Type Inference
Section and Practice Problems

Section 9

---

1 Type Inference

(a) Recall the constraint-based typing judgment $\Gamma \vdash e : \tau \triangleright C$. Give inference rules for products and sums. That is, for the following expressions.

- (e_1, e_2)
- $\#1\ e$
- $\#2\ e$
- $\mathsf{inl}_{\tau_1+\tau_2}\ e$
- $\mathsf{inr}_{\tau_1+\tau_2}\ e$
- $\mathsf{case}\ e_1\ \mathsf{of}\ e_2 \mid e_3$

Answer:

Note that in all of the rules below except for the rule for pairs (e_1, e_2), the types in the premise and conclusion are connected only through constraints. The reason for this is the same as in the typing rule for function application, and for addition: we may not be able to derive that the premise has the appropriate type, e.g., for a projection $\#1\ e$, we may not be able to derive that $\Gamma \vdash e : \tau_1 \times \tau_2 \triangleright C$. We instead use constraints to ensure that the derived type is appropriate.

$$\frac{\Gamma \vdash e_1 : \tau_1 \triangleright C_1 \qquad \Gamma \vdash e_2 : \tau_2 \triangleright C_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2 \triangleright C_1 \cup C_2}$$

$$\frac{\Gamma \vdash e : \tau \triangleright C}{\Gamma \vdash \#1\ e : X \triangleright C \cup \{\tau \equiv X \times Y\}}\ X, Y \textit{ are fresh} \qquad \frac{\Gamma \vdash e : \tau \triangleright C}{\Gamma \vdash \#2\ e : Y \triangleright C \cup \{\tau \equiv X \times Y\}}\ X, Y \textit{ are fresh}$$

$$\frac{\Gamma \vdash e : \tau \triangleright C}{\Gamma \vdash \mathit{inl}_{\tau_1+\tau_2}\ e : \tau_1 + \tau_2 \triangleright C \cup \{\tau \equiv \tau_1\}} \qquad \frac{\Gamma \vdash e : \tau \triangleright C}{\Gamma \vdash \mathit{inr}_{\tau_1+\tau_2}\ e : \tau_1 + \tau_2 \triangleright C \cup \{\tau \equiv \tau_2\}}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \triangleright C_1 \qquad \Gamma \vdash e_2 : \tau_2 \triangleright C_2 \qquad \Gamma \vdash e_3 : \tau_3 \triangleright C_3}{\Gamma \vdash \mathit{case}\ e_1\ \mathit{of}\ e_2 \mid e_3 : Z \triangleright C_1 \cup C_2 \cup C_3 \cup \{\tau_1 \equiv X + Y, \tau_2 \equiv X \to Z, \tau_3 \equiv Y \to Z\}}\ X, Y, Z \textit{ are fresh}$$

(b) Determine a set of constraints C and type τ such that

$$\vdash \quad \lambda x : A.\ \lambda y : B.\ (\#1\ y) + (x\ (\#2\ y)) + (x\ 2) \ \ : \tau \triangleright C$$

and give the derivation for it.

Answer:

$$C = \{B \equiv X \times Y\ ,\ X \equiv \textbf{\textit{int}}\ ,\ B \equiv Z \times W\ ,\ A \equiv W \rightarrow U\ ,\ U \equiv \textbf{\textit{int}}\ ,\ A \equiv \textbf{\textit{int}} \rightarrow V\ ,\ V \equiv \textbf{\textit{int}}\}$$
$$\tau \equiv A \rightarrow B \rightarrow \textbf{\textit{int}}$$

To see how we got these constraints, we will consider the subexpressions in turn (rather than trying to typeset a really really big derivation).

The expression $\#1\ y$ *requires us to add a constraint that the type of* y *(i.e.,* B*) is equal to a product type for some fresh variables* X *and* Y*, thus constraint* $B \equiv X \times Y$*. (And expression* $\#1\ y$ *has type* X*.)*

The expression $(\#2\ y)$ *similarly requires us to add a constraint that the type of* y *(i.e.,* B*) is equal to a product type for some fresh variables* Z *and* W*, thus constraint* $B \equiv Z \times W$*. (And expression* $\#2\ y$ *has type* W*.)*

The expression $x\ (\#2\ y)$ *requires us to add a constraint that unifies the type of* x *(i.e.,* A*) with a function type* $W \rightarrow U$ *(where* W *is the type of* $\#2\ y$ *and* U *is a fresh type variable).*

The expression $x\ 2$ *requires us to add a constraint that unifies the type of* x *(i.e.,* A*) with a function type* $\textbf{\textit{int}} \rightarrow V$ *(where* $\textbf{\textit{int}}$ *is the type of expression* 2 *and* V *is a fresh type).*

The addition operations leads us to add constraints $X \equiv \textbf{\textit{int}}$*,* $U \equiv \textbf{\textit{int}}$*, and* $V \equiv \textbf{\textit{int}}$ *(i.e., the types of expressions* $(\#1\ y)$*,* $(x\ (\#2\ y))$ *and* $(x\ 2)$ *must all unify with* $\textbf{\textit{int}}$*.*

(c) Recall the unification algorithm from Lecture 16. What is the result of $unify(C)$ for the set of constraints C from Question 1(b) above?

Answer: *The result is a substitution equivalent to*

$$[A \mapsto \textbf{\textit{int}} \rightarrow \textbf{\textit{int}}\ ,\ B \mapsto \textbf{\textit{int}} \times \textbf{\textit{int}}\ ,\ X \mapsto \textbf{\textit{int}}\ ,\ Y \mapsto \textbf{\textit{int}}\ ,\ Z \mapsto \textbf{\textit{int}}\ ,\ W \mapsto \textbf{\textit{int}}\ ,\ U \mapsto \textbf{\textit{int}}\ ,\ V \mapsto \textbf{\textit{int}}]$$