ES128: Homework 5
Solutions

Problem 1

For the beam shown in Fig. 1, compute the deflection at the element nodes. The modulus of elasticity is $E = 10 \times 10^6$ Pa and the cross section is as shown in Fig. 1. Also compute the maximum bending stress. Use the finite element method with the minimum number of elements.

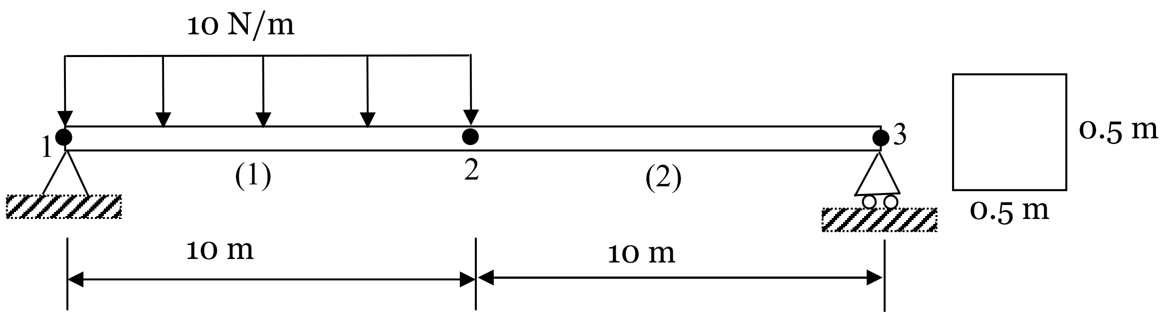

Fig. 1

Solution

The elements and nodes are shown in Fig. 1.

For both elements, E= 10^7 , $I = (0.5)^4/12$ =0.0052, and L=10.

For elements 1 and 2, the stiffness matrix is

$$K^{(1)} = K^{(2)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} = 10^4 \begin{bmatrix} 0.0625 & 0.3125 & -0.0625 & 0.3125 \\ 0.3125 & 2.0833 & -0.3125 & 1.0417 \\ -0.0625 & -0.3125 & 0.0625 & -0.3125 \\ 0.3125 & 1.0417 & -0.3125 & 2.0833 \end{bmatrix}.$$

For element 1, the force vector is

$$f^{(1)} = \frac{pL}{2}\begin{bmatrix} 1 \\ L/6 \\ 1 \\ -L/6 \end{bmatrix} + \begin{bmatrix} r_{y1} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -50 + r_{y1} \\ -83.3333 \\ -50 \\ 83.3333 \end{bmatrix}.$$

The global system of the equations is

$$\begin{matrix} [1u] & [3u] & [1\theta] & [2u] & [2\theta] & [3\theta] \end{matrix}$$

$$10^4 \begin{bmatrix} 0.0625 & 0 & 0.3125 & -0.0625 & 0.3125 & 0 \\ 0 & 0.0625 & 0 & -0.0625 & -0.3125 & -0.3125 \\ 0.3125 & 0 & 2.0833 & -0.3125 & 1.0417 & 0 \\ -0.0625 & -0.0625 & -0.3125 & 0.125 & 0 & 0.3125 \\ 0.3125 & -0.3125 & 1.0417 & 0 & 4.1666 & 1.0417 \\ 0 & -0.3125 & 0 & 0.3125 & 1.0417 & 2.0833 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \theta_1 \\ u_2 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -50 + r_{u1} \\ r_{u3} \\ -83.3333 \\ -50 \\ 83.3333 \\ 0 \end{bmatrix}.$$

Solution of the above system gives

$$\begin{bmatrix}\theta_1\\ u_2\\ \theta_2\\ \theta_3\end{bmatrix}=\begin{bmatrix}-0.0360\\ -0.200\\ 0.0040\\ 0.0280\end{bmatrix}.$$

For element 1,

$$\varepsilon_{xx}=-y\begin{bmatrix}\dfrac{d^2N_{u1}}{dx^2} & \dfrac{d^2N_{u1}}{dx^2} & \dfrac{d^2N_{u1}}{dx^2} & \dfrac{d^2N_{u1}}{dx^2}\end{bmatrix}\begin{bmatrix}0\\ \theta_1\\ u_2\\ \theta_2\end{bmatrix}$$

$$=-\frac{y}{10}\begin{bmatrix}0.6\xi & 3\xi-1 & -0.6\xi & 3\xi+1\end{bmatrix}\begin{bmatrix}0\\ -0.0360\\ -0.200\\ 0.0040\end{bmatrix}$$

$$=-\frac{y}{10}(0.0240\xi+0.0364),$$

where $-1\le\xi\le1$.

When $\xi=1$ and y= $\pm$ 0.25, ε_{xx} reaches the maximum value ($\pm$ 0.0015). Thus the maximum stress ($\sigma_{xx}=E\varepsilon_{xx}$) is $\pm1.5\times10^4$ Pa.

Problem 2

The two-dimensional frame structure shown in Fig. 2 (next page) is composed of two 2×4 m steel members ($E = 10\times 10^6$ Pa), and the 2 m dimension is perpendicular to the plane of loading . All connections are treated as welded joints. Using two beam-axial elements and the node number as shown, determine

a. The global stiffness matrix.
b. The global load vector.
c. The displacement components of node 2.
d. The reaction forces and moments at nodes 1 and 3.
e. Maximum stress in each element.

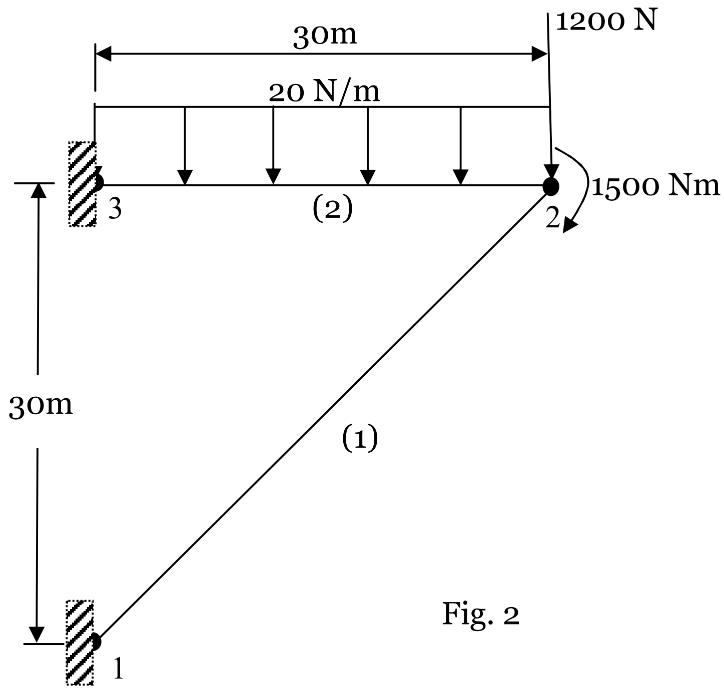

Fig. 2

Solution

The elements and nodes are shown in Fig. 2. For element 1, E= 10^7,
$I = 2(4)^3/12$ =10.6667, $\theta = \pi/4$, A=8, and L= $30\sqrt{2}$ =42.4264.

Thus, $\dfrac{EA}{L} = 1.8856\times 10^6$ =c1, $\dfrac{EI}{L^3} = 1.3968\times 10^3$ =c2, $\dfrac{EI}{L^2} = 5.9259\times 10^4$ =c3, and $\dfrac{EI}{L} = 2.5142\times 10^6$ =c4.

The local stiffness matrix is
$$\begin{bmatrix} c1 & 0 & 0 & -c1 & 0 & 0 \\ 0 & 12\times c2 & 6\times c3 & 0 & -12\times c2 & 6\times c3 \\ 0 & 6\times c3 & 4\times c4 & 0 & -6\times c3 & 2\times c4 \\ -c1 & 0 & 0 & c1 & 0 & 0 \\ 0 & -12\times c2 & -6\times c3 & 0 & 12\times c2 & -6\times c3 \\ 0 & 6\times c3 & 2\times c4 & 0 & -6\times c3 & 4\times c4 \end{bmatrix}.$$

The local-global transformation matrix is $\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

The global stiffness matrix of element 1 is

$$\begin{matrix} [1u_x] & [1u_y] & [1\theta] & [2u_x] & [2u_y] & [2\theta] \end{matrix}$$

$$10^7 \begin{bmatrix} 0.0951 & 0.0934 & -0.0251 & -0.0951 & -0.0934 & -0.0251 \\ 0.0934 & 0.0951 & 0.0251 & -0.0934 & -0.0951 & 0.0251 \\ -0.0251 & 0.0251 & 1.0057 & 0.0251 & -0.0251 & 0.5028 \\ -0.0951 & -0.0934 & 0.0251 & 0.0951 & 0.0934 & 0.0251 \\ -0.0934 & -0.0951 & -0.0251 & 0.0934 & 0.0951 & -0.0251 \\ -0.0251 & 0.0251 & 0.5028 & 0.0251 & -0.0251 & 1.0057 \end{bmatrix} \begin{bmatrix} 1u_x \\ 1u_y \\ 1\theta \\ 2u_x \\ 2u_y \\ 2\theta \end{bmatrix}.$$

For element 2, $\theta = 0$ and L= 30.

The global stiffness matrix of element 2 is

$$\begin{matrix} [3u_x] & [3u_y] & [3\theta] & [2u_x] & [2u_y] & [2\theta] \end{matrix}$$

$$10^7 \begin{bmatrix} 0.2667 & 0 & 0 & -0.2667 & 0 & 0 \\ 0 & 0.0047 & 0.0711 & 0 & -0.0047 & 0.0711 \\ 0 & 0.0711 & 1.4222 & 0 & -0.0711 & 0.7111 \\ -0.2667 & 0 & 0 & 0.2667 & 0 & 0 \\ 0 & -0.0047 & -0.0711 & 0 & 0.0047 & -0.0711 \\ 0 & 0.0711 & 0.7111 & 0 & -0.0711 & 1.4222 \end{bmatrix} \begin{bmatrix} 3u_x \\ 3u_y \\ 3\theta \\ 2u_x \\ 2u_y \\ 2\theta \end{bmatrix}.$$

The global stiffness matrix of the system is

$$\begin{matrix} [1u_x] & [1u_y] & [1\theta] & [3u_x] & [3u_y] & [3\theta] & [2u_x] & [2u_y] & [2\theta] \end{matrix}$$

$$K = 10^7 \begin{bmatrix} 0.0951 & 0.0934 & -0.0251 & 0 & 0 & 0 & -0.0951 & -0.0934 & -0.0251 \\ 0.0934 & 0.0951 & 0.0251 & 0 & 0 & 0 & -0.0934 & -0.0951 & 0.0251 \\ -0.0251 & 0.0251 & 1.0057 & 0 & 0 & 0 & 0.0251 & -0.0251 & 0.5028 \\ 0 & 0 & 0 & 0.2667 & 0 & 0 & -0.2667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0047 & 0.0711 & 0 & -0.0047 & 0.0711 \\ 0 & 0 & 0 & 0 & 0.0711 & 1.4222 & 0 & -0.0711 & 0.7111 \\ -0.0951 & -0.0934 & 0.0251 & -0.2667 & 0 & 0 & 0.3618 & 0.0934 & 0.0251 \\ -0.0934 & -0.0951 & -0.0251 & 0 & -0.0047 & -0.0711 & 0.0934 & 0.0998 & -0.0962 \\ -0.0251 & 0.0251 & 0.5028 & 0 & 0.0711 & 0.7111 & 0.0251 & -0.0962 & 2.4279 \end{bmatrix} \begin{bmatrix} 1u_x \\ 1u_y \\ 1\theta \\ 3u_x \\ 3u_y \\ 3\theta \\ 2u_x \\ 2u_y \\ 2\theta \end{bmatrix}.$$

The global force vector is f= $\begin{bmatrix} r_{x1} \\ r_{y1} \\ r_{\theta 1} \\ r_{x3} \\ r_{y3}-300 \\ r_{\theta 3}-1500 \\ 0 \\ -1500 \\ 0 \end{bmatrix}\begin{bmatrix} 1u_x \\ 1u_y \\ 1\theta \\ 3u_x \\ 3u_y \\ 3\theta \\ 2u_x \\ 2u_y \\ 2\theta \end{bmatrix}$.

Thus, $\mathrm{d_F} = \begin{bmatrix} u_{2x} \\ u_{2u} \\ \theta_2 \end{bmatrix}$,

$$\mathrm{K_{EF}} = 10^7\begin{bmatrix} -0.0951 & -0.0934 & -0.0251 \\ -0.0934 & -0.0951 & 0.0251 \\ 0.0251 & -0.0251 & 0.5028 \\ -0.2667 & 0 & 0 \\ 0 & -0.0047 & 0.0711 \\ 0 & -0.0711 & 0.7111 \end{bmatrix},$$

$$\mathrm{K_{FF}} = 10^7\begin{bmatrix} 0.3618 & 0.0934 & 0.0251 \\ 0.0934 & 0.0998 & -0.0962 \\ 0.0251 & -0.0962 & 2.4279 \end{bmatrix},$$

and $\mathrm{f_F} = \begin{bmatrix} 0 \\ -1500 \\ 0 \end{bmatrix}$.

Since $\mathrm{d_E} = 0$, $\mathrm{d_F} = \mathrm{K_{FF}^{-1} f_F} = \begin{bmatrix} 0.0005 \\ -0.0021 \\ -0.0001 \end{bmatrix}$.

Based on $\mathrm{f_E} = \mathrm{K_{EF} d_F}$, we can get
at node 1, $r_{x1} = 1.464\times10^3$, $r_{y1} = 1.4644\times10^3$, and $r_{\theta 1} = 0.2181\times10^3$,
at node 3, $r_{x3} = -1.464\times10^3$, $r_{y3} = 0.3356\times10^3$, and $r_{\theta 3} = 2.3621\times10^3$.

For element 1, the global displacement vector is [0, 0, 0, 0.0005, -0.0021,-0.0001]^T. With local-global transformation, we can get the local displacement vector [0, 0, 0, -0.0011, -0.0018,-0.0001]^T. The strain

$$\varepsilon_{xx} = -\frac{0.0011}{30\sqrt{2}} - y\begin{bmatrix} \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{u1}}{dx^2} \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix}$$

$$=-2.5927\times10^{-5} - \frac{y}{30\sqrt{2}}\begin{bmatrix} \frac{6\xi}{30\sqrt{2}} & 3\xi-1 & -\frac{6\xi}{30\sqrt{2}} & 3\xi+1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -0.0018 \\ -0.0001 \end{bmatrix}.$$

The maximum strain is -3.2785×10^{-5} when y=-2 and $\xi=1$. As a result, the maximum stress is -327.85 N.

For element 2; the local displacement vector is [0, 0, 0, 0.0005, -0.0021,-0.0001]^T, and the strain $\varepsilon_{xx} = \frac{0.0005}{30} - y\begin{bmatrix} \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{u1}}{dx^2} & \frac{d^2N_{u1}}{dx^2} \end{bmatrix}\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix}$

$$=1.6667\times10^{-5} - \frac{y}{30}\begin{bmatrix} \frac{6\xi}{30} & 3\xi-1 & -\frac{6\xi}{30} & 3\xi+1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -0.0021 \\ -0.0001 \end{bmatrix}.$$

The maximum strain is 3.1334×10^{-5} when y=2 and $\xi=-1$. As a result, the maximum stress is 313.34 N.

Problem 3

A square frame of length L = 1 m on each side is subject to equal and opposite horizontal loads P = 100 N. Each of the four beam members has a square cross-section of width a = 2 cm. The material is steel with Young's modulus E = 200GPa. Obtain the horizontal displacement, Δ, of joint A relative to joint C. (Hints: Make use of the two-fold symmetry to reduce the number of unknowns. Take the displacement components and rotation to vanish at the center of the frame. Symmetry dictates that the horizontal displacement and rotation vanish at B, while the vertical displacement and the rotation vanish at A.)

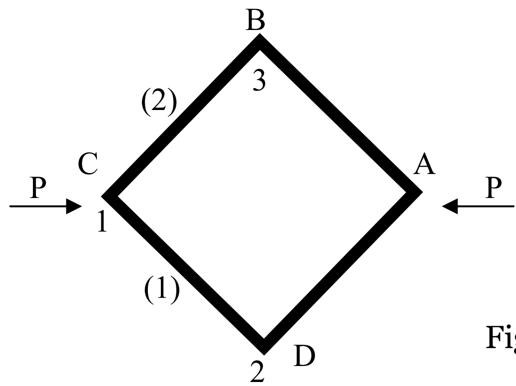

Fig. 3

Solution

Let us focus on the left part (CDB), and the nodes and elements are shown in Fig.3. For element 1, E= 10^{11}, $I = (2\times10^{-2})^4/12$ =1.3333×10^{-8}, A=4×10^{-4}, L= 1, and $\theta = -\pi/4$.
The global stiffness matrix of element 1 is

$$
\begin{matrix} [1u_x] & [1u_y] & [1\theta] & [2u_x] & [2u_y] & [2\theta] \end{matrix}
$$

$$
10^7\begin{bmatrix} 4.0016 & -3.9984 & 0.0011 & -4.0016 & 3.9984 & 0.0011 \\ -3.9984 & 4.0016 & 0.0011 & 3.9984 & -4.0016 & 0.0011 \\ 0.0011 & 0.0011 & 0.0011 & -0.0011 & -0.0011 & 0.0005 \\ -4.0016 & 3.9984 & -0.0011 & 4.0016 & -3.9984 & -0.0011 \\ 3.9984 & -4.0016 & -0.0011 & -3.9984 & 4.0016 & -0.0011 \\ 0.0011 & 0.0011 & 0.0005 & -0.0011 & -0.0011 & 0.0011 \end{bmatrix}\begin{bmatrix} 1u_x \\ 1u_y \\ 1\theta \\ 2u_x \\ 2u_y \\ 2\theta \end{bmatrix}.
$$

For element 2, $\theta = \pi / 4$, and the global stiffness matrix is

$$
\begin{matrix} [1u_x] & [1u_y] & [1\theta] & [3u_x] & [3u_y] & [3\theta] \end{matrix}
$$

$$
10^7 \begin{bmatrix} 4.0016 & 3.9984 & -0.0011 & -4.0016 & -3.9984 & -0.0011 \\ 3.9984 & 4.0016 & 0.0011 & -3.9984 & -4.0016 & 0.0011 \\ -0.0011 & 0.0011 & 0.0011 & 0.0011 & -0.0011 & 0.0005 \\ -4.0016 & -3.9984 & 0.0011 & 4.0016 & 3.9984 & 0.0011 \\ -3.9984 & -4.0016 & -0.0011 & 3.9984 & 4.0016 & -0.0011 \\ -0.0011 & 0.0011 & 0.0005 & 0.0011 & -0.0011 & 0.0011 \end{bmatrix} \begin{bmatrix} 1u_x \\ 1u_y \\ 1\theta \\ 3u_x \\ 3u_y \\ 3\theta \end{bmatrix}.
$$

The global stiffness matrix of the system is

$$
\begin{matrix} [1u_x] & [1u_y] & [1\theta] & [2u_x] & [2u_y] & [2\theta] & [3u_x] & [3u_y] & [3\theta] \end{matrix}
$$

$$
10^7 \begin{bmatrix} 8.0032 & 0 & 0 & -4.0016 & 3.9984 & 0.0011 & -4.0016 & -3.9984 & -0.0011 \\ 0 & 8.0032 & 0.0022 & 3.9984 & -4.0016 & 0.0011 & -3.9984 & -4.0016 & 0.0011 \\ 0 & 0.0022 & 0.0022 & -0.0011 & -0.0011 & 0.0005 & 0.0011 & -0.0011 & 0.0005 \\ -4.0016 & 3.9984 & -0.0011 & 4.0016 & -3.9984 & -0.0011 & 0 & 0 & 0 \\ 3.9984 & -4.0016 & -0.0011 & -3.9984 & 4.0016 & -0.0011 & 0 & 0 & 0 \\ 0.0011 & 0.0011 & 0.0005 & -0.0011 & -0.0011 & 0.0011 & 0 & 0 & 0 \\ -4.0016 & -3.9984 & 0.0011 & 0 & 0 & 0 & 4.0016 & 3.9984 & 0.0011 \\ -3.9984 & -4.0016 & -0.0011 & 0 & 0 & 0 & 3.9984 & 4.0016 & -0.0011 \\ -0.0011 & 0.0011 & 0.0005 & 0 & 0 & 0 & 0.0011 & -0.0011 & 0.0011 \end{bmatrix} \begin{bmatrix} 1u_x \\ 1u_y \\ 1\theta \\ 2u_x \\ 2u_y \\ 2\theta \\ 3u_x \\ 3u_y \\ 3\theta \end{bmatrix}.
$$

The displacement vector is $d = \begin{bmatrix} u_{1x} \\ 0 \\ 0 \\ 0 \\ u_{2y} \\ 0 \\ 0 \\ u_{3y} \\ 0 \end{bmatrix}$.

The force vector is $f = \begin{bmatrix} 100 \\ 0 \\ 0 \\ r_{x2} \\ 0 \\ r_{\theta 2} \\ r_{x3} \\ 0 \\ r_{\theta 3} \end{bmatrix}$.

Thus, $\mathrm{d_F} = \begin{bmatrix} u_{x1} \\ u_{y2} \\ u_{y3} \end{bmatrix}$,

$$\mathrm{K_F} = 10^7 \begin{bmatrix} 8.0032 & 3.9984 & -3.9984 \\ 3.9984 & 4.0016 & 0 \\ -3.9984 & 0 & 4.0016 \end{bmatrix},$$

and $\mathrm{f_F} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$.

Since $\mathrm{d_E} = 0$, $\mathrm{d_F} = \mathrm{K_F^{-1} f_F}$. Thus $\mathrm{d_F} = \begin{bmatrix} u_{x1} \\ u_{y2} \\ u_{y3} \end{bmatrix} = 10^{-3} \begin{bmatrix} 0.7816 \\ -0.7809 \\ 0.7809 \end{bmatrix}$.

The horizontal displacement of joint A relative to joint C is 1.5632×10^{-3} m.