1 Type Inference

(a) Recall the constraint-based typing judgment $\Gamma \vdash e : \tau \triangleright C$. Give inference rules for products and sums. That is, for the following expressions.

- $(e_1, e_2)$
- $\#1 e$
- $\#2 e$
- $\text{inl}_{\tau_1 + \tau_2} e$
- $\text{inr}_{\tau_1 + \tau_2} e$
- $\text{case } e_1 \text{ of } e_2 \mid e_3$

**Answer:**

Note that in all of the rules below except for the rule for pairs $(e_1, e_2)$, the types in the premise and conclusion are connected only through constraints. The reason for this is the same as in the typing rule for function application, and for addition: we may not be able to derive that the premise has the appropriate type, e.g., for a projection $\#1 e$, we may not be able to derive that $\Gamma \vdash e : \tau_1 \times \tau_2 \triangleright C$. We instead use constraints to ensure that the derived type is appropriate.

\[
\begin{align*}
\Gamma \vdash e_1 : \tau_1 \triangleright C_1 & \quad \Gamma \vdash e_2 : \tau_2 \triangleright C_2 \\
\hline
\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2 \triangleright C_1 \cup C_2
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : \tau \triangleright C & \quad X, Y \text{ are fresh} \\
\hline
\Gamma \vdash \#1 e : X \triangleright C \cup \{\tau \equiv X \times Y\}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : \tau \triangleright C & \quad X, Y \text{ are fresh} \\
\hline
\Gamma \vdash \#2 e : Y \triangleright C \cup \{\tau \equiv X \times Y\}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : \tau \triangleright C & \quad X, Y \text{ are fresh} \\
\hline
\Gamma \vdash \text{inl}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2 \triangleright C \cup \{\tau \equiv \tau_1\}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e : \tau \triangleright C & \quad X, Y, Z \text{ are fresh} \\
\hline
\Gamma \vdash \text{inr}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2 \triangleright C \cup \{\tau \equiv \tau_2\}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e_1 : \tau_1 \triangleright C_1 & \quad \Gamma \vdash e_2 : \tau_2 \triangleright C_2 \\
\hline
\Gamma \vdash \text{case } e_1 \text{ of } e_2 \mid e_3 : Z \triangleright C_1 \cup C_2 \cup C_3 \cup \{\tau_1 \equiv X + Y, \tau_2 \equiv X \rightarrow Z, \tau_3 \equiv Y \rightarrow Z\}
\end{align*}
\]

(b) Determine a set of constraints $C$ and type $\tau$ such that

\[
\vdash \lambda x : A. \lambda y : B. (\#1 y) + (x (\#2 y)) + (x 2) : \tau \triangleright C
\]

and give the derivation for it.
Answer:

\[ C = \{ B \equiv X \times Y , \ X \equiv \text{int} , \ B \equiv Z \times W , \ A \equiv W \to U , \ U \equiv \text{int} , \ A \equiv \text{int} \to V , \ V \equiv \text{int} \} \]

\[ \tau \equiv A \to B \to \text{int} \]

To see how we got these constraints, we will consider the subexpressions in turn (rather than trying to typeset a really really big derivation).

The expression \#1 \( y \) requires us to add a constraint that the type of \( y \) (i.e., \( B \)) is equal to a product type for some fresh variables \( X \) and \( Y \), thus constraint \( B \equiv X \times Y \). (And expression \#1 \( y \) has type \( X \).)

The expression \((\#2 \ y)\) similarly requires us to add a constraint that the type of \( y \) (i.e., \( B \)) is equal to a product type for some fresh variables \( Z \) and \( W \), thus constraint \( B \equiv Z \times W \). (And expression \#2 \( y \) has type \( W \).)

The expression \( x \ (\#2 \ y) \) requires us to add a constraint that unifies the type of \( x \) (i.e., \( A \)) with a function type \( W \to U \) (where \( W \) is the type of \#2 \( y \) and \( U \) is a fresh type variable).

The expression \( x \ 2 \) requires us to add a constraint that unifies the type of \( x \) (i.e., \( A \)) with a function type \( \text{int} \to V \) (where \( \text{int} \) is the type of expression \( 2 \) and \( V \) is a fresh type).

The addition operations leads us to add constraints \( X \equiv \text{int} \), \( U \equiv \text{int} \), and \( V \equiv \text{int} \) (i.e., the types of expressions \((\#1 \ y)\), \((x \ (\#2 \ y))\) and \((x \ 2)\) must all unify with \text{int}.

(c) Recall the unification algorithm from Lecture 14. What is the result of \text{unify}(C)\ for the set of constraints \( C \) from Question 1(b) above?

Answer: The result is a substitution equivalent to

\[ [A \mapsto \text{int} \to \text{int} , \ B \mapsto \text{int} \times \text{int} , \ X \mapsto \text{int} , \ Y \mapsto \text{int} , \ Z \mapsto \text{int} , \ W \mapsto \text{int} , \ U \mapsto \text{int} , \ V \mapsto \text{int}] \]