Supersymmetry and the Binding of a Magnetic Atom to a Filamentary Current

Lene Vestergaard Hau,[1,2] J. A. Golovchenko,[1,2] and Michael M. Burns[1]
[1]*Rowland Institute for Science, 100 Edwin H. Land Boulevard, Cambridge, Massachusetts 02142*
[2]*Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138*
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We suggest the binding of neutral atoms to a current carrying wire through the interaction between the atomic magnetic dipole moment and the wire's magnetic field. The theoretical description is based upon an extension of the concept of supersymmetry to multicomponent wave functions. A solution for spin $\frac{1}{2}$ particles is obtained directly in coordinate space. Spin 1 particles are considered as well. Experimentally, the system should be immediately realizable for 25 mK sodium atoms around a wire with a diameter of 0.5 μm and a current of 400 mA.

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The availability of cold atoms through the recent development of laser cooling techniques [1] has resulted in new possibilities for studying and utilizing de Broglie wave optical phenomena. Recent work has shown, for example, that long lived, localized bound states of atoms can exist around a sinusoidally driven charged wire [2].

In this paper we suggest binding of cold atoms to a thin wire with a steady current, through the interaction between the $1/r$ magnetic field and the magnetic dipole moment of the atom. The theoretical description of the system involves an extension of the concept of supersymmetry to multicomponent wave functions describing the dynamics of spin particles.

Blümel and Dietrich [3] considered the possibility of binding very cold neutrons to a current carrying wire. This proposal has yet to be realized due to the small magnetic moment of the neutron and the lack of appropriate cooling schemes. The theoretical description presented in Ref. [3] relied on a power series solution of the Schrödinger equation for a spin $\frac{1}{2}$ particle in the $1/r$ magnetic field. The problem was treated also by Pron'ko and Stroganov [4] and Voronin [5]. In Ref. [4], the energy spectrum was determined from the dynamical symmetry group and the wave functions were obtained in the momentum representation. In Ref. [5] it was discovered that the strongly coupled spinor component equations were related by supersymmetry in a momentum space representation, which allowed an analytical solution to the problem in terms of standard functions of analysis.

In the course of investigating the problem of an arbitrary spin particle in a $1/r$ magnetic field, we discovered a much simpler approach which can be carried out in the coordinate space of the particle and utilizes supersymmetry in a multicomponent representation. We used this method to solve the case of a spin $\frac{1}{2}$ particle and furthermore to find the spectrum and eigenstates for the spin 1 problem corresponding to bound states with vanishing total angular momentum along the wire direction. The suggested bound states can be immediately realized with laser cooled sodium atoms which can be considered to be spin 1 particles as discussed below.

Consider a magnetic moment μ in a magnetic field B. The stationary Schrödinger equation is given by

$$-\frac{\hbar^2}{2M}\Delta\psi - \boldsymbol{\mu}\cdot\mathbf{B}\psi = E\psi . \tag{1}$$

The magnetic moment can be written in terms of the spin operator S as

$$\boldsymbol{\mu} = -g\mu_0\mathbf{S}/\hbar , \tag{2}$$

with g the Landé g factor and μ_0 the Bohr magneton. We choose the wire current I to be directed along the x axis of a Cartesian system, and we write the magnetic field as

$$\mathbf{B} = \frac{2I}{cr}[\mathbf{e}_z\cos\theta - \mathbf{e}_y\sin\theta], \tag{3}$$

where r, θ denote polar coordinates in the transverse y-z plane, and $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors along the y and z directions, respectively.

In the adiabatic limit (for large B fields) where the Larmor precession frequency is large compared to the atomic orbital frequency, the projection of the dipole moment along the magnetic field is constant. As a result, the atom-wire interaction potential reduces to a scalar $1/r$ potential, and a hydrogenic energy spectrum is obtained. With this motivation, we perform a rotation of the spin quantization axes through the angle θ around the x direction with the operator $\exp(i\mathbf{S}_x\theta/\hbar)$. Performing a unitary transformation of the Schrödinger equation with this operator, the θ dependence in the magnetic dipole coupling is eliminated and the eigenstates for the resulting Hamiltonian may be written as

$$e^{i\mathbf{S}_x\theta/\hbar}\psi(\vec{r}) = e^{im\theta}\frac{1}{\sqrt{r}}\chi(r)e^{ikx}. \tag{4}$$

Here $\hbar m$ is an eigenvalue for the operator $-i\hbar(\partial/\partial\theta)$ representing the component of the total angular momentum along the wire direction.

The resulting radial equation for χ, defined through Eq. (4), is in reduced units

$$H_m\chi = -\frac{d^2}{d\rho^2}\chi + \frac{1}{\rho^2}\left[\left(\frac{S_x}{\hbar} - m\right)^2 - \frac{1}{4}\right]\chi + \frac{2}{\rho}\frac{S_z}{\hbar}\chi = \varepsilon\chi . \tag{5}$$

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The distance $\rho = r/r_0$ is measured in units of r_0,

$$r_0 = \frac{\hbar^2}{MC^2}, \qquad \text{where } C^2 = \frac{2g\mu_0 I}{c}. \tag{6}$$

Furthermore, the reduced transverse energy is defined as

$$\varepsilon = \frac{E - \hbar^2 k^2/2M}{C^2/2r_0}. \tag{7}$$

The solution must be normalizable and vanish at $\rho = 0$. In the following we assume that the quantum number m is non-negative since the eigenvalue spectrum for Eq. (5) is the same for $\pm m$.

For a spin $\frac{1}{2}$ particle, the problem can be solved through the use of supersymmetry which is a generalization of the operator technique used for the harmonic oscillator [6,7]. The Hamiltonian H_m in Eq. (5), corresponding to total angular momentum $m\hbar$, can be factored as

$$H_m = A_m^+ A_m + K_m, \tag{8}$$

where

$$A_m = \frac{\partial}{\partial\rho} + W_m(\rho), \qquad A_m^+ = -\frac{\partial}{\partial\rho} + W_m(\rho) \tag{9}$$

are given in terms of the superpotential

$$W_m(\rho) = \begin{pmatrix} -\dfrac{2m+1}{2\rho} - \dfrac{1}{2m+1} & \dfrac{1}{2\rho} \\ \dfrac{1}{2\rho} & -\dfrac{2m+1}{2\rho} + \dfrac{1}{2m+1} \end{pmatrix}, \tag{10}$$

in a spin matrix representation where S_z is diagonal. The factorization constant K_m is proportional to the identity operator

$$K_m = -\frac{1}{(2m+1)^2}\mathbf{1}. \tag{11}$$

Since $A_m^+ A_m$ is positive definite, the ground state $\chi_{m,0}$ for H_m is given as a normalizable solution to

$$A_m \chi_{m,0} = 0, \tag{12}$$

and the ground state energy is obtained from Eq. (11) as

$$\varepsilon_{m,0} = -\frac{1}{(2m+1)^2}. \tag{13}$$

The supersymmetric partner Hamiltonian H_m is defined as

$$H_m^+ = A_m A_m^+ + K_m, \tag{14}$$

which remarkably equals the Hamiltonian H_{m+1} for total angular momentum $(m+1)\hbar$.

Except for the ground state energy in Eq. (13), the spectra for H_m and H_m^+ are identical, and the eigenstates are connected by the A_m, A_m^+ operators. Hence the energy spectrum for the Hamiltonian H_{m_0} is obtained immediately as the series of ground state energy levels in Eq. (13) for $m \geq m_0$,

$$\varepsilon_{m_0,n} = -\frac{1}{[2(m_0+n)+1]^2}, \qquad n \geq 0. \tag{15}$$

If χ_{m+1} is an eigenstate for H_{m+1}, the state $A_m^+ \chi_{m+1}$ is an eigenstate for H_m with the same eigenvalue. Thereby the eigenstates for the Hamiltonian H_{m_0} are obtained from the solutions to the ground state equation (12) with $m \geq m_0$ by applying a succession of the A^+ operators.

To solve the ground state equation (12), we transform the wave function

$$\chi(\rho) = \Phi(\rho)\rho^m, \tag{16}$$

and change variables

$$\eta = \frac{\rho}{2m+1}. \tag{17}$$

With

$$\Phi = \begin{pmatrix} \Phi_{+1/2} \\ \Phi_{-1/2} \end{pmatrix}. \tag{18}$$

Equation (12) then leads to the following equations for the components of Φ:

$$-\frac{d^2\Phi_{-1/2}}{d\eta^2} - \frac{1}{\eta}\Phi_{-1/2} = -\Phi_{-1/2},$$
$$\Phi_{+1/2} = 2\eta\left[-\frac{d}{d\eta} + \frac{1}{2\eta} - 1\right]\Phi_{-1/2}. \tag{19}$$

Note that Eq. (19) is independent of m (shape invariance [8]), and that the differential equation for $\Phi_{-1/2}$ is equivalent to the eigenstate equation for a scalar hydrogen problem with the odd eigenvalue of -1. Unlike the ordinary hydrogen solution, we do not require that $\Phi_{-1/2}(0)$ is zero, and we can in fact find a normalizable solution

$$\Phi_{-1/2}(\eta) = k_1(\eta) = \frac{2\eta}{\pi}[K_1(\eta) - K_0(\eta)], \tag{20}$$

where we have expressed the Bateman function k_1 in terms of the more common modified Bessel functions K_0 and K_1.

Solving Eq. (1), we obtain for the wave function ψ [see Eqs. (4), (6), and (16)–(20)],

$$\psi(\vec{r}) = \left(\cos\frac{\theta}{2} - i\frac{2S_x}{\hbar}\sin\frac{\theta}{2}\right)e^{im\theta}e^{ikx}\frac{1}{\sqrt{r}}\left(\frac{r}{r_0}\right)^m \times \begin{pmatrix} 2\eta\left[-\dfrac{d}{d\eta} + \dfrac{1}{2\eta} - 1\right] \\ 1 \end{pmatrix} k_1(\eta) \tag{21}$$

evaluated at $\eta = (r/r_0)/(2m+1)$. A plot of the corresponding radial probability distribution for quantum number $m = 51/2$ is shown in Fig. 1. Clearly, a particle bound in such a state around a wire of radius less than $\approx 800 r_0$ will have a very long lifetime.

For a spin 1 particle, the eigenvalue problem for total angular momentum $m = 0$ reduces to a spin $\frac{1}{2}$ problem and a hydrogenic energy spectrum is obtained also in this case. For larger m values, an attempt to factor [as in Eq. (8)] and supersymmetrically connect the different Hamiltonians leads to factorization constants K_m, which are nondiagonal and therefore cannot be interpreted as energies. However, a diagonalization of K_m results in the eigenvalue zero (nondegenerate) and a doubly degenerate

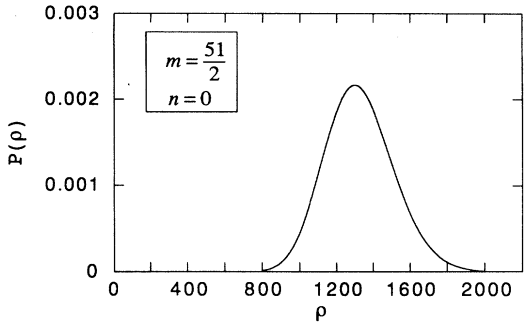

FIG. 1. Plot of the radial probability distribution corresponding to the solution in Eq. (21) with $m = 51/2$. The horizontal scale is in units of the "Bohr" radius r_0 [see Eq. (6)]. For a wire current of $I = 400$ mA, a Landé g factor with $|g| = \frac{1}{2}$ (as in the ground states of sodium), and a mass M equal to the sodium atomic mass, the Bohr radius $r_0 = 7.4$ Å. This distance is inversely proportional to M and gI. The corresponding binding energy for a spin $\frac{1}{2}$ particle in the state shown is $E_B = 6 \times 10^{-10}$ eV. This energy is directly proportional to the mass M and the square of gI (note that the approximate binding energy for a spin 1 particle with $m = 13$ is about 4 times as large [see Eq. (22)]). It is seen from the figure that the wave function will have a negligible overlap with a wire of diameter 0.5 μm corresponding to a radius $r = 340 r_0$.

value which can be written as twice the sum of two spin $\frac{1}{2}$ energies,

$$\varepsilon = -\frac{1}{2m^2} - \frac{1}{2(m+1)^2}, \qquad m \geq 1. \qquad (22)$$

Curiously, the spectrum in (22) agrees well with the energy levels for higher m values as obtained from numerical integration of the vector Schrödinger equation. These calculations show that the levels for different m values are nondegenerate and hence cannot be connected by supersymmetry [9]. We conjecture that there is a nearby problem which exhibits perfect supersymmetry and that this problem can be reduced into two spin $\frac{1}{2}$ problems. Whether the method of supersymmetry will yield complete solution to the spin 1 and higher spin problems is as yet unknown.

For an experimental realization of the bound states, there are two main requirements: The radius of the bound states must be larger than the radius of the current carrying wire, and the energy of binding of the magnetic dipole to the wire must be larger than typical thermal kinetic energies of the atoms. The quantities r_0 and $C^2/2r_0$ [see Eqs. (6) and (7)] set the scale for the radius and binding energy in the ground state. As an example, with a wire current of 400 mA we can bind sodium atoms with kinetic energies corresponding to a temperature of 25 mK in hydrogenic orbits of radius 0.5 μm. Sodium atoms can be laser cooled in polarization gradients to this temperature [1], and in our laboratory, we have, furthermore, succeeded in producing self-supporting, single crystalline copper whiskers [10] of lengths on the order of 1 cm and diameters of 2–15 μm. These whiskers are then thinned by ion milling to a thickness of 0.2 μm over a length of a few millimeters. Note that the ground state for sodium is a spin 1 system (the total internal angular momentum is 1). For the experimental situation described here, the atomic Zeeman shift is smaller than the internal fine structure splitting by 2 to 3 orders of magnitude—so it is a very good approximation to regard sodium as an elementary spin 1 particle. It should also be noted that we have verified experimentally that the thin wires can support the required current density of 2×10^5 A/cm^2. Furthermore, electrical power dissipation will cause a heating of the wire to temperatures less than 600 °C well below the melting point for copper at 1100 °C.

In this paper we have extended the concept of supersymmetry to describe spinor particles. With this formalism, the energy eigenvalue problem for a magnetic atom around a current carrying wire is solved easily for particles with spin $\frac{1}{2}$ as well as for spin 1 states with vanishing total angular momentum along the wire direction. A hydrogenic spectrum is identified in both cases, and the wave functions are obtained by solving one scalar ground state problem and operating on the resulting state by a series of first order differential operators. Progress has been made on the spin 1 problem also for higher total angular momentum states.

This problem also presents very interesting experimental challenges; the system may prove useful as an alternative to the Kapitza waveguide [2] for de Broglie waves with similar binding energies and radii. For the Kapitza waveguide, the states have a long but finite lifetime due to quantum dynamical instabilities. In the magnetic case, the lifetime is mainly limited by the wave function overlap with the finite sized wire, which can be minimized for higher total angular momentum states.

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