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Errata: Surface instabilities of constrained elastomeric layers subject to electro-static stressing by J. W. Hutchinson, J. Mech. Phys. Solids 153 (2021) 104,462.

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Thanks to the help of the coauthors of this erratum (RH & CML), it has been possible to identify and correct an error in the bifurcation analysis of the first author (JWH) for the constrained layer problems addressed in the paper cited above. The error is not obvious, and the results for this problem are anticipated to be important for subsequent work on electro-mechanical instabilities. Therefore, this note briefly presents the corrected results along with discussion to make it less likely the error will be repeated. The error in the bifurcation analysis propagates throughout the post-bifurcation and imperfection-sensitivity analysis in the paper cited above giving rise to additional changes. Corrections to the post-buckling behavior are made in the Supplementary Materials accompanying this erratum with one figure summarizing selected results included below.

The said paper (Hutchinson, 2021) considered strictly incompressible neoHookean materials (and the Gent extension). An infinite layer is subject to an equi-biaxial pre-stretch, λ_0, prior to being bonded to a rigid substrate (see Fig. 1a). In the pre-bifurcation state the layer is uniform with thickness h. In Problem I, the top and bottom surfaces of layer are conducting with a voltage V_0 imposed across them. The layer is dielectric with permittivity ε. In Problem II, a voltage difference V_0 is imposed between a rigid conducting electrode located a distance h above the conducting top surface of the layer. In II, the medium between the conducting surfaces (vacuum, gas or liquid) has permittivityε. In the notation and nondimensionalization of the paper, with $(x,y)=k(x_1,x_2)$, $k=2\pi/\ell$ and ℓ as the wavelength, the periodic displacements of the plane strain bifurcation modes of the layer have the separated form: $(u_1,u_2)=\ell(U(y)\sin(kx),V(y)\cos(kx))$. The corrected functional governing bifurcation for the Gent generalization of the incompressible neo-Hookean material is

$$\begin{aligned}\frac{\Delta\Psi_2}{\mu\ell^2}=&\frac{\pi}{2}\int_{-y_B}^{0}\frac{1}{\omega_0}\left\{\lambda_0^{\,2}\left(U^2+V^2\right)+\lambda_0^{\,-4}\left(U'^2+V'^2\right)-2\lambda_0^{\,-4}(UV)'+\frac{2}{\omega_0 J_L}\left(\lambda_0^{\,2}U+\lambda_0^{\,-4}V'\right)^2\right\}dy\\&-\frac{\pi}{2}\left[\Omega\,\mathrm{cotanh}(y_B)-\frac{2\pi\gamma}{\mu\ell}\right]V(0)^2\end{aligned} \tag{1}$$

with μ as the ground state shear modulus, γ as the surface energy, J_L as the Gent stiffening parameter, $(\ )'=d(\ )/dy$, $\omega_0=1-(2\lambda_0^{\,2}+\lambda_0^{\,-4}-3)/J_L$ and $y_B=2\pi h/\ell$. The dimensionless eigenvalue parameter determining the voltage at bifurcation is $\Omega=(\varepsilon/\mu)(V_0/h)^2$. The incompressibility constraint, $V'+U=0$, must be enforced using a Lagrange multiplier. In the limit $J_L\to\infty$ the neoHookean material is obtained. The functional is applicable to the bifurcation in both Problems I and II.

The error in the bifurcation functionals listed in Hutchinson (2021), i.e., (3.4), (4.1) and (4.4), is a single term uniquely identified by the factor $\tau\Omega$ (with $\tau=1$ for I and $\tau=-1$ for II). Although the form is somewhat different, (1) is equivalent to the functional (4.4) in the original paper when the term $\tau\Omega$ is deleted. The terms in (1) comprise all the quadratic changes in the elastic, electro-static and

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surface energies making up the free energy of the system. The term in the original paper involving $\tau\Omega$ arises from the hydrostatic stress in the pre-bifurcation state of the layer. It can be eliminated from the integral over the layer and expressed as a surface contribution to the energy functional, as it has in (4.4) of the paper. The term is not a contribution to the strain energy in layer, which depends on the deformation but not the hydrostatic stress. The surface contribution of this term is not compatible with the electro-static traction changes generated by the bifurcation displacements. The validity of this argument did not come directly. We became convinced there was an error in the critical voltage for the incompressible material when independent analyses first conducted by RH and CML for bifurcation in Problem I for compressible neoHookean materials clearly approached a limit different from that given in the paper when calculations for nearly incompressible materials were carried out. Here we will provide the results from one such calculation focusing on Problem I with no pre-stretch ($\lambda_0 = 1$) and for the short wavelength limit valid when ℓ/h is sufficiently small.

The analysis described next employs a version of an isotropic, finite strain compressible material whose incompressible limit coincides with the neoHookean material, as described by Boyce and Arruda (2000), c.f., their equations (35) and (36). The compressible analysis is carried out for Problem I in the reference configuration prior to the application of the voltage V_0. Coordinates in the reference configuration are denoted X_I, and those in the current configuration are denoted x_i. The deformation gradient is $F_{iJ} = \frac{\partial x_i}{\partial X_J} = x_{i,J}$. The nominal electric field is $E_I = -\frac{\partial \phi}{\partial X_I} = -\phi_{,I}$, where ϕ is the electric potential. The true electric field is $e_i = -\frac{\partial \phi}{\partial x_i} = -\phi_{,i} = -\frac{\partial \phi}{\partial X_J}\frac{\partial X_J}{\partial x_i} = F_{Ji}^{-1}E_J$. The form of the free energy used in this work is,

$$W = \frac{\mu}{2}(F_{iJ}F_{iJ} - 3) - \mu \ln J + \frac{\lambda}{2}(J-1)^2 - \frac{\varepsilon}{2} J F_{Ki}^{-1} F_{Li}^{-1} E_K E_L \tag{2}$$

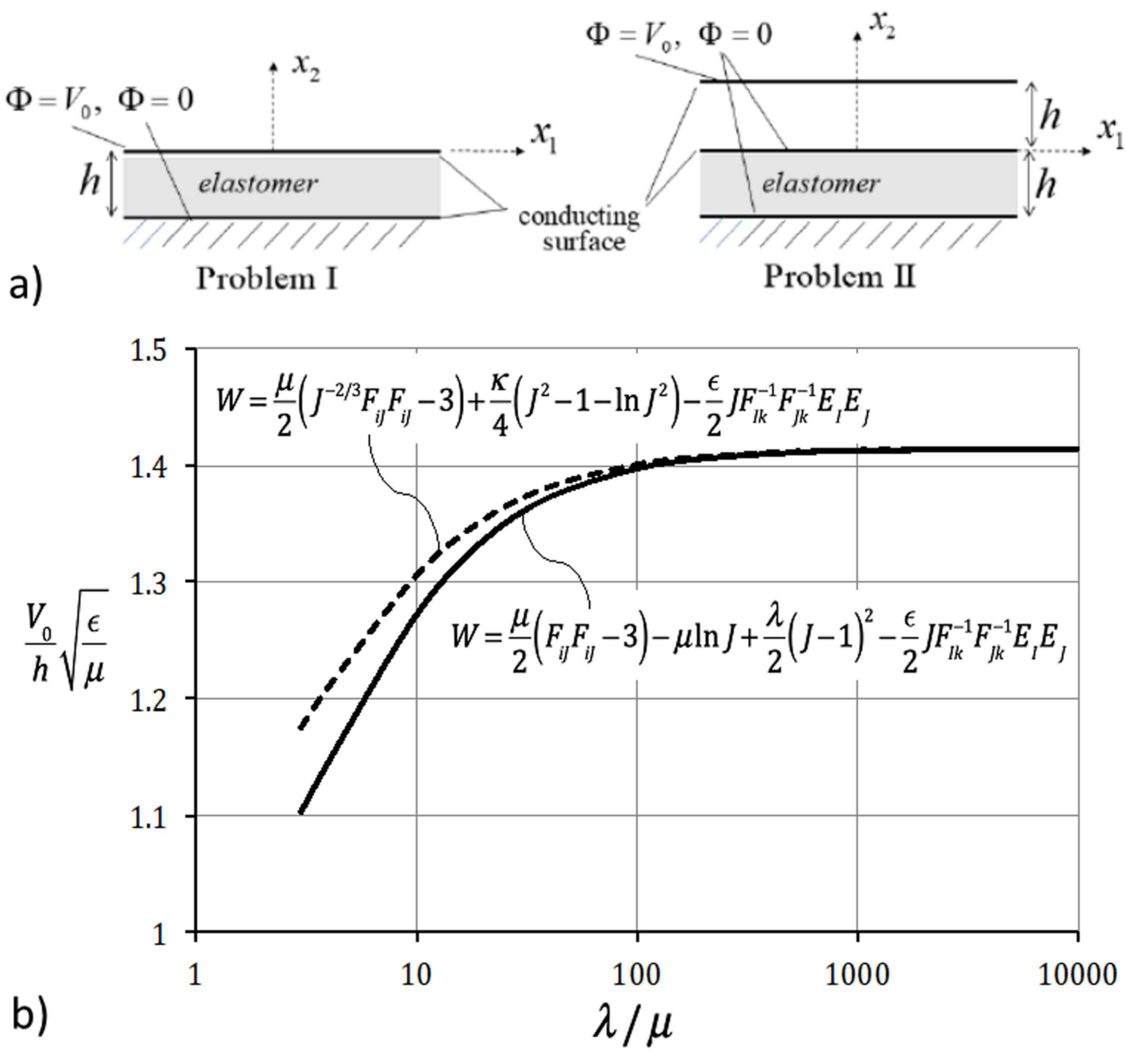

Fig. 1. a) Problems I and II. b) The dependence of the critical dimensionless voltage on the ratio of the two Lamé parameters showing the limit for Problem I as the material becomes incompressible for the material model discussed in the text (solid line) and for a second isotropic material model (dashed line and $\kappa = \lambda + 2\mu/3$) which also approaches the neoHookean model in the incompressible limit.

Here, μ and λ are the usual Lamé parameters when the material is subjected to infinitesimal deformation, ϵ is the dielectric permittivity, and $J = \det(F_{iJ})$ is the determinant of the deformation gradient. The analysis was also performed for a second compressible form shown in Fig. 1b, where the bulk modulus κ is related to the Lamé parameters as, $\kappa = \lambda + 2\mu/3$. We present the details for the form in Eq. (2) because of its simpler analytic solution. The governing equations of equilibrium and charge balance, i.e. Gauss' law, are given as,

$$P_{Ji,J} = 0 \text{ and } D_{I,I} = 0 \tag{3}$$

where $P_{Ji} = \frac{\partial W}{\partial F_{iJ}}$ is the first Piola-Kirchhoff stress, and $D_I = -\frac{\partial W}{\partial E_I}$ is the nominal electric displacement.

We look for perturbed solutions about a homogeneous state given by,

$$x_i = \delta_{iJ}X_J + \varepsilon_0\delta_{i2}X_2 + \alpha u_i(X_1, X_2) \tag{4}$$

$$\phi = \frac{V_0}{h}(X_2 + h) + \alpha\varphi(X_1, X_2) \tag{5}$$

where h is the film thickness in the reference state, V_0 is the voltage applied to the top surface at $X_2 = 0$, ε_0 is the film strain in the X_2 direction at the bifurcation voltage, $\alpha u_i(X_1, X_2)$ is the perturbed displacement field, $\alpha\varphi(X_1, X_2)$ is the perturbed electric potential field, and α is a small parameter. The homogeneous film strain satisfies the equation,

$$\frac{\epsilon}{2}\left(\frac{V_0}{h}\right)^2 + \lambda\varepsilon_0(1+\varepsilon_0)^2 + \mu(1+\varepsilon_0)^3 - \mu(1+\varepsilon_0) = 0 \tag{6}$$

The equilibrium equations and Gauss' law are expanded to order α^1 and yield equations governing the perturbed fields,

$$\frac{V_0}{(1+\varepsilon_0)h}u_{2,11} + \frac{V_0}{(1+\varepsilon_0)^3 h}u_{2,22} - \varphi_{,11} - \frac{1}{(1+\varepsilon_0)^2}\varphi_{,22} = 0 \tag{7}$$

$$\left[\lambda + \mu + \frac{\mu}{(1+\varepsilon_0)^2}\right]u_{2,22} + \mu u_{2,11} + \left[(1+\varepsilon_0)\lambda + \frac{\mu}{1+\varepsilon_0}\right]u_{1,21} = 0 \tag{8}$$

$$\left[(1+\varepsilon_0)^2\lambda + 2\mu\right]u_{1,11} + \mu u_{1,22} + \left[(1+\varepsilon_0)\lambda + \frac{\mu}{1+\varepsilon_0}\right]u_{2,21} = 0 \tag{9}$$

These equations are also subject to the voltage and traction-free boundary conditions on the surface of the film,

$$\varphi(X_1, 0) = 0 \tag{10}$$

$$\frac{\varepsilon V_0}{h(1+\varepsilon_0)}\varphi_{,1}(X_1,0) + \left[\frac{\mu}{1+\varepsilon_0} - \lambda\varepsilon_0 - \frac{\varepsilon {V_0}^2}{2h(1+\varepsilon_0)^2}\right]u_{2,1}(X_1,0) + \mu u_{1,2}(X_1,0) = 0 \tag{11}$$

$$\frac{\epsilon V_0}{h(1+\varepsilon_0)^2}\varphi_{,3}(X_1,0) + \left[\lambda + \frac{\mu}{1+\varepsilon_0} + \frac{\mu\varepsilon_0^2}{(1+\varepsilon_0)^2} - \frac{\epsilon {V_0}^2}{h(1+\varepsilon_0)^3}\right]u_{2,2}(X_1,0) + \lambda(2+\varepsilon_0)u_{1,1}(X_1,0) = 0 \tag{12}$$

Taking the wavenumber of the wrinkling oscillations to be unity, and the wavelength of the oscillations to be much smaller than h, the solution to these equations is,

$$\varphi(X_1, X_2) = \frac{V_0}{h(1+\varepsilon_0)}\{u_2(X_1,X_2) - U_2(1-A)\exp[(1+\varepsilon_0)X_2]\cos(X_1)\} \tag{13}$$

$$u_2(X_1, X_2) = U_2\{\exp[X_2] - A\exp[pX_2]\}\cos(X_1) \tag{14}$$

$$u_1(X_1, X_2) = -U_2\left\{\frac{1}{1+\varepsilon_0}\exp[X_2] - \frac{A(1+\varepsilon_0)}{p}\exp[pX_2]\right\}\sin(X_1) \tag{15}$$

$$p = (1+\varepsilon_0)\sqrt{\frac{2\mu + \lambda(1+\varepsilon_0)^2}{\mu + (\lambda+\mu)(1+\varepsilon_0)^2}} \tag{16}$$

$$A = \frac{\varepsilon\left(\frac{V_0}{h}\right)^2 + 2(1+\varepsilon_0)[\varepsilon_0(1+\varepsilon_0)\lambda - 2\mu]}{\varepsilon\left(\frac{V_0}{h}\right)^2 + 2(1+\varepsilon_0)[\varepsilon_0(\lambda-2\mu) + \varepsilon_0^2(\lambda-\mu) - 2\mu]} \tag{17}$$

The critical value for V_0/h and the associated value for A result from an eigenvalue problem required to satisfy the boundary conditions with a non-trivial solution.

$$\frac{V_0}{h} = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} \tag{18}$$

$$a = \frac{\varepsilon^2\left[(1+\varepsilon_0)^2 - p\right]}{4(1+\varepsilon_0)^2} \tag{19}$$

$$b = \frac{\varepsilon\left[(1+\varepsilon_0)^2[\varepsilon_0(1+\varepsilon_0)\lambda - 2\mu] - p\varepsilon_0(1+\varepsilon_0)\lambda + p\left(2+4\varepsilon_0+4\varepsilon_0^2+\varepsilon_0^3\right)\mu\right]}{1+\varepsilon_0} \tag{20}$$

$$c = (1+\varepsilon_0)^2[\varepsilon_0(1+\varepsilon_0)\lambda - 2\mu]^2 - p\left[\varepsilon_0(1+\varepsilon_0)\lambda - \left(2+2\varepsilon_0+\varepsilon_0^2\right)\mu\right]^2 \tag{21}$$

Eq. (18) provides V_0/h in terms of ε_0, which then casts (6) as a single nonlinear algebraic equation governing ε_0. In the limit of incompressible behavior $\lambda \to \infty$, $\varepsilon_0 \to 0$, $p \to 1$, $a \to 0$, $b \to 2\varepsilon\mu$, $c \to -4\mu^2$, and the solution for V_0/h becomes,

$$\frac{V_0}{h} = \sqrt{\frac{-c}{b}} = \sqrt{\frac{2\mu}{\varepsilon}} \quad \text{as } \lambda \to \infty \tag{22}$$

The plot of $\sqrt{\varepsilon/\mu}\,V_0/h$ as a function of λ/μ is given in Fig. 1b showing the approach to the limit $\sqrt{2}$. A second curve (dashed) has been included in Fig. 1b for another nonlinear isotropic elastic material whose incompressible limit is also the neoHookean material. Although the dependence on compressibility is different, the second model has the same limiting critical voltage as the first model. The free energy for the layer of second material is shown in the figure where $\kappa = \lambda + 2\mu/3$ is the bulk modulus.

In terms of the eigenvalue parameter introduced earlier, the incompressible limit is simply $\Omega = 2$ or $\sqrt{\varepsilon/\mu}\,V_0/h = \sqrt{2}$. The incorrect result obtained in the original paper is $\sqrt{\varepsilon/\mu}\,V_0/h = 1.287$. The correct result was first obtained by Huang (2005) who modeled the layer as an isotropic linearly elastic compressible material and took the result to the incompressible limit. Our subsequent work analyzing the non-objective linear elastic model of the layer within the finite strain context leads to the same incorrect result $\sqrt{\varepsilon/\mu}\,V_0/h = 1.287$ as in Hutchinson (2021). The inconsistency of the results for the linear elastic layer almost certainly stems from the fact that the linear elastic model is not objective in the finite strain context, whereas the two models used to generate the results in Fig. 1b are objective.

Results for the incompressible neoHookean layer from the corrected bifurcation functional (1) accounting for interaction with the bottom of the layer are presented in Fig. 2a for the case of no pre-stretch and several levels of surface energy measured by $\gamma/\mu h$. The lowest curve for a layer with no surface energy shows that the dimensionless critical voltage attains the short wavelength limit, $\sqrt{\Omega} = \sqrt{2}$, for all wavelengths satisfying $\ell/h < 1$. This plot also reveals the strong effect of the surface energy on the critical voltage at short wavelengths. Fig. 2b presents curves of the dimensionless critical voltage as a function of pre-stretch in the short wavelength limit for several values of the dimensionless surface energy, now as $\gamma/\mu\ell$, including the limit with $\gamma = 0$. An exact relatively simple analytical

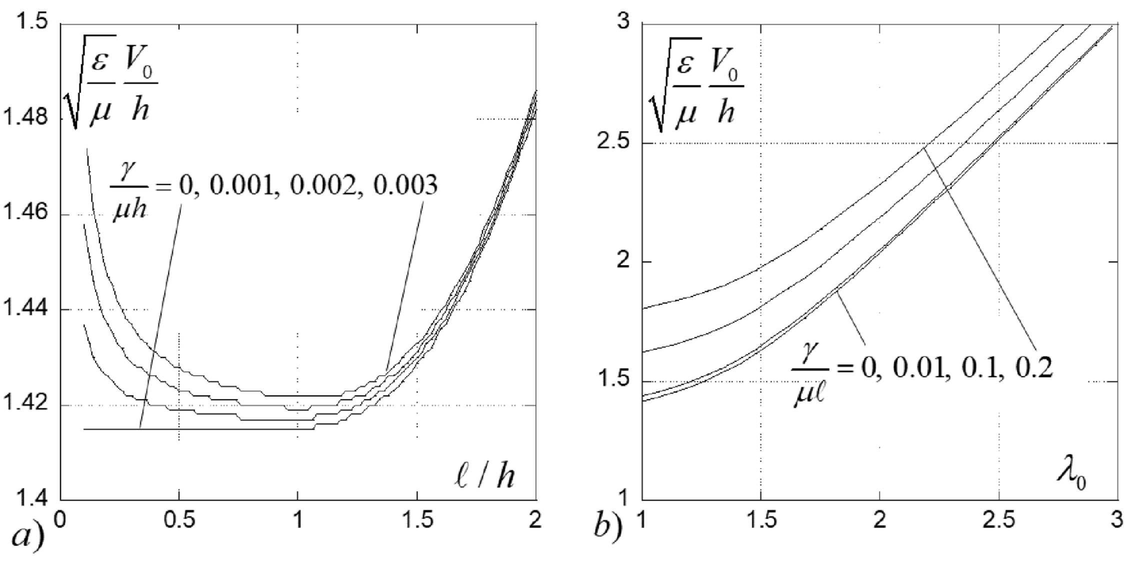

Fig. 2. Dimensionless voltage at bifurcation as dependent on wavelength, surface energy and pre-stretch for the incompressible neoHookean layer. a) As dependent on wavelength and surface energy with no pre-stretch. b) As dependent on pre-stretch and surface energy in the short wavelength limit when $\ell/h < 1$.

formula can be obtained for the short wavelength limit:

$$\Omega - \frac{2\pi\gamma}{\mu\ell} = \frac{{\lambda_0}^9 + {\lambda_0}^6 + 3{\lambda_0}^3 - 1}{{\lambda_0}^4\left({\lambda_0}^3 + 1\right)} \tag{23}$$

Stiffening greater than that predicted by the neoHookean material is commonly observed for some elastomers, which is expected to impact the effect of pre-stretch. The Gent generalization of the neoHookean material captures stiffening with a single additional parameter J_L, c.f., discussion in Section 6 and eq. (6.1) of the original paper. Fig. 3 displays the effect of the Gent parameter on the critical dimensionless voltage as dependent on pre-stretch in the short wavelength limit ($\ell/h < 1$) for layers with no surface energy computed using the corrected functional (1). The results in Fig. 2 and 3 apply to both Problems I & II.

The corrected bifurcation results for the critical voltage for the pre-stretched neoHookean material in Problem I are in much better agreement with the creasing experiments of Wang et al. (2011) than the previous erroneous ones. At a pre-stretch $\lambda_0 = 3$, the experimental critical voltages have a scatter of about 15% with the corrected bifurcation voltage falling in the center of the scatter. Because creasing is an unstable phenomenon and sensitive to small imperfections, one would expect the experimental data to consistently fall somewhat below the bifurcation prediction. This suggests that some stiffening effect such as that captured by the Gent model may be at play at the larger pre-stretches. In addition, the inconsistency noted in Section 6 of Hutchinson (2021) concerning the existing numerical prediction of the crease threshold for pre-stretched neoHookean materials still stands because that prediction falls above the corrected bifurcation result when $\lambda_0 \geq 2$.

The error in the bifurcation analysis carries into the post-bifurcation analysis and has a effect on the role of pre-stretch. The revision of the post-bifurcation analysis is presented in the Supplementary Materials attached to the errata. Here, we include Fig. 4 for Problem I which reveals that pre-stretch tends to reduce the level of instability at bifurcation. Let V_0^C be the voltage at bifurcation and V_0 be the voltage in the initial post-bifurcation regime. The initial post-bifurcation analysis determines the nonlinear coupling of the shortwave length modes and generates the lowest order dependence of quantities of interest on the eigenmodal mode amplitudes. Fig. 4a plots the lowest order asymptotic dependence of V_0/V_0^C on the maximum downward deflection of the layer normalized by ℓ, $-u_2(0,0)/\ell$, for seven values of the pre-stretch ranging from no pre-stretch to $\lambda_0 = 2.5$. In the uniform state the capacitance of a section of the layer of thickness h, length ℓ and unit depth is $C_0 = \varepsilon\ell/h$. The increased capacitance of this section in the bifurcation state is plotted in Fig. 4b for the same values of pre-stretch. The primary insight into the effect of pre-stretch on the level of instability is given by Fig. 4c where the voltage is plotted against the electrical charge in the section, c. For the uniform layer, $c = C_0 V_0$, and the value at bifurcation is $c_C = C_0 V_0^C$. With no pre-stretch, or relatively small pre-stretch, bifurcation is highly unstable under either prescribed voltage or charge. For increased levels pre-stretch above about $\lambda_0 = 2$ the initial post-bifurcation response falls less dramatically. It must be emphasized that these results are the result of an asymptotic analysis perturbing about the bifurcation point, and their range of applicability into the post-bifurcation regime is unknown. The corresponding results for Problem II are presented in the revised Supplementary Materials. A detailed numerical analysis of the post-bifurcation behavior is underway which is not subject to the limitations of the current asymptotic perturbation analysis.

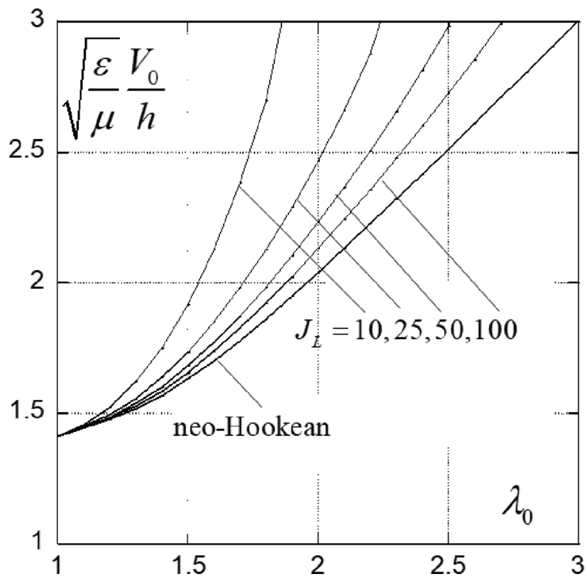

Fig. 3. The influence of the Gent stiffening parameter, J_L, on the critical dimensionless voltage as a function of pre-stretch in the short wavelength limit in the absence of surface energy.

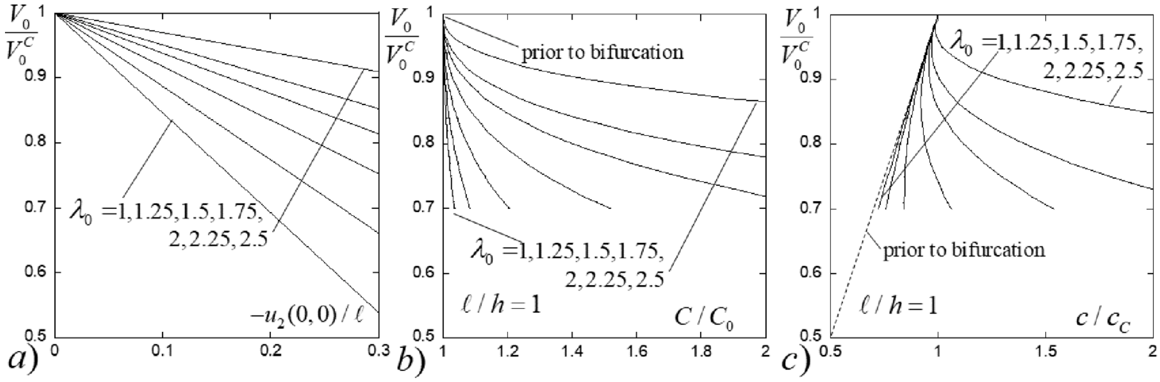

Fig. 4. The influence of pre-stretch, λ_0, on the initial post-bifurcation behavior of the constrained incompressible layer for Problem I. a) Voltage versus amplitude of the surface bifurcation deflection. b) Ratio of the capacitance in the initial post-bifurcation regime to the capacitance of the uniform layer. c) Voltage versus electrical charge in the initial post-bifurcation regime. These plots were computed using 6 modes; plots b) and c) were computed with $\ell/h = 1$.

Declaration of Competing Interest

We declare that the paper being submitted (see below) involves no conflict of interests on the part of the three authors.

Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.jmps.2022.104809.

References

Boyce, M.C., Arruda, E.M., 2000. Constitutive models for rubber elasticity: a review. Rub. Chem. Tech. 73, 504–523.
Huang, R., 2005. Electrically induced surface instability of a conductive thin film on a dielectric substrate. Appl. Phys. Lett. 87, 151911.
Hutchinson, J.W., 2021. Surface instabilities of constrained elastomeric layers subject to electro-static stressing. J. Mech. Phys. Solids 153, 104462.
Wang, Q., Tahir, M., Zhang, L., Zhao, X., 2011. Electro-creasing instability in deformed polymers: experiments and theory. Soft Mater. 7, 6583–6589.