Curry-Howard correspondence
Existential types

Zena M. Ariola
University Of Oregon

Outline

- Natural Deduction - Minimal Logic
- Simply Typed Lambda-calculus
- Soundness and Completeness of Inference Systems
- β-reduction and η-equality
- Intuitionistic and Classical Logic - jumps
- Forall quantification - Polymorphism
- Existential quantification - Packages

The foundational crisis of mathematics

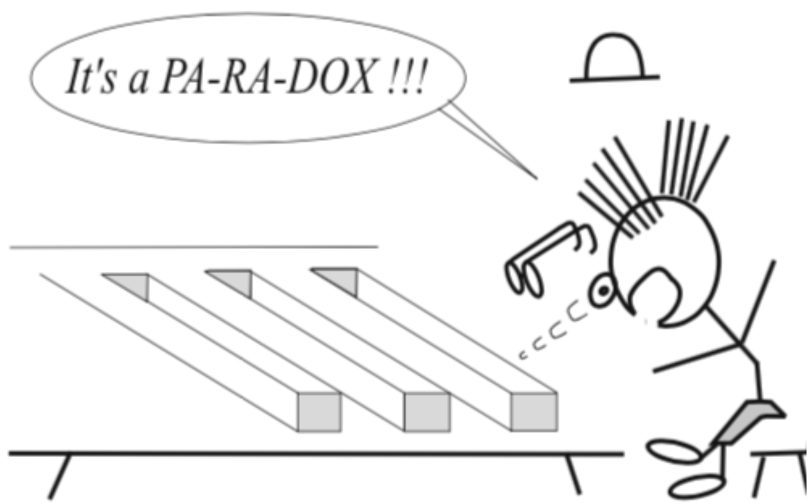
Georg Cantor (1845-1918)

At the beginning of the 19th century set theory was born

This theory was very promising because it offered a common foundation to all the fields of mathematics.

$$\mathscr{R} = \{x \mid x \notin x\}$$

$$\mathscr{R} \in \mathscr{R} \Longleftrightarrow \mathscr{R} \notin \mathscr{R}$$



Need of logic to define mathematics

David Hilbert
(1862-1943)

Let's devise a complete and finite set of axioms from which all true statements can be derived.

Gentzen introduced a formal system which modeled mathematical reasoning quite directly

Gerhard Gentzen
(1909-1945)

Kurt Gödel
(1906-1978)

Gödel in 1935 published his incompleteness result.

Natural Deduction

Natural Deduction - Gentzen (1935)

Formulae:

$$A, B ::= X \mid A \to B \mid A \wedge B \mid A \vee B$$

Judgement: $\vdash A$ $\Gamma \vdash A$

If all assumptions in Γ are true then **A** is true

Introduction and Elimination rules

Natural deduction

$$\frac{}{\Gamma, A \vdash A}\ Ax$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\wedge E_2$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\to_I \qquad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}\to_E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee I_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee I_2 \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\vee E$$

Exercises

$$A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$$

$$(A \vee B) \wedge (A \vee C) \vdash A \vee (B \wedge C)$$

$$A \rightarrow (B \rightarrow C) \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)$$

$$(A \rightarrow B) \rightarrow (A \rightarrow C) \vdash A \rightarrow (B \rightarrow C)$$

$$A \rightarrow (B \vee C) \vdash (A \rightarrow B) \vee (A \rightarrow C)$$

$$(A \rightarrow B) \vee (A \rightarrow C) \vdash A \rightarrow (B \vee C)$$

Formulae as Types

Formulae $A, B ::= X \mid A \to B \mid A \vee B \mid A \wedge B$

Types $A, B ::= X \mid A \to B \mid A + B \mid A \times B$

Formulae are not terms as in logic programming

Natural deduction

$$\frac{}{\Gamma, A \vdash A}\ Ax$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\wedge E_2$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\to_I \qquad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}\to_E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee I_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee I_2 \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\vee E$$

Proofs as programs

If π is a proof of formula A then π can be represented as a lambda-calculus term

$$\frac{}{\Gamma, x : A \vdash x : A}\ Ax$$

$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}\times_I \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \pi_1 e : A}\times_{E_1} \qquad \frac{\Gamma \vdash e : A \times B}{\Gamma \vdash \pi_2 : B}\times_{E_2}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}\to_I \qquad \frac{\Gamma \vdash e_1 : A \to B \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 e_2 : B}\to_E$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \mathit{inl}\ e : A + B}+_1 \qquad \frac{\Gamma \vdash e : B}{\Gamma \vdash \mathit{inr}\ e : A + B}+_2$$

$$\frac{\Gamma \vdash e : A + B \quad \Gamma, x : A \vdash e_1 : C \quad \Gamma, x : B \vdash e_2 : C}{\Gamma \vdash \mathit{case}\ e\ \mathit{of}\ \mathit{inl}\ x \to e_1 \mid \mathit{inr}\ x \to e_2 : C}+_E$$

Proofs as Programs

There is a one-to-one correspondence:

Typable terms ≈ minimal propositional logic proofs

$$x_1 : A_1, x_2 : A_2, \cdots, x_n : A_n \vdash e : A$$

e is a proof of A from assumptions $A_1, A_2, \cdots, A_n$

e is a program of type A with free variables $x_1, x_2, \cdots, x_n$ of type $A_1, A_2, \cdots, A_n$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (A \to B), A \vdash A \to (A \to B) \quad A \to (A \to B), A \vdash A}{A \to (A \to B), A \vdash A \to B} \qquad \dfrac{}{A \to (A \to B), A \vdash A}}{A \to (A \to B), A \vdash B}}{A \to (A \to B) \vdash A \to B}}{\vdash (A \to (A \to B)) \to A \to B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x : A{\to}(A{\to}B),\, y : A \vdash x : A{\to}(A{\to}B) \quad x : A{\to}(A{\to}B),\, y : A \vdash y : A}{x : A{\to}(A{\to}B),\, y : A \vdash xy : A{\to}B} \qquad \dfrac{}{x : A \to (A \to B),\, y : A \vdash y : A}}{x : A{\to}(A{\to}B),\, y : A \vdash xyy : B}}{x : A{\to}(A{\to}B) \vdash \lambda y.xyy : A{\to}B}}{\vdash \lambda x.\lambda y.xyy : (A{\to}(A{\to}B)) \to A{\to}B}$$

Harmony (Dummett 1976)

Local Soundness shows that the elimination rules are not too strong.

$$\frac{\Gamma \vdash A \vee B}{\Gamma \vdash B}$$

Local Completeness show that the elimination rules are not too weak

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

Local Soundness

Elimination rules applied to the result of an introduction rule do not produce new information.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma \vdash A} \qquad \dfrac{\mathcal{E}}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\wedge I}{\Gamma \vdash A}\wedge E \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}\\ \Gamma \vdash A\end{array}$$

Local Reductions

Local Soundness

Elimination rules applied to the result of an introduction rule do not produce new information.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, A \vdash B\end{array}}{\Gamma \vdash A \to B}\to_I \qquad \begin{array}{c}\mathcal{E}\\ \Gamma \vdash A\end{array}}{\Gamma \vdash B}\to_E \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}[\mathcal{E}/A]\\ \Gamma \vdash B\end{array}$$

Local Reductions

What does local soundness correspond to?

Howard (1980) observes that the elimination of a detour corresponds to β-reduction

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma, A \vdash B}}{\Gamma \vdash A \to B} \to_i \quad \dfrac{\mathcal{E}}{\Gamma \vdash A}}{\Gamma \vdash B} \to_e \qquad \Longrightarrow \qquad \dfrac{\mathcal{D}[\mathcal{E}/A]}{\Gamma \vdash B}$$

$$\dfrac{\dfrac{\Gamma, x : A \vdash e_1 : B}{\Gamma \vdash \lambda x : A.e_1 : A \to B} \quad \Gamma \vdash e_2 : A}{\Gamma \vdash (\lambda x : A.e_1)\; e_2 : B} \qquad \Longrightarrow \qquad \Gamma \vdash e_1[e_2/x] : B$$

$$(\lambda x \,.\, e_1)e_2 \to_\beta e_1[e_2/x]$$

Can all the detours be eliminated from a proof?

A term in normal form (i.e. without β-redexes) corresponds to a proof without detours

Local soundness for pairs

$$\dfrac{\dfrac{\dfrac{\mathcal{D}}{\Gamma \vdash A} \quad \dfrac{\mathcal{E}}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\wedge I}{\Gamma \vdash A}\wedge E \quad \Longrightarrow \quad \dfrac{\mathcal{D}}{\Gamma \vdash A}$$

$$\dfrac{\dfrac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}\times_I}{\Gamma \vdash \pi_1\,(e_1, e_2) : A}\times_E \quad \Longrightarrow \quad \Gamma \vdash e_1 : A$$

$$\pi_1(e_1, e_2) \rightarrow e_1$$
$$\pi_2(e_1, e_2) \rightarrow e_2$$

Local Completeness

The elimination rules are strong enough to get all the pieces out of a structured proposition so that they can be recomposed. A proof can always end with an introduction rule.

$$\begin{array}{c}\mathscr{D}\\ \Gamma \vdash A \wedge B\end{array} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathscr{D}\\ \Gamma \vdash A \wedge B\end{array}}{\Gamma \vdash A}\wedge E \qquad \dfrac{\begin{array}{c}\mathscr{D}\\ \Gamma \vdash A \wedge B\end{array}}{\Gamma \vdash B}\wedge E}{\Gamma \vdash A \wedge B}\wedge I$$

Local Expansion

Local completeness

The elimination rules are strong enough to get all the pieces out of a structured proposition so that they can be recomposed. A proof can always end with an introduction rule.

$$\begin{array}{c}\mathscr{D}\\ \Gamma \vdash A \to B\end{array} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathscr{D}\\ \Gamma, A \vdash A \to B\end{array} \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B}\to_E}{\Gamma \vdash A \to B}\to_I$$

Local Expansion

What does completeness correspond to?

$$\begin{array}{c}\mathscr{D}\\ \Gamma \vdash A \to B\end{array} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathscr{D}\\ \Gamma, A \vdash A \to B\end{array} \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B} \to_E}{\Gamma \vdash A \to B} \to_I$$

$$\Gamma \vdash e : A \to B \quad \Longrightarrow \quad \dfrac{\dfrac{\Gamma, x : A \vdash e : A \to B \qquad \Gamma, x : A \vdash x : A}{\Gamma, x : A \vdash e\, x : B} \to_E}{\Gamma \vdash \lambda x. e\, x : A \to B} \to_I$$

$$e =_\eta \lambda x \,.\, e\ x$$

Extensionality witnesses local completeness

Pairs are surjective

$$\dfrac{\mathscr{D}}{\Gamma \vdash A \wedge B} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\mathscr{D}}{\Gamma \vdash A \wedge B}}{\Gamma \vdash A}\wedge E \quad \dfrac{\dfrac{\mathscr{D}}{\Gamma \vdash A \wedge B}}{\Gamma \vdash B}\wedge E}{\Gamma \vdash A \wedge B}\wedge I$$

$$\Gamma \vdash e : A \times B \quad \Longrightarrow \quad \dfrac{\dfrac{\Gamma \vdash e : A \times B}{\pi_1\, e : A}\times_E \quad \dfrac{\Gamma \vdash e : A \times B}{\Gamma \vdash \pi_2\, e : B}\times_E}{\Gamma \vdash (\pi_1\, e, \pi_2\, e) : A \times B}\times_I$$

$$e =_\eta (\pi_1\ e, \pi_2\ e)$$

Disjunction and union

Logic

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee_i \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee_i \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\vee_e$$

Type theory

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash inj_1\ M : A + B}+_i \qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash inj_2\ M : A + B}+_i$$

$$\frac{\Gamma \vdash M : A + B \quad \Gamma, x_1 : A \vdash C \quad \Gamma, x_2 : B \vdash C}{\Gamma \vdash case\ M\ of\ inj_1\ x_1 : A \Rightarrow N_1 \mid inj_2 x_2 \Rightarrow N_2\ end : C}+_e$$

Local reductions

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \vee B}\vee I \quad \begin{array}{c}\mathcal{E}\\ \Gamma, A \vdash C\end{array} \quad \Gamma, B \vdash C}{\Gamma \vdash C}\vee E \implies \begin{array}{c}\mathcal{E}[\mathcal{D}/A]\\ \Gamma \vdash C\end{array}$$

$$\frac{\dfrac{\Gamma \vdash M : A}{\Gamma \vdash inj_1\ M : A + B}+I \quad \Gamma, x_1 : A \vdash N_1 : C \quad \Gamma, x_2 : B \vdash N_2 : C}{\Gamma \vdash case\ M\ of\ inj_1\ x_1 \Rightarrow N_1 \mid inj_2\ x_2 \Rightarrow N_2\ end : C}+E \implies \Gamma \vdash N_1[M/x_1]$$

Disjunction and union - Local expansion

Local Expansion

$$\begin{array}{c}\mathcal{D}\\ \Gamma\vdash A\vee B\end{array} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\vdash A\vee B\end{array} \qquad \dfrac{\Gamma, A\vdash A}{\Gamma, A\vdash A\vee B}\vee I \qquad \dfrac{\Gamma, B\vdash B}{\Gamma, B\vdash A\vee B}\vee I}{\Gamma\vdash A\vee B}\vee E$$

$$\Gamma\vdash M : A+B \Longrightarrow \dfrac{\Gamma\vdash M : A+B \qquad \dfrac{\Gamma, x_1 : A\vdash x_1 : A}{\Gamma, x_1 : A\vdash inj_1\ x_1 : A+B}+I \qquad \dfrac{\Gamma, x_2 : B\vdash x_2 : B}{\Gamma, x_2 : B\vdash inj_2\ x_2 : A+B}+I}{\Gamma\vdash case\ M\ of\ inj_1\ x_1 \Rightarrow inj_1\ x_1 \mid inj_2\ x_2 \Rightarrow inj_2\ x_2 : A+B}+E$$

$$M =_\eta case\ M\ of\ inj_1 x_1 \Rightarrow inj_1 x \mid inj_2 x_2 \Rightarrow inj_2 x$$

Note that the elimination follows the introduction, instead of the other way around!

Hilbert axioms

D. Hilbert
(1862-1943)

$$A \to A \qquad (1)$$

$$A \to B \to A \qquad (2)$$

$$(A \to B \to C) \to (A \to B) \to A \to C \qquad (3)$$

Modus Ponens

$$\vdash A \to ((B \to A) \to A) \qquad (2)$$

$$\vdash A \to ((B \to A) \to A) \to (A \to (B \to A)) \to A \to A \qquad (3)$$

$$\vdash (A \to (B \to A)) \to (A \to A) \qquad MP$$

$$\vdash A \to B \to A \qquad (2)$$

$$\vdash A \to A \qquad MP$$

Combinatory Logic

Haskell Curry
(1900-1982)

Terms $M ::= I \mid K \mid S \mid M\,M$

Axioms

$$I\,M = M$$

$$K\,M\,N = M$$

$$S\,M\,N\,P = (M\,P)(N\,P)$$

Types

$$I : A \to A$$

$$K : A \to B \to A$$

$$S : (A \to B \to C) \to (A \to B) \to A \to C$$

Hilbert axioms $\approx$ Typed Combinatory Logic

Proofs as Programs

$$\vdash K : A \to ((B \to A) \to A) \qquad (2)$$

$$\vdash S : A \to ((B \to A) \to A) \to (A \to (B \to A)) \to A \to A \qquad (3)$$

$$\vdash SK : (A \to (B \to A)) \to (A \to A) \qquad MP$$

$$\vdash K : A \to B \to A \qquad (2)$$

$$\vdash SKK : A \to A \qquad MP$$

Curry-Howard isomorphism

Logic	Type Theory
Formula	Type
Proof	Program
Detour eliminations	β-reduction
Expansions	Extensionality
Propositional logic	STLC

Is this a coincidence?

Intuitionistic Logic

Formulae:

$$A, B ::= X \mid A \to B \mid A \wedge B \mid A \vee B \mid \bot$$

$$\neg A \triangleq A \to \bot$$

Introduction rule?

Elimination rule?

Elimination of detour?

Computational Interpretation of IL

- What are terms of type $A \to \bot$? They are special functions: they never return! We call them continuations
- What about EFQ?

$$\dfrac{\dfrac{\Gamma, \mathbf{tp} : \neg A \vdash e : A}{\Gamma, \mathbf{tp} : \neg A \vdash \mathbf{tp}\ e : \bot}}{\Gamma, \mathbf{tp} : \neg A \vdash \mathit{jump}\ \mathbf{tp}\ e : B} \bot_E$$

$$\mathcal{A}bort\ e \triangleq \mathit{jump}\ \mathbf{tp}\ e$$

```
fun product  nil      = 0
 |  product  (x::xs) = if x=0 then Abort 0 else x*(prod xs)
```

Classical Logic and Control Operators

Classical Logic is obtained by adding one of the following axioms to Intuitionistic logic:

$$A \vee \neg A \quad \neg\neg A \to A \quad (\neg A \to \bot) \to A \quad ((A \to \bot) \to A) \to A$$

$$\frac{\Gamma \vdash M : \bot}{\Gamma \vdash \mathcal{A}bort\, M : A} \qquad \frac{\Gamma, k : \neg A \vdash M : A}{\Gamma \vdash \texttt{callcc}(\lambda k.M) : A}\ PL_{\bot}$$

$$\frac{\Gamma, k : \neg A \vdash M : \bot}{\Gamma \vdash \mathcal{C}(\lambda k.M) : A}\ PBC$$

Proofs-as-Programs for Classical Logic

Intuitionist Logic	=	Minimal Logic + EFQ
Minimal Classical Logic	=	Minimal Logic + Pierce Law
Classical Logic	=	Minimal Logic + Pierce Law + EFQ

Logic	Type Theory
Minimal Logic	λ-calculus
Intuitionistic Logic	λ-calculus + $\mathcal{A}bort$
Minimal Classical	λ-calculus + `callcc` + `throw`
Classical logic	λ-calculus + `callcc` + `throw` + **tp**

If A is provable in classical logic, then $[\![A]\!]$ is provable in IL

Negative translations and continuation-passing style transformations

Second-order quantification - Polymorphism

$$A, B ::= X \mid A \to B \mid A \wedge B \mid A \vee B \mid \forall X\,.\,A$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall X.A}\forall I \qquad \frac{\Gamma \vdash \forall X.A}{\Gamma \vdash A[B/X]}\forall E$$

John Reynold, 1974:

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \Lambda X.e : \forall X.A}\forall I \qquad \frac{\Gamma \vdash e : \forall X.A}{\Gamma \vdash e[B] : A[B/X]}\forall E$$

$$\dfrac{\dfrac{\dfrac{x : X \vdash x : X}{x : X \vdash \Lambda X.x : \forall X.X}\forall_I}{x : X \vdash (\Lambda X.x)[B] : B}\forall_E}{\vdash \lambda x : X.(\Lambda X.x)[B] : X \to B}\to_I$$

Where is the problem? $\dfrac{\Gamma \vdash A \qquad X \text{ does not occur free in } \Gamma}{\Gamma \vdash \forall X.A}\forall I$

Second-order quantification-Abstraction

$$A, B ::= X \mid A \to B \mid A \wedge B \mid A \vee B \mid \forall X\,.\,A \mid \exists X\,.\,A$$

$$\frac{\Gamma \vdash A[B/X]}{\Gamma \vdash \exists X.A}\,\exists_I \qquad \frac{\Gamma \vdash \exists X.A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,\exists_E$$

$$\frac{\Gamma \vdash e : A[B/X]}{\Gamma \vdash pack\ \{B, e\} : \exists X.A}\,\exists_I \qquad \frac{\Gamma \vdash e_1 : \exists X.A \quad \Gamma, x : A \vdash e_2 : B}{\Gamma \vdash unpack\ \{X, x\} = e_1\ in\ e_2 : B}\,\exists_E$$

$$\cfrac{\cfrac{\exists X.X \vdash \exists X.X \quad X \vdash X}{\exists X.X \vdash X}\,\exists_E}{\vdash \exists X.X \to X}\,\to_i$$

$$\frac{\Gamma \vdash \exists X.A \quad \Gamma, A \vdash B \quad X \notin FV(\Gamma, B)}{\Gamma \vdash B}\,\exists_E$$