Large-step semantics

CS 1520 (Spring 2025)

Harvard University

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Today, we learn to

- define and use large-step operational semantics
  (also known as big-step semantics)
- prove equivalence between semantics
  (large-step and small-step)
- discuss the advantages
  of each style of semantics

Small-step semantics (Review)

$$\longrightarrow \subseteq \textbf{Config} \times \textbf{Config}$$

$$< e, \sigma > \longrightarrow^* < n, \sigma' >$$

Large-step semantics

$$\Downarrow \subseteq \textbf{Config} \times \textbf{FinalConfig}$$

where

$$\textbf{Config} = \textbf{Exp} \times \textbf{Store}$$
$$\text{and } \textbf{FinalConfig} = \textbf{Int} \times \textbf{Store} \subseteq \textbf{Config}.$$

We write $< e, \sigma > \Downarrow < n, \sigma' >$ to mean that $(< e, \sigma >, < n, \sigma' >) \in \Downarrow$

Small-step vs Large-step

$$< e, \sigma > \Downarrow < n, \sigma' >$$

$$< e, \sigma > \longrightarrow^{*} < n, \sigma' >$$

Small-step vs Large-step

$$< e, \sigma > \Downarrow < n, \sigma' >$$

$$< e, \sigma > \longrightarrow^{*} < n, \sigma' >$$

$$< e, \sigma > \longrightarrow < e_1, \sigma_1 > \ldots \longrightarrow < e_k, \sigma_k > \longrightarrow < n, \sigma' >$$

Semantic Rules (1)

$$\textsc{Int}_{\textsc{Lrg}}\ \frac{}{< n, \sigma > \Downarrow < n, \sigma >}$$

$$\textsc{Var}_{\textsc{Lrg}}\ \frac{}{< x, \sigma > \Downarrow < n, \sigma >} \text{ where } n = \sigma(x)$$

$$\textsc{Add}_{\textsc{Lrg}}\ \frac{\begin{array}{c} < e_1, \sigma > \Downarrow < n_1, \sigma'' > \\ < e_2, \sigma'' > \Downarrow < n_2, \sigma' > \\ \text{where } n \text{ is the sum of } n_1 \text{ and } n_2 \end{array}}{< e_1 + e_2, \sigma > \Downarrow < n, \sigma' >}$$

Semantic Rules (2)

$$\textsc{Mul}_{\textsc{Lrg}}\ \frac{\begin{array}{c} < e_1, \sigma > \Downarrow < n_1, \sigma'' > \\ < e_2, \sigma'' > \Downarrow < n_2, \sigma' > \\ \text{where } n \text{ is the product of } n_1 \text{ and } n_2 \end{array}}{< e_1 \times e_2, \sigma > \Downarrow < n, \sigma' >}$$

$$\textsc{Asg}_{\textsc{Lrg}}\ \frac{\begin{array}{c} < e_1, \sigma > \Downarrow < n_1, \sigma'' > \\ < e_2, \sigma''[x \mapsto n_1] > \Downarrow < n_2, \sigma' > \end{array}}{< x := e_1; e_2, \sigma > \Downarrow < n_2, \sigma' >}$$

Example

$< \text{foo} := 3; \text{foo} \times \text{bar}, \sigma > \Downarrow < 21, \sigma' >$ for a store σ such that $\sigma(\text{bar}) = 7$, and $\sigma' = \sigma[\text{foo} \mapsto 3]$.

Example Proof Tree

$$
\dfrac{
\textsc{Int}_{\textsc{Lrg}}\ \dfrac{}{< 3, \sigma > \Downarrow < 3, \sigma >}
\qquad
\textsc{Mul}_{\textsc{Lrg}}\ \dfrac{
\textsc{Var}_{\textsc{Lrg}}\ \dfrac{}{< \mathsf{foo}, \sigma' > \Downarrow < 3, \sigma' >}
\quad
\textsc{Var}_{\textsc{Lrg}}\ \dfrac{}{< \mathsf{bar}, \sigma' > \Downarrow < 7, \sigma' >}
}{< \mathsf{foo} \times \mathsf{bar}, \sigma' > \Downarrow < 21, \sigma' >}
}{< \mathsf{foo} := 3; \mathsf{foo} \times \mathsf{bar}, \sigma > \Downarrow < 21, \sigma' >}
$$

Equivalence of semantics

For all expressions e, stores σ and σ', and integers n, we have:

$$< e, \sigma > \Downarrow < n, \sigma' > \iff < e, \sigma > \longrightarrow^* < n, \sigma' > .$$

$\Longrightarrow$

$$P(e) = \forall \sigma, \sigma' \in \textbf{Store}.\ , \forall n \in \textbf{Int}.$$

$$< e, \sigma > \Downarrow < n, \sigma' >$$

$$\Longrightarrow$$

$$< e, \sigma > \longrightarrow^{*} < n, \sigma' >$$

Proof by structural induction on expression e.

$\Longrightarrow$ Case $e \equiv x$

We have

$$< x, \sigma > \Downarrow < n, \sigma' >$$

and need to show

$$< x, \sigma > \longrightarrow^{*} < n, \sigma' >$$

$\Longrightarrow$ Case $e \equiv x$

- By hypothesis, $< x, \sigma > \Downarrow < n, \sigma' >$.
- By inversion (only VAR_{LRG} applies), $n = \sigma(x)$ and $\sigma = \sigma'$.
- By VAR, $< x, \sigma > \longrightarrow < n, \sigma >$.
- We can conclude $< x, \sigma > \longrightarrow^{*} < n, \sigma >$.

$\Longrightarrow$ Case $e \equiv x$

$$\textsc{Var}_{\textsc{Lrg}} \; \frac{}{< x, \sigma > \Downarrow < n, \sigma >} \; \text{where } n = \sigma(x)$$

$$\textsc{Var} \; \frac{}{< x, \sigma > \longrightarrow < n, \sigma >} \; \text{where } n = \sigma(x)$$

- By hypothesis, $< x, \sigma > \Downarrow < n, \sigma' >$.
- By inversion (only $\textsc{Var}_{\textsc{Lrg}}$ applies), $n = \sigma(x)$ and $\sigma = \sigma'$.
- By $\textsc{Var}$, $< x, \sigma > \longrightarrow < n, \sigma >$.
- We can conclude $< x, \sigma > \longrightarrow^{*} < n, \sigma >$.

$\Longrightarrow$ Case $e \equiv n$

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- By hypothesis and inversion, we know that

$$< n, \sigma > \Downarrow < n, \sigma >$$

$\Longrightarrow$ Case $e \equiv n$

- By hypothesis and inversion, we know that

$$< n, \sigma > \Downarrow < n, \sigma >$$

- By reflexivity

$$< n, \sigma > \longrightarrow^{*} < n, \sigma >$$

$\Longrightarrow$ Case $e \equiv e_1 + e_2$

$\Longrightarrow$ Case $e \equiv e_1 + e_2$

$$P(e_1) = \forall n, \sigma, \sigma' : \quad < e_1, \sigma > \Downarrow < n, \sigma' > \Longrightarrow < e_1, \sigma > \longrightarrow^* < n, \sigma' >$$

$$P(e_2) = \forall n, \sigma, \sigma' : \quad < e_2, \sigma > \Downarrow < n, \sigma' > \Longrightarrow < e_2, \sigma > \longrightarrow^* < n, \sigma' >$$

$$P(e) = \forall n, \sigma, \sigma' : \quad < e_1 + e_2, \sigma > \Downarrow < n, \sigma' > \Longrightarrow < e_1 + e_2, \sigma > \longrightarrow^* < n, \sigma' >$$

Lemma 1 (Needed)

If $< e, \sigma > \longrightarrow^* < n, \sigma' >$ then for all n_1, e_2 the following hold.

- $< e + e_2, \sigma > \longrightarrow^* < n + e_2, \sigma' >$
- $< e \times e_2, \sigma > \longrightarrow^* < n \times e_2, \sigma' >$
- $< n_1 + e, \sigma > \longrightarrow^* < n_1 + n, \sigma' >$
- $< n_1 \times e, \sigma > \longrightarrow^* < n_1 \times n, \sigma' >$

$\Longrightarrow$ Case $e \equiv e_1 \times e_2$

$\Longrightarrow$ Case $e \equiv x := e_1; e_2$

Need lemma similar to lemma 1:

$$
\begin{aligned}
\forall e \in \textbf{Exp}.\ \forall \sigma, \sigma' \in \textbf{Store}, \forall n \in \textbf{Int}. \\
\text{if } \ < e, \sigma > \longrightarrow^* < n, \sigma' > \ \text{ then} \\
\forall x \in \textbf{Var}.\ \forall e_2 \in \textbf{Exp}. \\
< x := e; e_2, \sigma > \longrightarrow^* < e_2, \sigma'[x \mapsto n] > .
\end{aligned}
$$

Equivalence of semantics (Recall)

For all expressions e, stores σ and σ', and integers n, we have:

$$< e, \sigma > \Downarrow < n, \sigma' > \iff < e, \sigma > \longrightarrow^* < n, \sigma' > .$$

$\Longleftarrow$

Proof by mathematical induction on the number of steps $< e, \sigma > \longrightarrow^* < n, \sigma' >$.

$\Longleftarrow$

$$P(k) = \forall e \in \textbf{Exp}, \sigma, \sigma' \in \textbf{Store}.\ , \forall n \in \textbf{Int}.$$

$$< e, \sigma > \Downarrow < n, \sigma' >$$

$$\Longleftarrow$$

$$< e, \sigma > \longrightarrow^{k} < n, \sigma' >$$

Proof by mathematical induction on the number of steps k.

$\Longleftarrow$ Base case

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If $< e, \sigma > \longrightarrow^* < n, \sigma' >$ in zero steps, then we must have $e \equiv n$ and $\sigma' = \sigma$. By $\textsc{Int}_{\textsc{Lrg}}$, $< n, \sigma > \Downarrow < n, \sigma >$.

$\Longleftarrow$ Inductive case

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- Assume $< e, \sigma > \longrightarrow < e'', \sigma'' > \longrightarrow^* < n, \sigma' >$.
- Inductive Hypothesis: $< e'', \sigma'' > \Downarrow < n, \sigma' >$.

Lemma 2 (Needed)

For all e, e', σ, and n, if $< e, \sigma > \longrightarrow < e', \sigma'' >$ and $< e', \sigma'' > \Downarrow < n, \sigma' >$, then $< e, \sigma > \Downarrow < n, \sigma' >$.

Lemma 1

If $< e, \sigma > \longrightarrow^* < n, \sigma' >$ then for all n_1, e_2 the following hold.

- $< e + e_2, \sigma > \longrightarrow^* < n + e_2, \sigma' >$
- $< e \times e_2, \sigma > \longrightarrow^* < n \times e_2, \sigma' >$
- $< n_1 + e, \sigma > \longrightarrow^* < n_1 + n, \sigma' >$
- $< n_1 \times e, \sigma > \longrightarrow^* < n_1 \times n, \sigma' >$

Proof of Lemma 1

By mathematical induction on the number of evaluation steps in $\longrightarrow^*$.

Proof of Lemma 1 (Base case)

Immediate by reflexivity for 0 number of steps.

Proof of Lemma 1 (Inductive case)

- Suppose $< e, \sigma > \longrightarrow < e_1, \sigma_1 > \longrightarrow^* < n, \sigma' >$.
- Show $< e + e_2, \sigma > \longrightarrow^* < n + e_2, \sigma' >$.
- By induction, $< e_1 + e_2, \sigma_1 > \longrightarrow^* < n + e_2, \sigma'' >$.
- By LADD, $< e + e_2, \sigma > \longrightarrow < e_1 + e_2, \sigma_1 >$.
- Result follows by transitivity:

$$< e + e_2, \sigma > \longrightarrow < e_1 + e_2, \sigma_1 > \longrightarrow^*$$

$$< n + e_2, \sigma'' >$$

Lemma 2

For all e, e', σ, and n, if $< e, \sigma > \longrightarrow < e', \sigma'' >$ and $< e', \sigma'' > \Downarrow < n, \sigma' >$, then $< e, \sigma > \Downarrow < n, \sigma' >$.

Proof of Lemma 2

(part of homework)

Break: Easier in Small-step or Large-step?

- Proving totality
- Dealing with non-termination
- Expressing interleaving / concurrency
- Your ideas?

Totality

$$\forall e \in \textbf{Exp}, \sigma \in \textbf{Store},$$
$$\exists n \in \textbf{Int}, \sigma' \in \textbf{Store},$$
$$< e, \sigma > \Downarrow < n, \sigma' >$$

Stuck vs Non-Terminating

$$e ::= \ldots \mid \mathsf{stop} \mid \mathsf{loop}$$

$$\textsc{Loop}\ \frac{}{<\mathsf{loop}, \sigma> \longrightarrow <\mathsf{loop}, \sigma>}$$

$$\textsc{Loop}_{\textsc{Lrg}}\ \frac{<loop, \sigma> \Downarrow <n, \sigma'>}{<loop, \sigma> \Downarrow <n, \sigma'>}$$

Rules Add (Original)

$$\textsc{LAdd}\ \frac{< e_1, \sigma > \longrightarrow < e_1', \sigma' >}{< e_1 + e_2, \sigma > \longrightarrow < e_1' + e_2, \sigma' >}$$

$$\textsc{RAdd}\ \frac{< e_2, \sigma > \longrightarrow < e_2', \sigma' >}{< n + e_2, \sigma > \longrightarrow < n + e_2', \sigma' >}$$

$$\textsc{Add}\ \frac{\text{where } p \text{ is the sum of } n \text{ and } m}{< n + m, \sigma > \longrightarrow < p, \sigma >}$$

Rules Add (Interleaving)

$$\textsc{LAdd}\ \frac{< e_1, \sigma > \longrightarrow < e_1', \sigma' >}{< e_1 + e_2, \sigma > \longrightarrow < e_1' + e_2, \sigma' >}$$

$$\textsc{RAdd}\ \frac{< e_2, \sigma > \longrightarrow < e_2', \sigma' >}{< e_1 + e_2, \sigma > \longrightarrow < e_1 + e_2', \sigma' >}$$

$$\textsc{Add}\ \frac{\text{where } p \text{ is the sum of } n \text{ and } m}{< n + m, \sigma > \longrightarrow < p, \sigma >}$$

Concurrency

$$\textsc{LCon}\ \frac{< e_1, \sigma > \longrightarrow < e_1', \sigma' >}{< e_1 || e_2, \sigma > \longrightarrow < e_1' || e_2, \sigma' >}$$

$$\textsc{RCon}\ \frac{< e_2, \sigma > \longrightarrow < e_2', \sigma' >}{< e_1 || e_2, \sigma > \longrightarrow < e_1 || e_2', \sigma' >}$$

$$\textsc{Con}\ \frac{\text{where } p \text{ is the join of } n \text{ and } m}{< n || m, \sigma > \longrightarrow < p, \sigma >}$$