Maxwell Mesh Method:

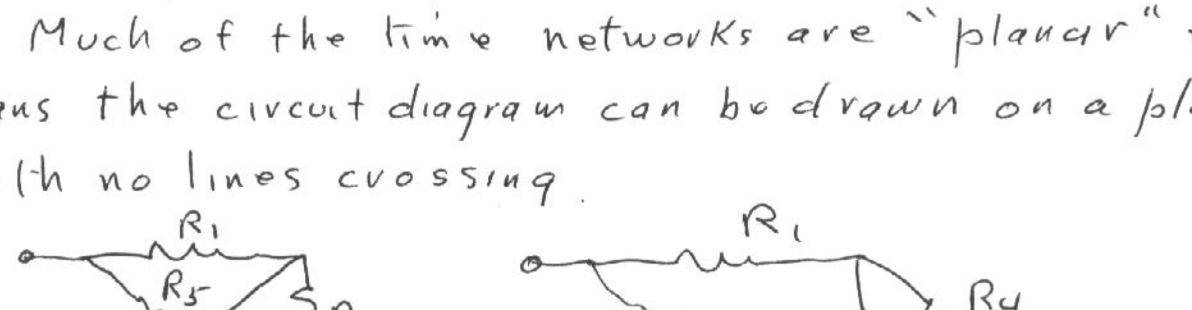
Instead of using branch currents as the variables a new set is introduced. These are a set of "circulating" or "mesh" currents which flow along the contour of each of a set of independent loops. KVL is then applied around each loop to yield a set of independent eqn's. (independent ⇒ each has something the others don't have).

How many eqn's (loops) are needed? For simple ckts get by inspection. However, there is a theorem from network topology that says if B is the number of branches, N is the # of nodes, then the # of independent loops (for a connected network) is

$$L = B - (N-1)$$

This is the # of ~~ta~~ closed paths (loops) needed to get an independent set — but it doesn't say how to choose the loops. There is a convenient out:

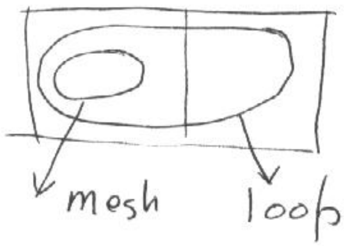
Much of the time networks are "planar" which means the circuit diagram can be drawn on a plane with no lines crossing.



looks nonplanar but can be drawn as ↗

The diagram of a planar network shows a bunch of "holes" which are the "windows" of the network or "meshes" of the network.

(Strictly a mesh is a loop such that no other subsidiary loop is inside it)

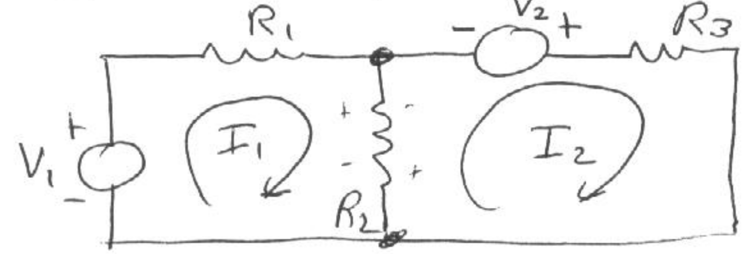


The meshes of a planar network are a suitable set to write the requisite set of independent eqns.

The Maxwell mesh procedure is then:

1. Convert any current sources to voltage sources.
2. Choose set of L independent loops (for planar networks this is easy).
3. Assign a circulating current around each mesh - usually called a mesh current. Reference direction usually all clockwise ↻ or ↺.
(In special cases may be a reason for having some ↻ and som ↺.)
4. Write KVL around each loop using terminal relation for each element explicitly. Pay attention to assigned reference directions.

Ex1 - mesh analysis



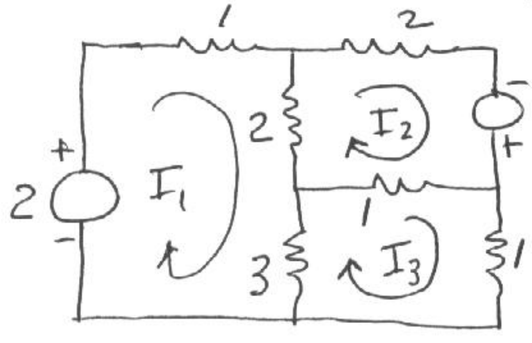
note: $B = 3$
$N = 2$
$L = 3-(2-1) = 3-1 =$
2 eqns $\Rightarrow$
2 meshes

mesh1: $$-V_1 + I_1 R_1 + I_1 R_2 - I_2 R_2 = 0$$
or $$(R_1+R_2) I_1 - R_2 I_2 = V_1 \qquad (1)$$

mesh2: $$-V_2 + I_2 R_3 + I_2 R_2 - I_1 R_2 = 0$$
or $$-R_2 I_1 + (R_2+R_3) I_2 = V_2 \qquad (2)$$

so get 2 eqns in 2 unknowns. Knowing I_1, I_2 all branch currents are determined.

A simple numerical example:



$$(1+2+3) I_1 - 2I_2 - 3I_3 = 2$$
$$6 I_1 - 2I_2 - 3I_3 = 2 \qquad (1)$$

$$-2I_1 + 5 I_2 - 1.I_3 = 1 \qquad (2)$$

$$-3I_1 - 1.I_2 + 5I_3 = 0 \qquad (3)$$

By Cramer:

$$I_1 = \frac{\begin{vmatrix} 2 & -2 & -3 \\ 1 & 5 & -1 \\ 0 & -1 & +5 \end{vmatrix}}{\begin{vmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{vmatrix}} = \frac{61}{67},$$ and similarly $I_2 = \frac{47}{67}$

$$I_3 = \frac{46}{67}$$

General form of the mesh eqns:

$$Z_{11} I_1 + Z_{12} I_2 + Z_{13} I_3 = E_1$$

$$Z_{21} I_1 + Z_{22} I_2 + Z_{23} I_3 = E_2$$

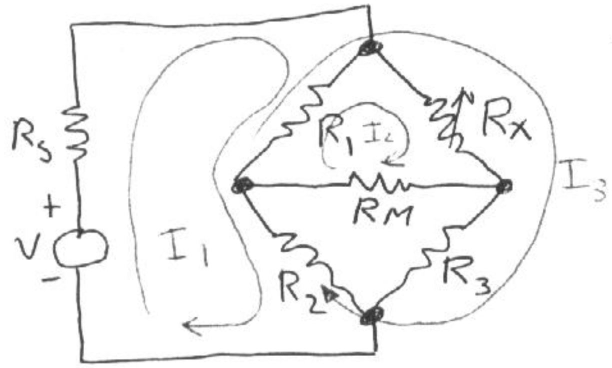
$$Z_{31} I_1 + Z_{32} I_2 + Z_{33} I_3 = E_3$$

Z_{ii} = self resistance of mesh i = sum of the resistances along contour of mesh i. These are the diagonal terms.

Z_{ij} = mutual resistance of meshes i and j = sum of resistances thru which both mesh current i and mesh current j flow, with proper sign. Note: if <u>all</u> the mesh currents have been assigned $\circlearrowright$ reference direction then the Z_{ij} terms are $-$(sum of common R's).

E_i = sum of voltage sources around mesh i. Positive if "aiding" mesh current, negative if "opposing" mesh current.

Choice of loops can be adapted to requirements, e.g. say want current thru load on a Wheatstone Bridge as fnc of unbalance



$$N = 4, \; B = 6, \; L = B-(N-1)$$
$$= 6-(4-1)$$
$$= 3$$

by formula or by inspection

Note have chosen loop currents so that only one goes thru load (RM). Need to pay attention to mutual terms.

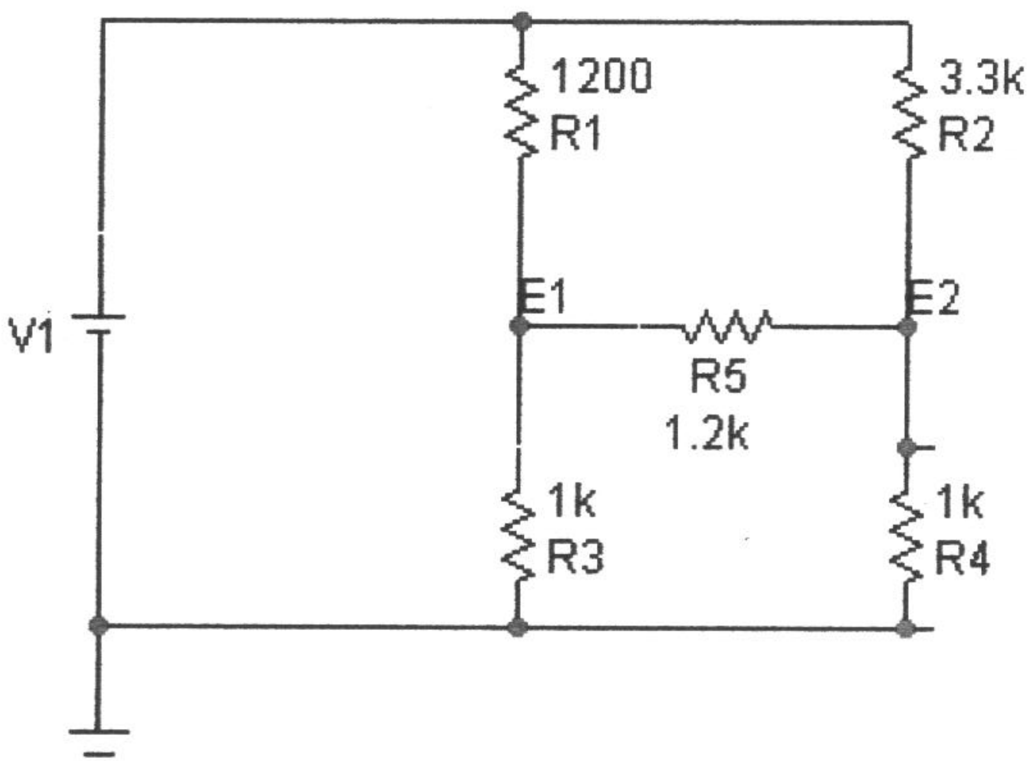
$$\begin{bmatrix} (R_S + R_1 + R_2) & -R_1 & -(R_1+R_2) \\ -R_1 & (R_1 + R_X + R_M) & +(R_1 + R_x) \\ -(R_1+R_2) & +(R_1 + R_x) & (R_1 + R_x + R_3 + R_2) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{so } I_2 = \frac{\begin{vmatrix} R_s + R_1 + R_2 & V & -(R_1+R_2) \\ -R_1 & 0 & (R_1 + R_x) \\ -R_1 + R_2 & 0 & R_1 + R_x + R_3 + R_2 \end{vmatrix}}{\Delta} = \frac{V(R_1 R_3 - R_2 R_x)}{\Delta}$$

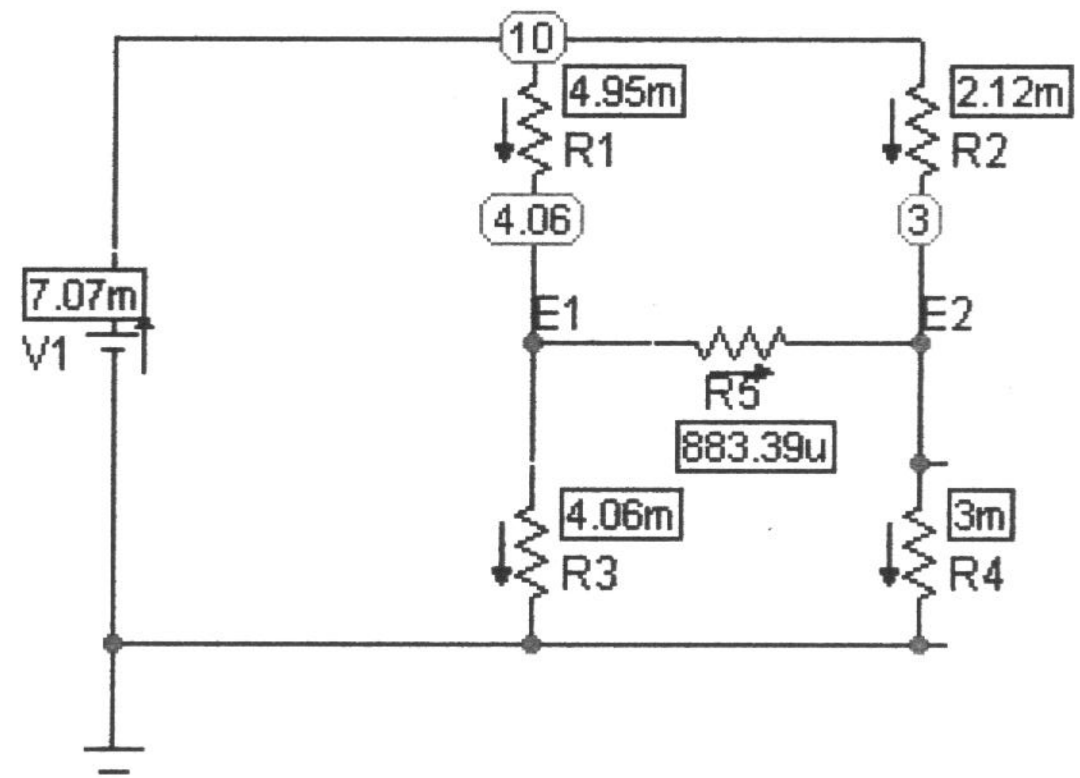
Note this gives balance condition ($I_2 = 0$ for $R_1 R_3 = R_2 R_x$) but is general for unbalance condition.

(MicroCAP does this easily).

Bridge Circuit-get conditions



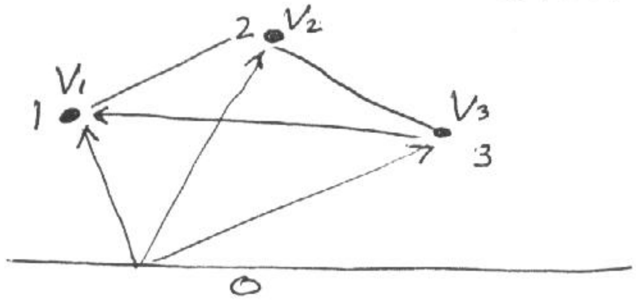
Node Voltages-Branch currents



Nodal analysis

In the mesh method the independent variables are a set of mesh currents which automatically satisfy KCL.

In the nodal method the independent variables are the voltages from each node to a reference (AKA datum) node which is usually just the ckt ground. This ensures that KVL is satisfied around any closed path.



KVL around loop is

$$V_{12} + V_{23} + V_{31} =$$

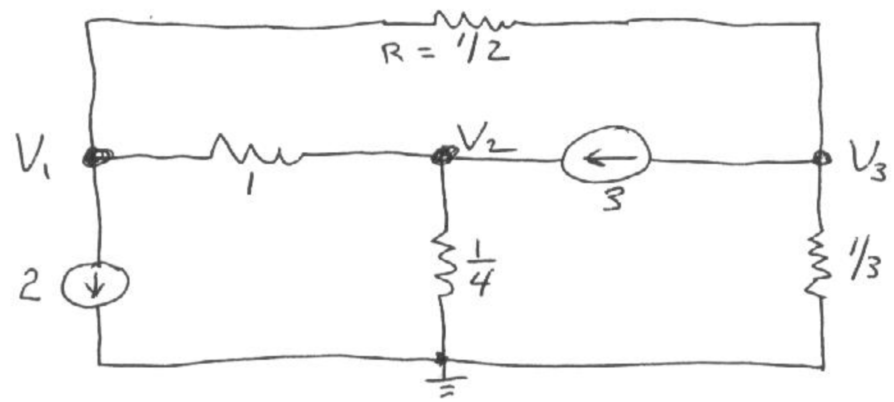
$$= \left[(V_1 - V_0) - (V_2 - V_0)\right] + \left[(V_2 - V_0) - (V_3 - V_0)\right] + \left[(V_3 - V_0) - (V_1 - V_0)\right]$$

$$= 0$$

Nodal procedure:

1. Convert all sources to current sources
2. Choose a reference node ("ground").
3. Write KCL at each node. Common convention is to say current leaving thru passive elements = current entering from current sources. Reference polarity at each node is +w/r to datum.

Example, nodal analysis:



node 1: $$(V_1 - V_3)\frac{1}{1/2} + (V_1 - V_2)\frac{1}{1} = -2$$

or $$3V_1 - 1V_2 - 2V_2 = -2 \qquad (1)$$

node 2 $$(V_2 - V_1)\frac{1}{1} + V_2\frac{1}{1/4} = 3$$

or $$-V_1 + 5V_2 + 0\cdot V_3 = 3 \qquad (2)$$

node 3: $$(V_3 - V_1)\frac{1}{1/2} + V_3\frac{1}{1/3} = -3$$

or $$-2V_1 + 0\cdot V_2 + 5V_3 = -3 \qquad (3)$$

so e.g.

$$V_1 = \frac{\begin{vmatrix} -2 & -1 & -2 \\ 3 & +5 & 0 \\ -3 & 0 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -2 \\ -1 & +5 & 0 \\ -2 & 0 & 5 \end{vmatrix}} = -\frac{65}{50}$$

Note general form of eqns:

$$Y_{11} V_1 + Y_{12} V_2 + Y_{13} V_3 = I_1$$

$$Y_{21} V_1 + Y_{22} V_2 + Y_{23} V_3 = I_2$$

$$Y_{31} V_1 + Y_{32} V_2 + Y_{33} V_3 = I_3$$

where:

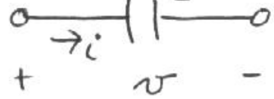
Y_{ii} = self admittance of node i = sum of all admittances (reciprocal resistances) at node i
= diagonal terms

Y_{ij} = mutual admittance of nodes i and j = minus sum of admittances between nodes i and j
= off diagonal terms

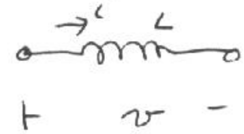
I_i = sum of currents from current sources feeding node i, + if entering, − if leaving

a-c circuits (s-s)
(sinusoidal excitation)

Consider:



$i = C\dfrac{dv}{dt}$ or



$v = L\dfrac{di}{dt}$

if $v = V_m \cos\omega t$
then $i = -\omega C V_m \sin\omega t$

if $i = I_m \cos\omega t$
then $v = -\omega L I_m \sin\omega t$

$$\text{but } -\sin x = \cos(x + \pi/2)$$

so

$$i = \omega C V_m \cos(\omega t + \pi/2)$$

and

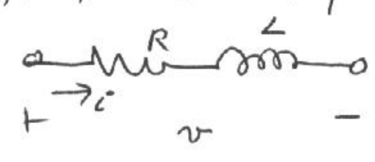
$$v = \omega L I_m \cos(\omega t + \pi/2)$$

In a capacitor <u>i leads v</u> by $\pi/2$ (phase shift)

In an inductor <u>v leads i</u> by $\pi/2$ (phase shift)

No phase shift for a resistor since $v(t) = R\, i(t)$

Staying with trig forms soon becomes cumbersome, e.g.



take $i = I_m \cos\omega t$

$$\text{then } v = v_R + v_L = R I_m \cos\omega t - \omega L I_m \sin\omega t$$
$$= I_m\left[R\cos\omega t - \omega L \sin\omega t\right]$$

$$\text{but } a\cos x - b\sin x = (a^2+b^2)^{1/2}\cos(x+\alpha)\,;\ \alpha = \tan^{-1}(b/a)$$

$$\text{so } v = I_m\left[R^2 + (\omega L)^2\right]^{1/2}\cos(\omega t + \alpha)\,;\ \alpha = \tan^{-1}\frac{\omega L}{R}$$

To avoid dealing with trig forms a different (but equivalent) approach is taken = <u>phasors</u>

Sin and Cos fnc's can be expressed as projections on horizontal + vertical axes of a rotating vector. Have correspondence: length of vector = amplitude of sinusoid

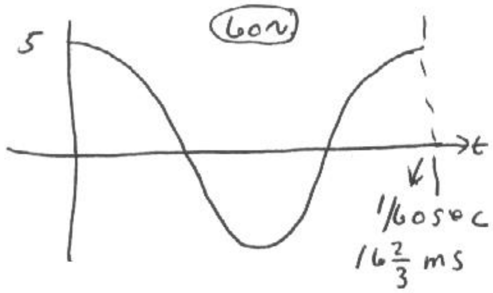
revolutions/unit time = frequency of sinusoid

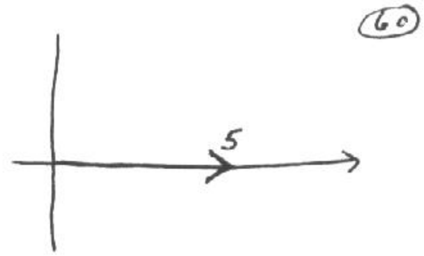
(cycles/sec = hertz = f)

(angular freq = $2\pi f$ rad/sec)

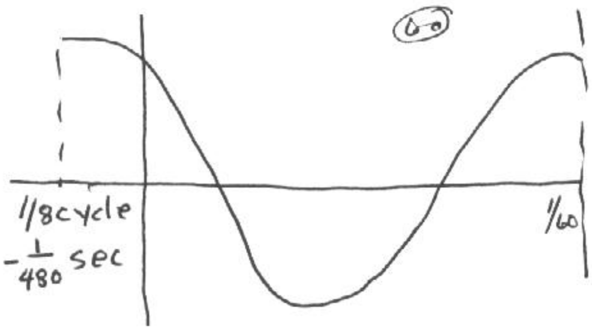
starting angle @ $t=0$ = phase angle of sinusoid

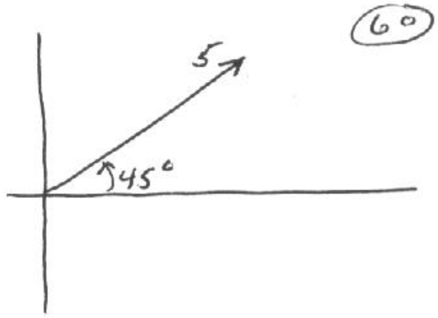
In a-c ckts, the vector as it exists at reference time (usually $t=0$) is called the phasor representation of the associated sinusoidal fnc:





$5\cos(377t)$ $\rightarrow$ phasor form





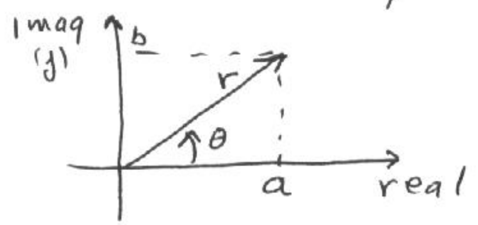
$5\cos(377t + 45°)$

$\uparrow$ or $\pi/4$

Property of linear systems is that responses do not contain any frequencies which are not in the excitation source. Hence all currents + voltages have same frequency as the source (can be phase shifts). Each v or i has a corresponding phasor representation and so all can be put on a phasor diagram. The vectors maintain their relative positions as they rotate. A snapshot yields a "still life" which has all the information. Snapshot taken at a convenient time - usually when source phasor has reference angle zero.

Phasor diagrams simplify computations because instead of working with the actual time varying sinusoids we can now combine phasors using complex numbers. Problem is thus transformed to a different domain.

Reminders: equivalent forms



polar form $r\angle\theta$ (or $r\measuredangle\theta$)

complex number form $a + jb$

$a = r\cos\theta \qquad b = r\sin\theta$

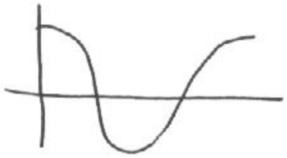
$\theta = \tan^{-1} b/a$

Eulers Formula:

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta \quad \text{(stationary)}$$

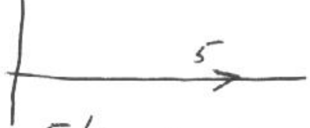
$$\epsilon^{j\omega t} = \cos\omega t + j\sin\omega t \quad \text{(rotating)}$$

So for example:

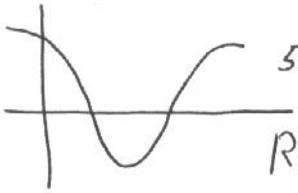


$5\cos 377t$

$Re\left[5\epsilon^{j377t}\right] \rightarrow$

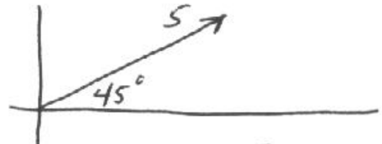


$5\angle 0,\ 5+j0,\ 5\epsilon^{j0}$



$5\cos(377t+45^\circ)$

$Re\left[5\epsilon^{j(377t+45^\circ)}\right] \rightarrow$

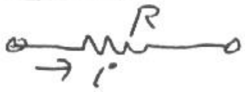


$5\angle 45^\circ,\ \frac{5}{\sqrt{2}}+j\frac{5}{\sqrt{2}},\ 5\epsilon^{j\pi/4}$

etc

Reinterpret basic relations in terms of phasors:

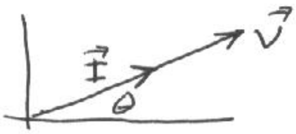
Resistor



$+\ v = Ri\ -$

for $i = I_m\cos(\omega t+\theta) = \vec{I} = I_m\angle\theta$

have $v = R I_m \cos(\omega t+\theta) = \vec{V}$



v and i in phase

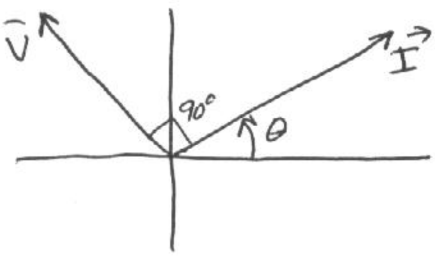
$\vec{V} = R\vec{I} = R I_m\angle\theta$

Inductor

$+\ v = L\frac{di}{dt}\ -$

for $i = I_m\cos(\omega t+\theta) = \vec{I} = I_m\angle\theta$

have $v = \omega L I_m\cos(\omega t+\theta+90^\circ)$



$$\vec{V} = \omega L I_m\angle\theta+90^\circ$$

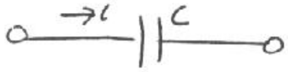
$$= \omega L \underbrace{(I_m\angle\theta)}_{\vec{I}}\ \underbrace{(1\angle 90^\circ)}_{0+j1=j}$$

so $\vec{V} = j\omega L\vec{I}$

(note: j can be interpreted as an operator which rotates phasor by $+90^\circ$)

Capacitor:

$\rightarrow i \quad C$



$+ \quad i = C\frac{dv}{dt} \quad -$

For $v = V_m \cos(\omega t + \theta)$ $\quad \vec{V} = V_m \angle \theta$

have $i = \omega C V_m \cos(\omega t + \theta + 90°)$

$$\vec{I} = \omega C V_m \angle \theta + 90°$$

$$= \omega C \underbrace{(V_m \angle \theta)}_{\vec{V}} \underbrace{(1 \angle 90°)}_{j}$$

or $\vec{I} = j \omega C \vec{V}$

or $\vec{V} = \frac{1}{j\omega C} \vec{I} = j\left(\frac{-1}{\omega C}\right)\vec{I} = -j\left(\frac{1}{\omega C}\right)\vec{I}$

Note that we now have a generalization of Ohm's Law for resistors applied to L's and C's

$$\vec{V} = R\vec{I} \qquad \vec{V} = (j\omega L)\vec{I} \qquad \vec{V} = \left(\frac{1}{j\omega C}\right)\vec{I}$$

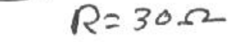
Hence in all cases V & I are related by an "impedance" which to this point is purely real (resistors) or purely imaginary (L's & C's) but in general will be complex when we have combinations of R, L, C's

Kirchoff's laws, divider rules etc still apply but now all operations are done using complex numbers.

Examples: (assume $\omega = 10$ rad/sec)

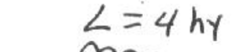
Series

$R = 30\Omega$



$Z = 30 + j0$

$L = 4$ hy

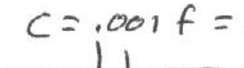


$Z = 0 + j(10)4$

$= 0 + j40$

$(Z_L = j\omega L)$

$C = .001\,f = 10^{-3} f = 1000\,\mu f$

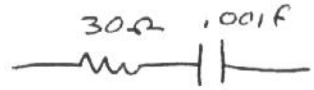


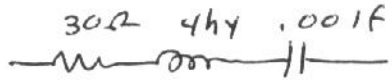
$Z = 0 + \frac{1}{j(10)(10^{-3})}$

$= 0 - j100$

$\left(Z_C = \frac{1}{j\omega C}\right)$

Combine:



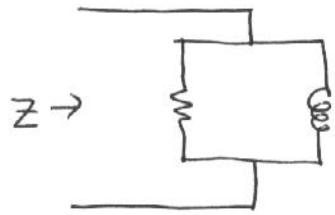


$Z = 30 + j40$

$Z = 30 - j100$

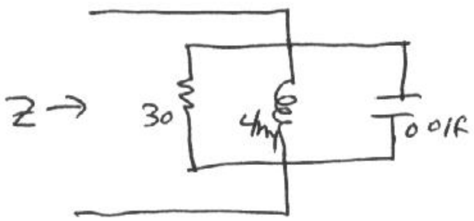
$Z = 30 + j40 - j100$
$= 30 - j60$

Parallel combinations:



$$Z = \frac{(30)(j40)}{30 + j40} = \frac{1200\angle 90^\circ}{50\angle 53^\circ} = 24\angle 37^\circ$$

$$= 19.2 + j14.4 \quad (= 24\cos 37^\circ + j24\sin 37^\circ)$$



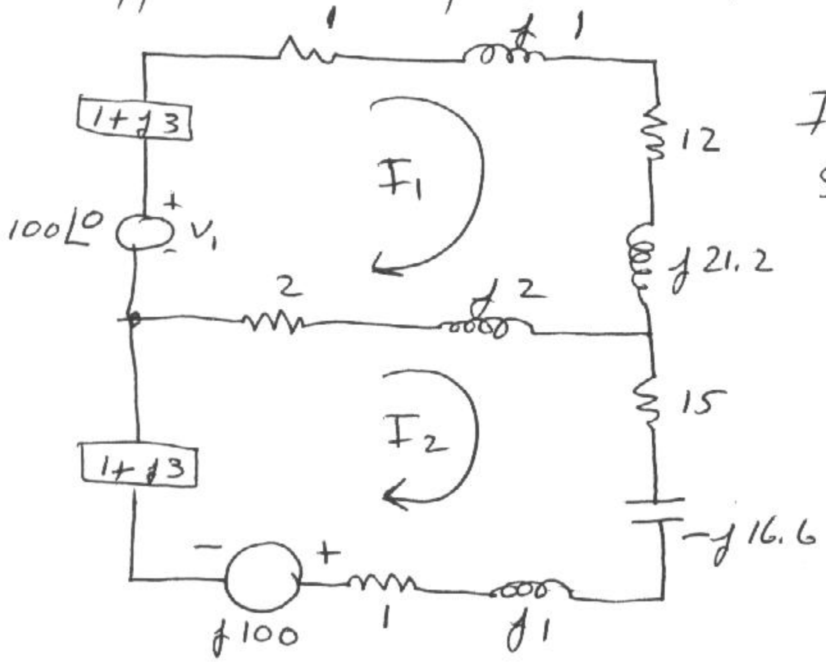
$$Z = 24\angle 37^\circ \,//\, (-j100)$$

$$= \frac{(24\angle 37)(100\angle -90)}{19.2 + \underbrace{j14.4 - j100}_{-j85.6}}$$

$$= \frac{2400\angle -53}{87.9\angle -77.4} = 27.3\angle 24.4$$

$$= 24.9 + j11.3$$

Typical a-c ckt problem: find I_1



Impedances already gotten
Source impedance shown without showing internal details. One source taken as reference ($\angle 0$) so sources are actually
$100 \cos \omega t$ and
$100 \cos(\omega t + 90°)$

so Z_{11} = self impedance for loop 1:

$$= 1+j3+1+j1+12+j21.2+j2+2 = 16+j27.2$$

$$Z_{22} = 2+j2+15-j16.6+j1+1+1+j3 = 19-j10.6$$

$$Z_{12} = \text{mutual impedance loops } 1+2 = 2+j2 = Z_{21}$$

so eqns are $\quad (16+j27.2)I_1 - (2+j2)I_2 = 100 \quad (1)$

$$-(2+j2)I_1 + (19-j10.6)I_2 = -j100 \quad (2)$$

Solve by Cramer or multiply (2) by $\frac{2+j2}{19-j10.6}$ and add to (1)

$$\left[16+j27.2-(2+j2)\frac{2+j2}{19-j10.6}\right]I_1 = 100 - j100\,\frac{2+j2}{19-j10.6}$$

result is $I_1 = 1.75 - j3.05 = 3.48\angle -59.8$

acceptable answer but note represents

$$i_1(t) = 3.48\cos(\omega t - 59.8°)$$