sigma'\rangle$. That means that there is some derivation using the axioms and inference rules of the large-step operational semantics, whose conclusion is $\langle x, \sigma\rangle \Downarrow \langle n, \sigma'\rangle$. There is only one rule whose conclusion could look like this, the rule Var_{Lrg}. That rule requires that $n = \sigma(x)$, and that $\sigma' = \sigma$.

    (This reasoning is an example of *inversion*: using the inference rules in reverse. That is, we know that some conclusion holds—$\langle x, \sigma\rangle \Downarrow \langle n, \sigma'\rangle$—and we examine the inference rules to determine which rule must have been used in the derivation, and thus which premises must be true, and which side conditions satisfied.)

    Since $n = \sigma(x)$ we know that $\langle x, \sigma\rangle \longrightarrow \langle n, \sigma\rangle$ also holds, by using the small-step axiom $\textsc{Var}$. So we can conclude that $\langle x, \sigma\rangle \longrightarrow^* \langle n, \sigma\rangle$ holds, which is what we needed to show.

- **Case** $e \equiv e_1 + e_2$**.**
  This is an inductive case. Expressions e_1 and e_2 are subexpressions of e, and so we can assume that $P(e_1)$ and $P(e_2)$ hold. We need to show that $P(e)$ holds. Let's write out $P(e_1)$, $P(e_2)$, and $P(e)$ explicitly.

$$\begin{aligned}
P(e_1) &= \forall n, \sigma, \sigma' : \langle e_1, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \Longrightarrow \langle e_1, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle \\
P(e_2) &= \forall n, \sigma, \sigma' : \langle e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \Longrightarrow \langle e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle \\
P(e) &= \forall n, \sigma, \sigma' : \langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \Longrightarrow \langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle
\end{aligned}$$

  Assume that for some σ, σ' and n we have $\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$. We now need to show that $\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$.
  We assumed that $\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$. Let's use inversion again: there is some derivation whose conclusion is $\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$. By looking at the large-step semantic rules, we see that only one rule could possible have a conclusion of this form: the rule ADD_{LRG}. So that means that the last rule use in the derivation was ADD_{LRG}. But in order to use the rule ADD_{LRG}, it must be the case that $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$ and $\langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle$ hold for some n_1 and n_2 such that $n = n_1 + n_2$ (i.e., there is a derivation whose conclusion is $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$ and a derivation whose conclusion is $\langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle$).
  Using the inductive hypothesis $P(e_1)$, since $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$, we must have $\langle e_1, \sigma \rangle \longrightarrow^* \langle n_1, \sigma'' \rangle$. Similarly, by $P(e_2)$, we have $\langle e_2, \sigma'' \rangle \longrightarrow^* \langle n_2, \sigma \rangle$. By Lemma 1 below, we have

$$\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n_1 + e_2, \sigma'' \rangle$$

  and by another application of Lemma 1 we have

$$\langle n_1 + e_2, \sigma'' \rangle \longrightarrow^* \langle n_1 + n_2, \sigma' \rangle$$

  and by the rule ADD we have

$$\langle n_1 + n_2, \sigma' \rangle \longrightarrow \langle n, \sigma' \rangle.$$

  Thus, we have $\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$, which proves this case.
- **Case** $e \equiv e_1 \times e_2$**.** Similar to the case $e = e_1 + e_2$ above.
- **Case** $e \equiv x := e_1; e_2$**.** Omitted. Try it as an exercise.

• $\Longleftarrow$. We proceed by mathematical induction on the number of steps $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$.

  - **Base case.** If $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ in zero steps, then we must have $e \equiv n$ and $\sigma' = \sigma$. Then, $\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle$ by the large-step operational semantics rule INT_{LRG}.
  - **Inductive case.** Assume that $\langle e, \sigma \rangle \longrightarrow \langle e'', \sigma'' \rangle \longrightarrow^* \langle n, \sigma' \rangle$, and that (the inductive hypothesis) $\langle e'', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$. That is, $\langle e'', \sigma'' \rangle \longrightarrow^* \langle n, \sigma' \rangle$ takes m steps, and we assume that the property holds for it ($\langle e'', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$), and we are considering $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$, which takes $m+1$ steps. We need to show that $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$. This follows immediately from Lemma 2 below.

□

Lemma 1. *If* $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ *then for all* n_1, e_2 *the following hold.*

- $\langle e + e_2, \sigma \rangle \longrightarrow^* \langle n + e_2, \sigma' \rangle$
- $\langle e \times e_2, \sigma \rangle \longrightarrow^* \langle n \times e_2, \sigma' \rangle$
- $\langle n_1 + e, \sigma \rangle \longrightarrow^* \langle n_1 + n, \sigma' \rangle$
- $\langle n_1 \times e, \sigma \rangle \longrightarrow^* \langle n_1 \times n, \sigma' \rangle$

Proof. By (mathematical) induction on the number of evaluation steps in $\longrightarrow^*$. □

Lemma 2. *For all* e, e', σ, *and* n, *if* $\langle e, \sigma \rangle \longrightarrow \langle e', \sigma'' \rangle$ *and* $\langle e', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$, *then* $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$.