Curry-Howard Isomorphism; Existential Types

CS 152 (Spring 2021)

Harvard University

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Today, we will learn about

- Curry-Howard Isomorphism
- Existential types

Curry-Howard Isomorphism

Conjunction = Product

Disjunction = Sum

Function Types?

Parametric Polymorphism

What about False?

Example 1: From Formula to Type

Example 2: From Type to Formula

Negation and Continuations

Existential Types

Syntax

$$
\begin{aligned}
e ::= {} & x \mid \lambda x\!:\!\tau.\, e \mid e_1\ e_2 \mid n \mid e_1 + e_2 \\
& \mid \{\ l_1 = e_1, \ldots, l_n = e_n\ \} \mid e.l \\
& \mid \text{pack } \{\tau_1, e\} \text{ as } \exists X.\ \tau_2 \\
& \mid \text{unpack } \{X, x\} = e_1 \text{ in } e_2 \\
v ::= {} & n \mid \lambda x\!:\!\tau.\, e \mid \{\ l_1 = v_1, \ldots, l_n = v_n\ \} \\
& \mid \text{pack } \{\tau_1, v\} \text{ as } \exists X.\ \tau_2 \\
\tau ::= {} & \textbf{int} \mid \tau_1 \to \tau_2 \mid \{\ l_1\!:\!\tau_1, \ldots, l_n\!:\!\tau_n\ \} \mid X \mid \exists X.\ \tau
\end{aligned}
$$

Example: Counter ADT

$$
\begin{aligned}
&\text{let } \mathit{counterADT} = \\
&\qquad \text{pack} \\
&\qquad\quad \{\textbf{int}, \{\ \text{new} = 0, \\
&\qquad\qquad\quad \text{get} = \lambda i : \textbf{int}.\ i, \\
&\qquad\qquad\quad \text{inc} = \lambda i : \textbf{int}.\ i + 1\ \}\ \} \\
&\qquad \text{as} \\
&\qquad\quad \exists \textbf{Counter}.\ \{\ \text{new} : \textbf{Counter}, \\
&\qquad\qquad\quad \text{get} : \textbf{Counter} \to \textbf{int}, \\
&\qquad\qquad\quad \text{inc} : \textbf{Counter} \to \textbf{Counter}\ \} \\
&\text{in } \ldots
\end{aligned}
$$

Example: Counter ADT, ctd

$$\text{unpack } \{C, x\} = \mathit{counterADT} \text{ in let } y\!:\!C = x.\text{new in}$$
$$x.\text{get } (x.\text{inc } (x.\text{inc } y))$$

Operational Semantics

$$E ::= \cdots \mid \text{pack } \{\tau_1, E\} \text{ as } \exists X.\ \tau_2$$
$$\mid \text{unpack } \{X, x\} = E \text{ in } e$$

$$\frac{}{\text{unpack } \{X, x\} = (\text{pack } \{\tau_1, v\} \text{ as } \exists Y.\ \tau_2) \text{ in } e \longrightarrow e\{v/x\}\{\tau_1/X\}}$$

Typing rules

$$\frac{\Delta, \Gamma \vdash e : \tau_2\{\tau_1/X\}}{\Delta, \Gamma \vdash \mathsf{pack}\ \{\tau_1, e\}\ \mathsf{as}\ \exists X.\ \tau_2 : \exists X.\ \tau_2}$$

$$\frac{\Delta, \Gamma \vdash e_1 : \exists X.\ \tau_1 \quad X \notin \Delta \qquad \Delta \cup \{X\}, \Gamma, x : \tau_1 \vdash e_2 : \tau_2 \quad \Delta \vdash \tau_2\ \mathsf{ok}}{\Delta, \Gamma \vdash \mathsf{unpack}\ \{X, x\} = e_1\ \mathsf{in}\ e_2 : \tau_2}$$

$$\frac{\Delta \cup \{X\} \vdash \tau\ \mathsf{ok}}{\Delta \vdash \exists X.\ \tau\ \mathsf{ok}}$$