Denotational Semantics

CS 152 (Spring 2020)

Harvard University

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Today, we will learn about

- Modeling programs as functions from input stores to output stores
- Denotational model
- Fixed point of a function

Program models

- Operational model
- Denotational model

Programs as functions

Programs as functions

For a program c, we write $\mathcal{C}[\![c]\!]$ for the *denotation* of c, that is, the mathematical function that c represents:

$$\mathcal{C}[\![c]\!] : \textbf{Store} \rightharpoonup \textbf{Store}.$$

Programs as functions

For a program c, we write $\mathcal{C}[\![c]\!]$ for the *denotation* of c, that is, the mathematical function that c represents:

$$\mathcal{C}[\![c]\!] : \textbf{Store} \rightharpoonup \textbf{Store}.$$

We write $\mathcal{C}[\![c]\!]\sigma$ for the result of applying this function to to the store σ.

Expressions as functions

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Similarly, $\mathcal{A}[\![a]\!]$ and $\mathcal{B}[\![b]\!]$ are denotations for arithmetic and boolean expressions.

$$\mathcal{A}[\![a]\!] : \textbf{Store} \to \textbf{Int}$$
$$\mathcal{B}[\![b]\!] : \textbf{Store} \to \{\textbf{true}, \textbf{false}\}$$

Expressions as functions

Note that $\mathcal{A}[\![a]\!]$ and $\mathcal{B}[\![b]\!]$ are total functions.

$$\mathcal{A}[\![a]\!] : \textbf{Store} \to \textbf{Int}$$
$$\mathcal{B}[\![b]\!] : \textbf{Store} \to \{\textbf{true}, \textbf{false}\}$$

Arithmetic expressions

$$\mathcal{A}[\![n]\!] = \{(\sigma, n)\}$$

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$$\mathcal{A}[\![x]\!] = \{(\sigma, \sigma(x))\}$$

Arithmetic expressions

$$
\begin{aligned}
\mathcal{A}[\![n]\!] &= \{(\sigma, n)\} \\
\mathcal{A}[\![x]\!] &= \{(\sigma, \sigma(x))\} \\
\mathcal{A}[\![a_1 + a_2]\!] &= \{(\sigma, n) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \\
&\qquad \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \\
&\qquad \wedge n = n_1 + n_2\}
\end{aligned}
$$

Arithmetic expressions

$$
\begin{aligned}
\mathcal{A}[\![n]\!] &= \{(\sigma, n)\} \\
\mathcal{A}[\![x]\!] &= \{(\sigma, \sigma(x))\} \\
\mathcal{A}[\![a_1 + a_2]\!] &= \{(\sigma, n) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \\
&\qquad \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \\
&\qquad \wedge n = n_1 + n_2\} \\
\mathcal{A}[\![a_1 \times a_2]\!] &= \{(\sigma, n) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \\
&\qquad \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \\
&\qquad \wedge n = n_1 \times n_2\}
\end{aligned}
$$

Boolean expressions

$$\mathcal{B}[\![\textbf{true}]\!] = \{(\sigma, \textbf{true})\}$$
$$\mathcal{B}[\![\textbf{false}]\!] = \{(\sigma, \textbf{false})\}$$

Boolean expressions

$$
\begin{aligned}
\mathcal{B}[\![\textbf{true}]\!] &= \{(\sigma, \textbf{true})\} \\
\mathcal{B}[\![\textbf{false}]\!] &= \{(\sigma, \textbf{false})\} \\
\mathcal{B}[\![a_1 < a_2]\!] &= \{(\sigma, \textbf{true}) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \\
&\qquad\quad \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \wedge n_1 < n_2\} \\
&\qquad\quad \cup\ \{(\sigma, \textbf{false}) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \\
&\qquad\quad \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \\
&\qquad\quad \wedge n_1 \geq n_2\}
\end{aligned}
$$

Denotations of some programs

Denotations of some programs

$$\mathcal{C}[\![\textsf{\textbf{skip}}]\!] = \{(\sigma, \sigma)\}$$

$$\mathcal{C}[\![x := a]\!] = \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in \mathcal{A}[\![a]\!]\}$$

$$\mathcal{C}[\![c_1; c_2]\!] = \{(\sigma, \sigma') \mid \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c_1]\!] \\ \wedge (\sigma'', \sigma') \in \mathcal{C}[\![c_2]\!])\}$$

Composition of relations

Suppose $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$.
The *composition* of relations $R_2 \circ R_1 \subseteq A \times C$ is

$$R_2 \circ R_1 = \{(a, c) \mid \exists b \in B.\, (a, b) \in R_1 \wedge (b, c) \in R_2\}.$$

Composition of relations

We have $\mathcal{C}[\![c_1; c_2]\!] = \mathcal{C}[\![c_2]\!] \circ \mathcal{C}[\![c_1]\!]$, where $\circ$ is the composition of relations.

Definition of a function

A function is a set of input-output pairs s.t. each input has a unique output.

if b **then** c_1 **else** c_2

The mathematical function represented by **if** b **then** c_1 **else** c_2 is the set

$$\begin{aligned}\mathcal{C}[\![\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2]\!] = \{(\sigma,\sigma') \mid (\sigma,\textbf{true}) \in \mathcal{B}[\![b]\!] \\ \wedge (\sigma,\sigma') \in \mathcal{C}[\![c_1]\!]\} \ \cup \\ \{(\sigma,\sigma') \mid (\sigma,\textbf{false}) \in \mathcal{B}[\![b]\!] \\ \wedge (\sigma,\sigma') \in \mathcal{C}[\![c_2]\!]\}\end{aligned}$$

These are the input-output pairs of our function.

$\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!]$

$\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!]$

$$\begin{aligned}\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] = \{(\sigma,\sigma) \mid (\sigma,\textbf{false}) \in \mathcal{B}[\![b]\!]\} \;\; &\cup \\ \{(\sigma,\sigma') \mid (\sigma,\textbf{true}) \in \mathcal{B}[\![b]\!] & \\ \wedge \exists \sigma''.\ ((\sigma,\sigma'') \in \mathcal{C}[\![c]\!] & \\ \wedge (\sigma'',\sigma') \in \mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!])\} &\end{aligned}$$

Computing $f = \mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!]$

So far we only have a recursive equation

$$\begin{aligned} f = \{(\sigma, \sigma) \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!]\} \;\; \cup \\ \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \\ \wedge \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c]\!] \wedge (\sigma'', \sigma') \in f)\} \end{aligned}$$

What is f, as a subset $\textbf{Store} \times \textbf{Store}$?

A simpler example

A simpler example

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1) + 2x - 1 & \text{otherwise} \end{cases} \tag{1}$$

$$
\begin{aligned}
f_0 &= \emptyset \\
f_1 &= \begin{cases} 0 & \text{if } x = 0 \\ f_0(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0,0)\} \\
f_2 &= \begin{cases} 0 & \text{if } x = 0 \\ f_1(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0,0),(1,1)\} \\
f_3 &= \begin{cases} 0 & \text{if } x = 0 \\ f_2(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0,0),(1,1),(2,4)\} \\
&\vdots
\end{aligned}
$$

Consider this higher-order function F:

$$\begin{aligned} F &: (\mathbb{N} \rightharpoonup \mathbb{N}) \to (\mathbb{N} \rightharpoonup \mathbb{N}) \\ F(g) &= g' \end{aligned}$$

where

$$g'(x) = \begin{cases} 0 & \text{if } x = 0 \\ g(x-1) + 2x - 1 & \text{otherwise} \end{cases}$$

- E.g. $F(\emptyset) = \{(0,0)\}$,
  $F(\{(1,0)\}) = \{(0,0),(2,1)\}$, and
  $F(\{(3,1),(0,1)\}) = \{(0,0),(4,8),(1,0)\}$.
- Note, however, that $F(f) = f$.
- In other words, f is a *fixed point* of F.

$$
\begin{aligned}
f &= \mathrm{fix}(F) \\
&= f_0 \cup f_1 \cup f_2 \cup f_3 \cup \ldots \\
&= \emptyset \cup F(\emptyset) \cup F(F(\emptyset)) \cup F(F(F(\emptyset))) \cup \ldots \\
&= \bigcup_{i \geq 0} F^i(\emptyset)
\end{aligned}
$$

Fixed-point semantics for loops

$$
\begin{aligned}
F_{b,c} &: (\textbf{Store} \rightharpoonup \textbf{Store}) \to (\textbf{Store} \rightharpoonup \textbf{Store}) \\
F_{b,c}(f) &= \{(\sigma, \sigma) \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!]\} \;\; \cup \\
&\quad \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \\
&\quad \wedge \exists \sigma''.\; ((\sigma, \sigma'') \in \mathcal{C}[\![c]\!] \\
&\quad \wedge (\sigma'', \sigma') \in f)\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] &= \bigcup_{i \geq 0} {F_{b,c}}^{i}(\emptyset) \\
&= \emptyset \cup F_{b,c}(\emptyset) \cup F_{b,c}(F_{b,c}(\emptyset)) \\
&\qquad \cup F_{b,c}(F_{b,c}(F_{b,c}(\emptyset))) \cup \ldots \\
&= \mathrm{fix}(F_{b,c})
\end{aligned}
$$

$$
\begin{aligned}
F_{b,c}(\emptyset) &= \{(\sigma,\sigma) \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!]\} \ \ \cup \\
&\quad\ \ \{(\sigma,\sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \\
&\quad\ \ \ \wedge \exists \sigma''.\ ((\sigma,\sigma'') \in \mathcal{C}[\![c]\!] \wedge (\sigma'',\sigma') \in \emptyset)\} \\
&= \{(\sigma,\sigma) \mid \sigma(\mathsf{foo}) \geq \sigma(\mathsf{bar})\}
\end{aligned}
$$

$$
\begin{aligned}
{F_{b,c}}^2(\emptyset) = {} & \{(\sigma,\sigma) \mid (\sigma,\textbf{false}) \in \mathcal{B}[\![b]\!]\} \;\; \cup \\
& \quad \{(\sigma,\sigma') \mid (\sigma,\textbf{true}) \in \mathcal{B}[\![b]\!] \\
& \quad\; \wedge \exists \sigma''.\ ((\sigma,\sigma'') \in \mathcal{C}[\![c]\!] \wedge (\sigma'',\sigma') \in F_{b,c}(\emptyset))\} \\
= {} & \{(\sigma,\sigma) \mid \sigma(\mathsf{foo}) \geq \sigma(\mathsf{bar})\} \;\; \cup \\
& \quad \{(\sigma,\sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo})+1]) \mid \sigma(\mathsf{foo}) < \sigma(\mathsf{bar}) \\
& \quad\; \wedge \sigma(\mathsf{foo})+1 \geq \sigma(\mathsf{bar})\}
\end{aligned}
$$

But $\sigma(\mathsf{foo}) < \sigma(\mathsf{bar}) \wedge \sigma(\mathsf{foo})+1 \geq \sigma(\mathsf{bar})$ if and only if $\sigma(\mathsf{foo})+1 = \sigma(\mathsf{bar})$ so we get:

$$
\begin{aligned}
= {} & \{(\sigma,\sigma) \mid \sigma(\mathsf{foo}) \geq \sigma(\mathsf{bar})\} \;\; \cup \\
& \quad \{(\sigma,\sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo})+1]) \mid \sigma(\mathsf{foo})+1 = \sigma(\mathsf{bar})\}
\end{aligned}
$$

$$
\begin{aligned}
{F_{b,c}}^3(\emptyset) = \{(\sigma,\sigma) \mid (\sigma,\mathbf{false}) \in \mathcal{B}[\![b]\!]\} \;\; \cup \\
\{(\sigma,\sigma') \mid (\sigma,\mathbf{true}) \in \mathcal{B}[\![b]\!] \land \exists\sigma''.\; ((\sigma,\sigma'') \in \mathcal{C}[\![c]\!] \\
\land (\sigma'',\sigma') \in {F_{b,c}}^2(\emptyset))\} \\
= \{(\sigma,\sigma) \mid \sigma(\mathsf{foo}) \geq \sigma(\mathsf{bar})\} \;\; \cup \\
\{(\sigma,\sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo})+1]) \mid \sigma(\mathsf{foo})+1 = \sigma(\mathsf{bar})\} \;\; \cup \\
\{(\sigma,\sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo})+2]) \mid \sigma(\mathsf{foo})+2 = \sigma(\mathsf{bar})\}
\end{aligned}
$$

$$
\begin{aligned}
{F_{b,c}}^4(\emptyset) = \{(\sigma,\sigma) \mid \sigma(\mathsf{foo}) \geq \sigma(\mathsf{bar})\} \;\; \cup \\
\{(\sigma,\sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo})+1]) \mid \sigma(\mathsf{foo})+1 = \sigma(\mathsf{bar})\} \;\; \cup \\
\{(\sigma,\sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo})+2]) \mid \sigma(\mathsf{foo})+2 = \sigma(\mathsf{bar})\} \;\; \cup \\
\{(\sigma,\sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo})+3]) \mid \sigma(\mathsf{foo})+3 = \sigma(\mathsf{bar})\}
\end{aligned}
$$

$$\begin{aligned}\mathcal{C}[\![\textbf{while}\ \mathsf{foo} < \mathsf{bar}\ \textbf{do}\ \mathsf{foo} := \mathsf{foo} + 1]\!] = \\ \{(\sigma, \sigma) \mid \sigma(\mathsf{foo}) \geq \sigma(\mathsf{bar})\} \ \cup \\ \{(\sigma, \sigma[\mathsf{foo} \mapsto \sigma(\mathsf{foo}) + n]) \mid \sigma(\mathsf{foo}) + n = \sigma(\mathsf{bar}) \wedge n \geq 1\}\end{aligned}$$