

Lambda Calculus

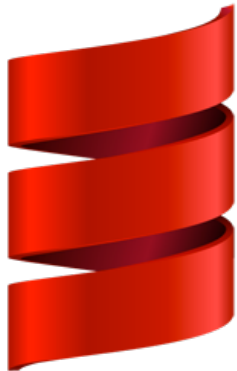
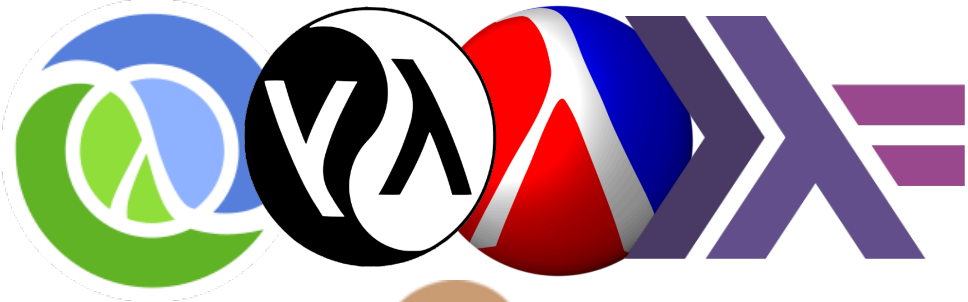
CS 152 (Spring 2024)

Harvard University

Tuesday, February 13, 2024

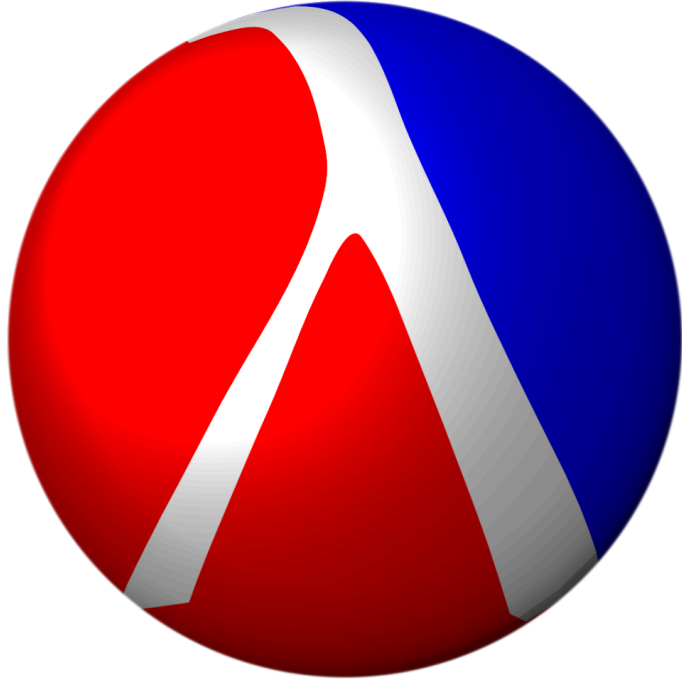
Today, we will learn about

- ▶ Lambda calculus
- ▶ α -equivalence
- ▶ β -reduction
- ▶ Call-by-value semantics
- ▶ Call-by-name semantics





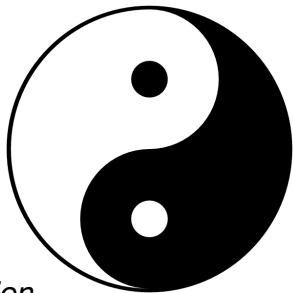






λ -calculus

- universal model of computation
- $e := x$ *// variable*
 - | $\lambda x.e$ *// function abstraction*
 - | $e e$ *// function application*



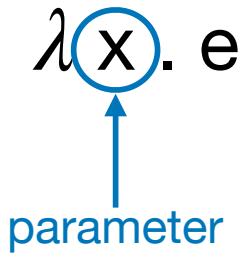
$\lambda x . e$

function



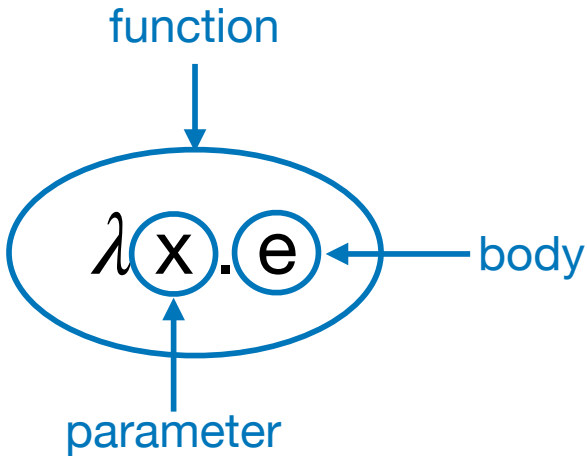
$\lambda x . e$

λ **x** . e



parameter

$\lambda x . \textcircled{e}$ ← body



Lambda calculus: Intuition

A function is a rule for determining a value from an argument. Some examples of functions in mathematics are

$$f(x) = x^3$$

$$g(y) = y^3 - 2y^2 + 5y - 6.$$

Lambda calculus: Intuition

A function is a rule for determining a value from an argument. Some examples of functions in mathematics are

$$f = \lambda x.x^3$$

$$g = \lambda y.y^3 - 2y^2 + 5y - 6.$$

Pure vs Applied Lambda Calculus

- ▶ The pure λ -calculus contains just function definitions (called *abstractions*), variables, and function *applications*.
- ▶ If we add additional data types and operations (such as integers and addition), we have an *applied* λ -calculus.

Pure Lambda Calculus: Syntax

$e ::= x$	variable
$\lambda x. e$	abstraction
$e_1 e_2$	application

Abstractions

Abstractions

- ▶ An abstraction $\lambda x. e$ is a function
- ▶ Variable x is the *parameter*
- ▶ Expression e is the *body* of the function.
- ▶ The expression $\lambda y. y \times y$ is a function that takes an argument y and returns square of y .

Applications

- ▶ An application $e_1 e_2$ requires that e_1 is (or evaluates to) a function, and then applies the function to the expression e_2 .
- ▶ For example, $(\lambda y. y \times y) 5$ is 25

Examples

$\lambda x. x$ a lambda abstraction called the *identity function*

$\lambda x. (f (g x))$ another abstraction

$(\lambda x. x) 42$ an application

$\lambda y. \lambda x. x$ an abstraction, ignores its argument
and returns the identity function

Lambda expressions extend as far to the right as possible

$\lambda x. x \lambda y. y$ is the same as $\lambda x. (x (\lambda y. y))$, and is not the same as $(\lambda x. x) (\lambda y. y)$.

Application is left-associative

$e_1 e_2 e_3$ is the same as $(e_1 e_2) e_3$.

Use parentheses!

In general, use parentheses to make the parsing of a lambda expression clear if you are in doubt.

Variable binding

- ▶ An occurrence of a variable x in a term is bound if there is an enclosing $\lambda x. e$; otherwise, it is *free*.
- ▶ A *closed term* is one in which all identifiers are bound.

Variable binding: $\lambda x. (x (\lambda y. y a) x) y$

Variable binding: $\lambda x. (x (\lambda y. y a) x) y$

- ▶ Both occurrences of x are bound
- ▶ The first occurrence of y is bound
- ▶ The a is free
- ▶ The last y is also free, since it is outside the scope of the λy .

Binding operator

The symbol λ is a *binding operator*: variable x is bound in e in the expression $\lambda x. e$.

α -equivalence

- ▶ $\lambda x. x$ is the same function as $\lambda y. y$.
- ▶ Expressions e_1 and e_2 that differ only in the name of bound variables are called *α -equivalent* (“alpha equivalent”)
- ▶ Sometimes written $e_1 =_{\alpha} e_2$.

Quiz: α -equivalence

- ▶ Are $\lambda x. \lambda y. x y$ and $\lambda y. \lambda x. y x$ α -equivalent?

Higher-order functions

- ▶ In lambda calculus, functions are values.
- ▶ In the pure lambda calculus, every value is a function, and every result is a function!

Higher-order functions

$\lambda f. f\ 42$

Higher-order functions

$$\lambda v. \lambda f. (f v)$$

Takes an argument v and returns a function that applies its own argument (a function) to v .

Semantics

β -equivalence

- ▶ We would like to regard $(\lambda x. e_1) e_2$ as equivalent to e_1 where every (free) occurrence of x is replaced with e_2 .
- ▶ E.g. we would like to regard $(\lambda y. y \times y) 5$ as equivalent to 5×5 .

Substitution $e_1\{e_2/x\}$

- ▶ We write $e_1\{e_2/x\}$ to mean expression e_1 with all free occurrences of x replaced with e_2 .
- ▶ We call $(\lambda x. e_1) e_2$ and $e_1\{e_2/x\}$ *β -equivalent*.
- ▶ Rewriting $(\lambda x. e_1) e_2$ into $e_1\{e_2/x\}$ is called a *β -reduction*.
- ▶ This corresponds to executing a lambda calculus expression.

Different semantics for the lambda calculus

$(\lambda x. x + x) ((\lambda y. y) 5)$

Different semantics for the lambda calculus

$$(\lambda x. x + x) ((\lambda y. y) 5)$$

We could use β -reduction to get either $((\lambda y. y) 5) + ((\lambda y. y) 5)$ or $(\lambda x. x + x) 5$.

Evaluation strategies: Full β -reduction

Allows $(\lambda x. e_1) e_2$ to step to $e_1\{e_2/x\}$ at any time.

Full β -reduction: small-step operational semantics

$$\beta\text{-REDUCTION} \frac{}{(\lambda x. e_1) e_2 \longrightarrow e_1\{e_2/x\}}$$

Full β -reduction: small-step operational semantics

$$\beta\text{-REDUCTION} \frac{}{(\lambda x. e_1) e_2 \longrightarrow e_1\{e_2/x\}}$$

$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

Full β -reduction: small-step operational semantics

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$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

$$\frac{e_2 \longrightarrow e'_2}{e_1 e_2 \longrightarrow e_1 e'_2}$$

Full β -reduction: small-step operational semantics

$$\beta\text{-REDUCTION} \frac{}{(\lambda x. e_1) e_2 \longrightarrow e_1\{e_2/x\}}$$

$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

$$\frac{e_2 \longrightarrow e'_2}{e_1 e_2 \longrightarrow e_1 e'_2}$$

$$\frac{e \longrightarrow e'}{\lambda x. e \longrightarrow \lambda x. e'}$$

Normal form

A term e is said to be in *normal form* when there is no e' such that $e \longrightarrow e'$.

Not every term has a normal form under full β -reduction.

Consider $\Omega = (\lambda x. x x) (\lambda x. x x)$.

$$\Omega = (\lambda x. x x) (\lambda x. x x) \longrightarrow (\lambda x. x x) (\lambda x. x x) = \Omega$$

It's an infinite loop!

Well-behaved nondeterminism

$$(\lambda x. \lambda y. y) \Omega (\lambda z. z)$$

Well-behaved nondeterminism

$$(\lambda x. \lambda y. y) \Omega (\lambda z. z)$$

This term has two redexes in it, the one with abstraction λx , and the one inside Ω .

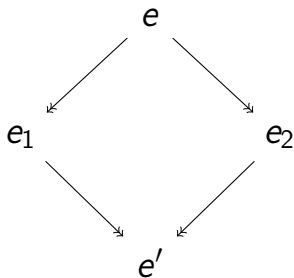
Well-behaved nondeterminism

- ▶ The full β -reduction strategy is non-deterministic.
- ▶ When a term has a normal form, however, it never has more than one.

Full β -reduction is confluent

Theorem (Confluence)

If $e \longrightarrow^ e_1$ and $e \longrightarrow^* e_2$ then there exists e' such that $e_1 \longrightarrow^* e'$ and $e_2 \longrightarrow^* e'$.*



Full β -reduction is confluent

Corollary

If $e \longrightarrow^ e_1$ and $e \longrightarrow^* e_2$ and both e_1 and e_2 are in normal form, then $e_1 = e_2$.*

Proof.

An easy consequence of confluence. □

Normal Order Evaluation

- ▶ *Normal order evaluation* uses the full β -reduction rules, except the left-most redex is always reduced first.
- ▶ Will eventually yield the normal form, if one exists.
- ▶ Allows reducing redexes inside abstractions

Call-by-value

- ▶ *Call-by-value* only allows an application to reduce after its argument has been reduced to a value and does not allow evaluation under a λ .
- ▶ Given an application $(\lambda x. e_1) e_2$, CBV semantics makes sure that e_2 is a value before calling the function.
- ▶ A value is an expression that can not be reduced/executed/simplified any further.

CBV: Small step operational semantics

$$\beta\text{-REDUCTION} \frac{}{(\lambda x. e) v \longrightarrow e\{v/x\}}$$

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$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

CBV: Small step operational semantics

$$\beta\text{-REDUCTION} \frac{}{(\lambda x. e) v \longrightarrow e\{v/x\}}$$

$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

$$\frac{e \longrightarrow e'}{v e \longrightarrow v e'}$$

CBV: Examples

$$\begin{aligned}(\lambda x. \lambda y. y x) (5 + 2) \lambda x. x + 1 &\longrightarrow (\lambda x. \lambda y. y x) 7 \lambda x. x + 1 \\ &\longrightarrow (\lambda y. y 7) \lambda x. x + 1 \\ &\longrightarrow (\lambda x. x + 1) 7 \\ &\longrightarrow 7 + 1 \\ &\longrightarrow 8\end{aligned}$$

$$\begin{aligned}(\lambda f. f 7) ((\lambda x. x x) \lambda y. y) &\longrightarrow (\lambda f. f 7) ((\lambda y. y) (\lambda y. y)) \\ &\longrightarrow (\lambda f. f 7) (\lambda y. y) \\ &\longrightarrow (\lambda y. y) 7 \\ &\longrightarrow 7\end{aligned}$$

Call-by-name semantics

- ▶ Applies the function as soon as possible.
- ▶ No need to ensure that the expression to which a function is applied is a value.

Call-by-name semantics

$$\beta\text{-REDUCTION} \frac{}{(\lambda x. e_1) e_2 \longrightarrow e_1\{e_2/x\}}$$

Call-by-name semantics

$$\beta\text{-REDUCTION} \frac{}{(\lambda x. e_1) e_2 \longrightarrow e_1\{e_2/x\}}$$

$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

Call-by-name semantics: example

$$\begin{aligned}(\lambda x. \lambda y. y x) (5 + 2) \lambda x. x + 1 &\longrightarrow (\lambda y. y (5 + 2)) \lambda x. x + 1 \\ &\longrightarrow (\lambda x. x + 1) (5 + 2) \\ &\longrightarrow (5 + 2) + 1 \\ &\longrightarrow 7 + 1 \\ &\longrightarrow 8\end{aligned}$$

compare to CBV:

$$\begin{aligned}(\lambda x. \lambda y. y x) (5 + 2) \lambda x. x + 1 &\longrightarrow (\lambda x. \lambda y. y x) 7 \lambda x. x + 1 \\ &\longrightarrow (\lambda y. y 7) \lambda x. x + 1 \\ &\longrightarrow (\lambda x. x + 1) 7 \\ &\longrightarrow 7 + 1 \\ &\longrightarrow 8\end{aligned}$$

Call-by-name semantics: example

$$\begin{aligned}(\lambda f. f\ 7)\ ((\lambda x. x\ x)\ \lambda y. y) &\longrightarrow ((\lambda x. x\ x)\ \lambda y. y)\ 7 \\ &\longrightarrow ((\lambda y. y)\ (\lambda y. y))\ 7 \\ &\longrightarrow (\lambda y. y)\ 7 \\ &\longrightarrow 7\end{aligned}$$

compare to CBV:

$$\begin{aligned}(\lambda f. f\ 7)\ ((\lambda x. x\ x)\ \lambda y. y) &\longrightarrow (\lambda f. f\ 7)\ ((\lambda y. y)\ (\lambda y. y)) \\ &\longrightarrow (\lambda f. f\ 7)\ (\lambda y. y) \\ &\longrightarrow (\lambda y. y)\ 7 \\ &\longrightarrow 7\end{aligned}$$

CBV vs CBN

One way in which CBV and CBN differ is when arguments to functions have no normal forms.

$$(\lambda x. (\lambda y. y)) \Omega$$

Under CBV semantics, this term does not have a normal form. If we use CBN semantics, then we have

$$(\lambda x. (\lambda y. y)) \Omega \longrightarrow_{\text{CBN}} \lambda y. y$$

CBV and CBN

- ▶ CBV and CBN are common evaluation orders
- ▶ Many programming languages use CBV semantics
- ▶ “Lazy” languages, such as Haskell, typically use Call-by-need semantics, a more efficient semantics similar to Call-by-name in that it does not evaluate actual arguments unless necessary
- ▶ However, Call-by-value semantics ensures that arguments are evaluated at most once.

Break

- ▶ If possible, give a program that cannot reduce in CBN and CBV, but reduces in full β -reduction.
- ▶ If possible, give a program that steps to the same expression in CBN and CBV.
- ▶ Formulate the rules of CBV in big-step style.
- ▶ How would you create a let-binding in lambda calculus?
- ▶ How do we define $e_1\{e_2/x\}$ formally?

CBV in big-step

$$\frac{}{\lambda x. e \Downarrow \lambda x. e}$$

$$\frac{e_1 \Downarrow \lambda x. e_{12} \quad e_2 \Downarrow v_2 \quad e_{12}\{v_2/x\} \Downarrow v}{e_1 e_2 \Downarrow v}$$

$e_1\{e_2/x\}$ formally

$$x\{e/x\} = e$$

$$y\{e/x\} = y$$

if $y \neq x$

$$(\lambda y. e_1)\{e/x\} = \lambda y. (e_1\{e/x\})$$

if $y \neq x$ and $y \notin FV(e)$

$$(e_1 e_2)\{e/x\} = (e_1\{e/x\}) (e_2\{e/x\})$$

$e_1\{e_2/x\}$ (almost) formally

$$x\{e/x\} = e$$

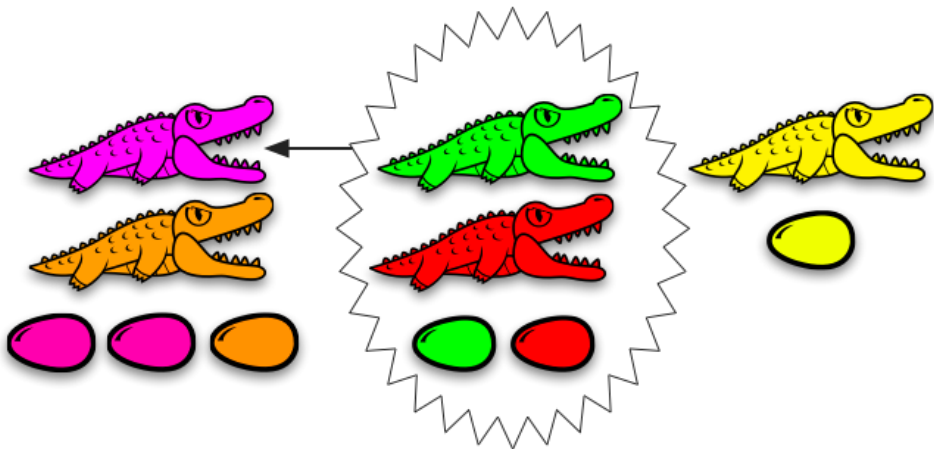
$$y\{e/x\} = y$$

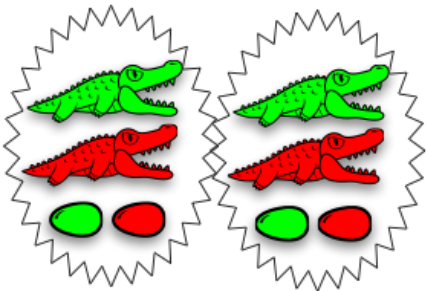
if $y \neq x$

$$(\lambda y. e_1)\{e/x\} = \lambda y. (e_1\{e/x\})$$

if $y \neq x$ and $y \notin FV(e)$

$$(e_1 e_2)\{e/x\} = (e_1\{e/x\}) (e_2\{e/x\})$$





Rules

- α -conversion

$$\lambda x.e[x] \rightarrow \lambda y.e[y]$$

- β -reduction

$$(\lambda x.e_1) e_2 \rightarrow e_1\{e_2/x\}$$

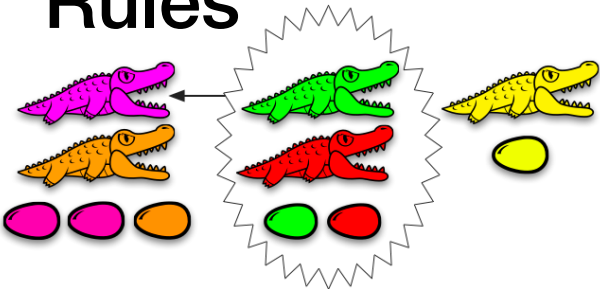
- η -conversion

$$(\lambda x.e x) \rightarrow e \text{ if } x \text{ does not occur free in } e$$

Rules

- α -conversion

$$\lambda x.e[x] \rightarrow \lambda y.e[y]$$



- β -reduction

$$(\lambda x.e1) e2 \rightarrow e1\{e2/x\}$$

- η -conversion

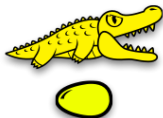
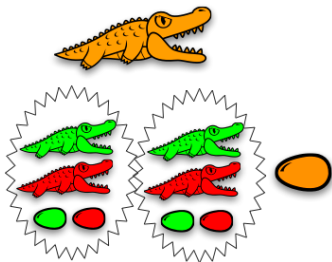
$$(\lambda x.e x) \rightarrow e \text{ if } x \text{ does not occur free in } e$$

Rules

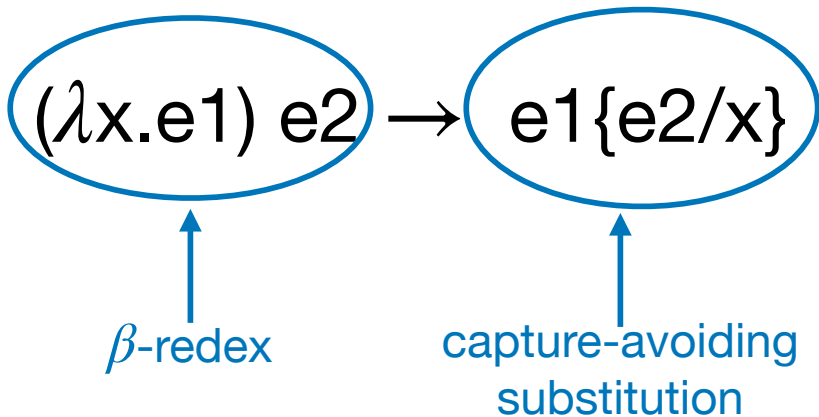
- α -conversion
 $\lambda x.e[x] \rightarrow \lambda y.e[y]$

- β -reduction
 $(\lambda x.e1) e2 \rightarrow e1\{e2/x\}$

- η -conversion
 $(\lambda x.e x) \rightarrow e$ if x does not occur free in e



β -reduction



β -reduction

$$(\lambda x. e1) e2 \rightarrow e1\{e2/x\}$$

β -reduction

The diagram illustrates the β -reduction rule. It shows the expression $(\lambda x. e1) e2$ on the left, which is circled in blue. An arrow points from the text β -redex below to the circled expression. An arrow points from the circled expression to the right, leading to the result $e1\{e2/x\}$.

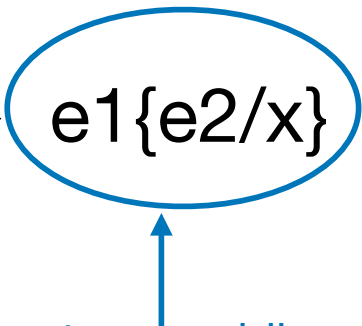
$$(\lambda x. e1) e2 \rightarrow e1\{e2/x\}$$

β -redex

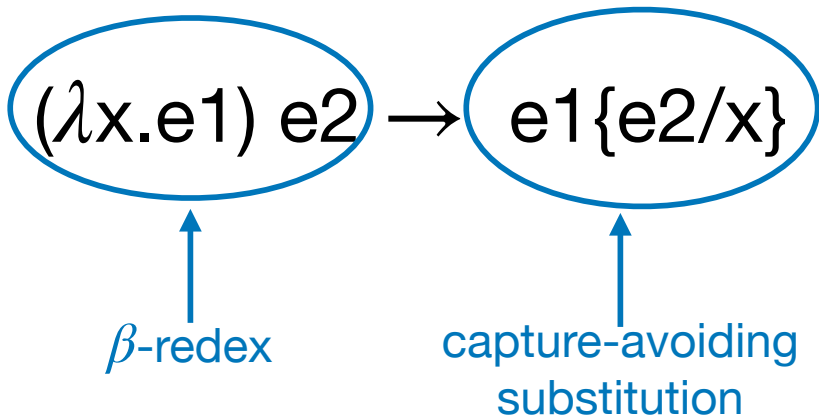
β -reduction

$$(\lambda x. e1) e2 \rightarrow e1\{e2/x\}$$

capture-avoiding substitution



β -reduction



$(\lambda x.x) ((\lambda x.x) (\lambda z.(\lambda x.x) z))$

$(\lambda x.x) ((\lambda x.x) (\lambda z.(\lambda x.x) z))$

$(\lambda x.x) ((\lambda x.x) (\lambda z.(\lambda x.x) z))$

↑
CBN

$(\lambda x.x) ((\lambda x.x) (\lambda z.(\lambda x.x) z))$

↑
CBV

$(\lambda x.x) ((\lambda x.x) (\lambda z.(\lambda x.x) z))$

only allowed in
full

$(\lambda x.x) ((\lambda x.x) (\lambda z.(\lambda x.x) z))$

↑
CBN

↑
CBV

↑
only allowed in
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