

## Large-step semantics

Lecture 4

Tuesday, February 3, 2022

### 1 Large-step semantics

So far we have defined the small step evaluation relation  $\longrightarrow \subseteq \mathbf{Config} \times \mathbf{Config}$  for our simple language of arithmetic expressions, and used its transitive and reflexive closure  $\longrightarrow^*$  to describe the execution of multiple steps of evaluation. In particular, if  $\langle e, \sigma \rangle$  is some start configuration, and  $\langle n, \sigma' \rangle$  is a final configuration, the evaluation  $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$  shows that by executing expression  $e$  starting with the store  $\sigma$ , we get the result  $n$ , and the final store  $\sigma'$ .

*Large-step semantics* is an alternative way to specify the operational semantics of a language. Large-step semantics directly give the final result.

We'll use the same configurations as before, but define a large step evaluation relation:

$$\Downarrow \subseteq \mathbf{Config} \times \mathbf{FinalConfig}$$

where

$$\begin{aligned} \mathbf{Config} &= \mathbf{Exp} \times \mathbf{Store} \\ \text{and } \mathbf{FinalConfig} &= \mathbf{Int} \times \mathbf{Store} \subseteq \mathbf{Config}. \end{aligned}$$

We write  $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$  to mean that  $(\langle e, \sigma \rangle, \langle n, \sigma' \rangle) \in \Downarrow$ . In other words, configuration  $\langle e, \sigma \rangle$  evaluates in one big step directly to final configuration  $\langle n, \sigma' \rangle$ . In general, the big step semantics takes a configuration to an “answer”. For our language of arithmetic expressions, “answers” are a subset of configurations, but this is not always true in general.

The large step semantics boils down to defining the relation  $\Downarrow$ . We use inference rules to inductively define the relation  $\Downarrow$ , similar to how we specified the small-step operational semantics  $\longrightarrow$ .

$$\begin{aligned} \text{INT}_{\text{LRG}} & \frac{}{\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle} & \text{VAR}_{\text{LRG}} & \frac{}{\langle x, \sigma \rangle \Downarrow \langle n, \sigma \rangle} \text{ where } \sigma(x) = n \\ \text{ADD}_{\text{LRG}} & \frac{\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \quad \langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle}{\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle} \text{ where } n \text{ is the sum of } n_1 \text{ and } n_2 \\ \text{MUL}_{\text{LRG}} & \frac{\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \quad \langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle}{\langle e_1 \times e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle} \text{ where } n \text{ is the product of } n_1 \text{ and } n_2 \\ \text{ASG}_{\text{LRG}} & \frac{\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle \quad \langle e_2, \sigma''[x \mapsto n_1] \rangle \Downarrow \langle n_2, \sigma' \rangle}{\langle x := e_1; e_2, \sigma \rangle \Downarrow \langle n_2, \sigma' \rangle} \end{aligned}$$

To see how we use these rules, here is a proof tree that shows that  $\langle \text{foo} := 3; \text{foo} \times \text{bar}, \sigma \rangle \Downarrow \langle 21, \sigma' \rangle$  for a store  $\sigma$  such that  $\sigma(\text{bar}) = 7$ , and  $\sigma' = \sigma[\text{foo} \mapsto 3]$ .

$$\text{ASG}_{\text{LRG}} \frac{\text{INT}_{\text{LRG}} \frac{}{\langle 3, \sigma \rangle \Downarrow \langle 3, \sigma \rangle} \quad \text{MUL}_{\text{LRG}} \frac{\text{VAR}_{\text{LRG}} \frac{}{\langle \text{foo}, \sigma' \rangle \Downarrow \langle 3, \sigma' \rangle} \quad \text{VAR}_{\text{LRG}} \frac{}{\langle \text{bar}, \sigma' \rangle \Downarrow \langle 7, \sigma' \rangle}}{\langle \text{foo} \times \text{bar}, \sigma' \rangle \Downarrow \langle 21, \sigma' \rangle}}{\langle \text{foo} := 3; \text{foo} \times \text{bar}, \sigma \rangle \Downarrow \langle 21, \sigma' \rangle}}$$

A closer look to this structure reveals the relation between small step and large-step evaluation: a depth-first traversal of the large-step proof tree yields the sequence of one-step transitions in small-step evaluation.

## 2 Equivalence of semantics

So far, we have specified the semantics of our language of arithmetic expressions in two different ways: small-step operational semantics and large-step operational semantics. Are they expressing the same meaning of arithmetic expressions? Can we show that they express the same thing?

**Theorem** (Equivalence of semantics). *For all expressions  $e$ , stores  $\sigma$  and  $\sigma'$ , and integers  $n$ , we have:*

$$\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \iff \langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle.$$

*Proof sketch.*

- $\implies$ . We proceed by structural induction on expressions  $e$ . The property we will prove by induction is:

$$P(e) = \forall \sigma, \sigma' \in \mathbf{Store}. \forall n \in \mathbf{Int}. \langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \implies \langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

We have to consider each of the possible axioms and inference rules for constructing an expression.

- **Case**  $e \equiv x$ .

Here, we are considering the case where the expression  $e$  is equal to some variable  $x$ . Assume that for some  $\sigma, \sigma'$ , and  $n$  we have  $\langle x, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ . That means that there is some derivation using the axioms and inference rules of the large-step operational semantics, whose conclusion is  $\langle x, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ . There is only one rule whose conclusion could look like this, the rule  $\text{Var}_{\text{LRG}}$ . That rule requires that  $n = \sigma(x)$ , and that  $\sigma' = \sigma$ .

(This reasoning is an example of *inversion*: using the inference rules in reverse. That is, we know that some conclusion holds— $\langle x, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ —and we examine the inference rules to determine which rule must have been used in the derivation, and thus which premises must be true, and which side conditions satisfied.)

Since  $n = \sigma(x)$  we know that  $\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle$  also holds, by using the small-step axiom  $\text{VAR}$ . So we can conclude that  $\langle x, \sigma \rangle \longrightarrow^* \langle n, \sigma \rangle$  holds, which is what we needed to show.

- **Case**  $e \equiv n$ .

Here, we consider the case where expression  $e$  is equal to some integer  $n$ . Assume that for some  $\sigma, \sigma'$ , and  $n'$  we have  $\langle n, \sigma \rangle \Downarrow \langle n', \sigma' \rangle$ . Like the case above, by inversion, we know that the rule  $\text{Int}_{\text{LRG}}$  was used to conclude that  $\langle n, \sigma \rangle \Downarrow \langle n', \sigma' \rangle$ , and so  $n' = n$  and  $\sigma' = \sigma$ .

So we need to show that  $\langle n, \sigma \rangle \longrightarrow^* \langle n, \sigma \rangle$ . But this holds trivially because of reflexivity of  $\longrightarrow^*$ .

- **Case**  $e \equiv e_1 + e_2$ .

This is an inductive case. Expressions  $e_1$  and  $e_2$  are subexpressions of  $e$ , and so we can assume that  $P(e_1)$  and  $P(e_2)$  hold. We need to show that  $P(e)$  holds. Let's write out  $P(e_1)$ ,  $P(e_2)$ , and  $P(e)$  explicitly.

$$\begin{aligned} P(e_1) &= \forall n, \sigma, \sigma' : \langle e_1, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \implies \langle e_1, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle \\ P(e_2) &= \forall n, \sigma, \sigma' : \langle e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \implies \langle e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle \\ P(e) &= \forall n, \sigma, \sigma' : \langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle \implies \langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle \end{aligned}$$

Assume that for some  $\sigma, \sigma'$  and  $n$  we have  $\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ . We now need to show that  $\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ .

We assumed that  $\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ . Let's use inversion again: there is some derivation whose conclusion is  $\langle e_1 + e_2, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ . By looking at the large-step semantic rules, we see that only one rule could possibly have a conclusion of this form: the rule  $\text{ADD}_{\text{LRG}}$ . So that means that the last rule used in the derivation was  $\text{ADD}_{\text{LRG}}$ . But in order to use the rule  $\text{ADD}_{\text{LRG}}$ , it must be the case that  $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$  and  $\langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle$  hold for some  $n_1$  and  $n_2$  such that  $n = n_1 + n_2$  (i.e., there is a derivation whose conclusion is  $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$  and a derivation whose conclusion is  $\langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma' \rangle$ ).

Using the inductive hypothesis  $P(e_1)$ , since  $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$ , we must have  $\langle e_1, \sigma \rangle \longrightarrow^* \langle n_1, \sigma'' \rangle$ . Similarly, by  $P(e_2)$ , we have  $\langle e_2, \sigma'' \rangle \longrightarrow^* \langle n_2, \sigma' \rangle$ . By Lemma 1 below, we have

$$\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n_1 + e_2, \sigma'' \rangle$$

and by another application of Lemma 1 we have

$$\langle n_1 + e_2, \sigma'' \rangle \longrightarrow^* \langle n_1 + n_2, \sigma' \rangle$$

and by the rule  $\text{ADD}$  we have

$$\langle n_1 + n_2, \sigma' \rangle \longrightarrow \langle n, \sigma' \rangle.$$

Thus, we have  $\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ , which proves this case.

- **Case**  $e \equiv e_1 \times e_2$ . Similar to the case  $e = e_1 + e_2$  above.
- **Case**  $e \equiv x := e_1; e_2$ . Omitted. Try it as an exercise.
- $\Leftarrow$ . We proceed by mathematical induction on the number of steps  $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ .
  - **Base case.** If  $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$  in zero steps, then we must have  $e \equiv n$  and  $\sigma' = \sigma$ . Then,  $\langle n, \sigma \rangle \Downarrow \langle n, \sigma \rangle$  by the large-step operational semantics rule  $\text{INT}_{\text{LRG}}$ .
  - **Inductive case.** Assume that  $\langle e, \sigma \rangle \longrightarrow \langle e'', \sigma'' \rangle \longrightarrow^* \langle n, \sigma' \rangle$ , and that (the inductive hypothesis)  $\langle e'', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$ . That is,  $\langle e'', \sigma'' \rangle \longrightarrow^* \langle n, \sigma' \rangle$  takes  $m$  steps, and we assume that the property holds for it ( $\langle e'', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$ ), and we are considering  $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ , which takes  $m+1$  steps. We need to show that  $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ . This follows immediately from Lemma 2 below.

□

**Lemma 1.** If  $\langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$  then for all  $n_1, e_2$  the following hold.

- $\langle e + e_2, \sigma \rangle \longrightarrow^* \langle n + e_2, \sigma' \rangle$
- $\langle e \times e_2, \sigma \rangle \longrightarrow^* \langle n \times e_2, \sigma' \rangle$
- $\langle n_1 + e, \sigma \rangle \longrightarrow^* \langle n_1 + n, \sigma' \rangle$
- $\langle n_1 \times e, \sigma \rangle \longrightarrow^* \langle n_1 \times n, \sigma' \rangle$

*Proof.* By (mathematical) induction on the number of evaluation steps in  $\longrightarrow^*$ . □

**Lemma 2.** For all  $e, e', \sigma$ , and  $n$ , if  $\langle e, \sigma \rangle \longrightarrow \langle e', \sigma'' \rangle$  and  $\langle e', \sigma'' \rangle \Downarrow \langle n, \sigma' \rangle$ , then  $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ .

## 2.1 Again: structural induction vs rule induction

Like in the previous lecture, here we can also prove that large-step evaluation implies small-step evaluation by induction on the large-step derivation rather than structural induction on the expression.

⇒. We proceed by induction on the derivation of  $\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle$ . The property we wish to show is

$$P(\langle e, \sigma \rangle \Downarrow \langle n, \sigma' \rangle) = \langle e, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$$

Suppose we have a derivation of  $\langle e, \sigma \rangle \longrightarrow \langle n, \sigma' \rangle$  for some  $e, \sigma, n,$  and  $\sigma'$ . Assume that the inductive hypothesis holds for any subderivation  $\langle e_0, \sigma_0 \rangle \longrightarrow \langle n_0, \sigma'_0 \rangle$  used in the derivation of  $\langle e, \sigma \rangle \longrightarrow \langle n, \sigma' \rangle$ . Consider the last rule used in the derivation of  $\langle e, \sigma \rangle \longrightarrow \langle n, \sigma' \rangle$ .

- $\text{INT}_{\text{LRG}}$ . Here,  $e \equiv n$  and  $\sigma' = \sigma$ . The result holds trivially, as  $\langle n, \sigma \rangle \longrightarrow^* \langle n, \sigma \rangle$ .
- $\text{VAR}_{\text{LRG}}$ . Here  $e \equiv x$  for some variable  $x$ , and  $\sigma' = \sigma$  and  $n = \sigma(x)$ . Using the small step rule VAR, we can derive  $\langle x, \sigma \rangle \longrightarrow \langle n, \sigma \rangle$ , and thus  $\langle x, \sigma \rangle \longrightarrow^* \langle n, \sigma \rangle$ .
- $\text{ADD}_{\text{LRG}}$ . Here  $e \equiv e_1 + e_2$  for some expressions  $e_1$  and  $e_2$ , and  $n = n_1 + n_2$  for some  $n_1$  and  $n_2$ , and we have derivations for  $\langle e_1, \sigma \rangle \Downarrow \langle n_1, \sigma'' \rangle$  and  $\langle e_2, \sigma'' \rangle \Downarrow \langle n_2, \sigma_1 \rangle$ .

By Lemma 1, we have

$$\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n_1 + e_2, \sigma'' \rangle$$

and by another application of Lemma 1 we have

$$\langle n_1 + e_2, \sigma'' \rangle \longrightarrow^* \langle n_1 + n_2, \sigma' \rangle$$

and by the rule ADD we have

$$\langle n_1 + n_2, \sigma' \rangle \longrightarrow \langle n, \sigma' \rangle.$$

Thus, we have  $\langle e_1 + e_2, \sigma \rangle \longrightarrow^* \langle n, \sigma' \rangle$ , which proves this case.

- $\text{MUL}_{\text{LRG}}$ . Similar to the case  $\text{ADD}_{\text{LRG}}$  above.
- $\text{ASG}_{\text{LRG}}$ . Omitted. Try it as an exercise.