PLASTIC STRESS–STRAIN RELATIONS OF F.C.C POLYCRYSTALLINE METALS HARDENING ACCORDING TO TAYLOR'S RULE*

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SUMMARY

SEVERAL stress–strain relations, including simple tension and pure shear, are calculated for face-centred cubic aggregates of single crystal's which harden according to Taylor's rule. The calculations are based on the theoretical model suggested by BUDIANSKY and WU (1962) and KRÖNER (1961). A simple correspondence is demonstrated between this model and LIN's (1957) extension to Taylor's model.

1. INTRODUCTION

TAYLOR'S (1938) model for predicting the stress–strain relation of f.c.c. poly-crystalline metals subject to large plastic strains was extended by BISHOP and HILL (1951) to polyaxial stress states and was later extended by LIN (1957) to predict the stress–strain relation for both large and small strains. Lin retained Taylor's assumption that the strain in each grain was homogeneous and equal to the macro-scopic polycrystalline strain. This assumption is abandoned in the model used in the present calculations and is replaced by what is perhaps a more intuitive model of the interacting single crystals. This model was suggested by BUDIANSKY and WU (1962) and KRÖNER (1961).

The single crystals comprizing the polycrystalline aggregate are assumed to harden according to Taylor's rule and for calculation purposes a linear version of ᶞhis rule is chosen. Both simple tension and pure shear stress–strain relations are calculated. In addition, a Bauschinger effect calculation is presented. In con-junction with this latter calculation it is noted, as CZYZAK, Bow and PAYNE (1961) had reported, that small amounts of plastic strain are predicted upon diminishing the tensile load. This region is usually considered elastic. Furthermore, a calcula-tion revealed that an accumulation of positive plastic strain is predicted if the tensile specimen is first loaded to an initial stress into the plastic region and then cyclically unloaded and reloaded to the initial tensile stress. In Section 6 a remarkably simple correspondence between the Budiansky–Wu model and Lin's extension to Taylor's model is demonstrated. This correspondence, which was apparently unnoticed by Budiansky and Wu, enables the stress-strain curve as predicted by one theory to be determined directly from the other.

2. The Polycrystalline Deformation Model

Consider a polycrystalline metal, of randomly oriented single crystals, deformed plastically. While there is a macroscopic stress and strain associated with the polycrystalline aggregate, the local stresses and strains vary not only from crystal to crystal, but also from point to point within the crystals. Each crystal is surrounded by crystals of different orientation and at the crystal boundaries single crystal slip is interrupted. The Budiansky–Wu theory is an idealized model of this interaction between the slipping crystals. The stress and strain in a single crystal surrounded by its plastically deformed neighbours are approximately obtained by considering the deformation of a spherical single crystal embedded in an infinite matrix. The matrix is subject to a stress at infinity which is equal to the macroscopic aggregate stress, and the plastic strain in the matrix is uniform and equal to the aggregate plastic strain. The stress and plastic strain in the spherical grain are simply related through a modified form of an ellipsoidal inclusion solution obtained by Eshelby (1957). Budiansky, Hashin and Sanders (1960) first employed this solution to solve for the stress and strain in a spherical crystal embedded in an elastic matrix. Kröner (1961) modified the Eshelby solution even further to include the effect of a plastic strain in the surrounding matrix and, thus, to account for the plastic grain interaction. Later, using somewhat different reasoning, Budiansky and Wu arrived at the same relation between the stress and plastic strain in the spherical crystal that Kröner had obtained. This formula is introduced in the following paragraph.

An elastically isotropic spherical single crystal is embedded in an infinite matrix which is subject to a stress, $\sigma^0{}_{ij}$, at infinity. The matrix is elastically isotropic with the same elastic moduli as the spherical grain. The plastic strain in the matrix, $E^p{}_{ij}$, is uniform and is a deviator strain, that is $E^p{}_{ii} = 0$. If the plastic strain in the spherical grain, $\epsilon^p{}_{ij}$, is uniform and a deviator, the stress in the grain is also uniform and is given by

$$\sigma_{ij} = \sigma^0{}_{ij} - \frac{2\,(7 - 5\nu)}{15\,(1 - \nu)}\,G\,(\epsilon^p{}_{ij} - E^p{}_{ij}) \tag{1}$$

where ν and G are the Poisson ratio and elastic shear modulus of the material.

The stress imposed on the matrix at infinity, $\sigma^0{}_{ij}$, corresponds to the macroscopic stress of the polycrystal and the plastic strain, $E^p{}_{ij}$, is the macroscopic plastic strain. This strain is defined as the average of the plastic strains in all the grains. The stress–strain history of the spherical embedded grain is dependent on the orientation of the grain axes relative to the specimen axes; and since the orientation distribution is assumed random, the polycrystalline plastic strain is the average over all orientations of the plastic strains in the spherical crystal. Thus

$$E^p{}_{ij} = (\epsilon^p{}_{ij})_{ave} = \frac{1}{4\pi^2} \int\!\!\int_H \sin \eta \, d\eta \, d\beta \int_{-\pi}^{\pi} d\phi \; \epsilon^p{}_{ij} \tag{2}$$

where η, β and ϕ are the Euler angles defined in Fig. 1 which fix the grain axes with respect to the specimen axes and where H is the hemisphere defined in that figure. Equation (1) is consistent with the assumption that macroscopic stress

is equal to the average of the stresses in all the grains. For since $E^p{}_{ij}$ is defined by $E^p{}_{ij} = (\epsilon^p{}_{ij})_{ave}$, it is noted that $\sigma^o{}_{ij} = (\sigma_{ij})_{ave}$.

In what follows it is convenient to introduce the two stress deviator tensors, $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{pp}\delta_{ij}$ and $S^o{}_{ij} = \sigma^o{}_{ij} - \frac{1}{3}\sigma^o{}_{pp}\delta_{ij}$, and to write (1) in deviator form as

$$s_{ij} = S^o{}_{ij} + \frac{2\,(7 - 5\nu)}{15\,(1 - \nu)}\, GE^p{}_{ij} - \frac{2\,(7 - 5\nu)}{15\,(1 - \nu)}\, G\epsilon^p{}_{ij}. \tag{3}$$

In simple tension $S^o{}_{11} = \frac{2}{3}\sigma$, $S^o{}_{22} = S^o{}_{33} = -\frac{1}{3}\sigma$ and $E^p{}_{11} = -2E^p{}_{22} = -2E^p{}_{33}$ (other components of these two tensors are zero); and thus it is possible to define a deviator tensor, $R^o{}_{ij}$, so that

$$R^o{}_{ij} = S^o{}_{ij} + \frac{2\,(7 - 5\nu)}{15\,(1 - \nu)}\, GE^p{}_{ij} \tag{4}$$

where $R^o{}_{11} = -2R^o{}_{22} = -2R^o{}_{33}$ (the other components of this tensor being zero). Similarly in simple shear since $S^o{}_{12}\,(= S^o{}_{21})$ and $E^p{}_{12}\,(= E^p{}_{21})$ are the only non-zero components of the polycrystalline stress and plastic strain tensors, it is possible to define another such $R^o{}_{ij}$ where only the component $R^o{}_{12}\,(= R^o{}_{21})$ is non-zero. For these two stress histories with $R^o{}_{ij}$ defined as above, (3) can be written as

$$s_{ij} = R^o{}_{ij} - \frac{2\,(7 - 5\nu)}{15\,(1 - \nu)}\, G\epsilon^p{}_{ij}. \tag{5}$$

In this equation the aggregate plastic strain does not explicity appear. With the constitutive relations specified, the stress–strain history of each grain is determined as a function of $R^o{}_{ij}$, then the polycrystalline plastic strain is found as a function of $R^o{}_{ij}$ and, finally, the polycrystalline stress is found from (4). Since the term $E^p{}_{ij}$ in (3) is dependent on the plastic strain in each of the grains the calculation procedure for an arbitrary stress–strain history is not as straightforward as described above. For any history other than simple tension or simple shear the calculation must proceed in an iterative fashion to determine the average quantities as well as the individual grain histories in incremental steps.

An f.c.c. single crystal can slip on each of its four (111) crystallographic planes and on each of these planes in the three (110) type directions. With n_j as the normal to the slip plane and m_j as one of the slip directions associated with this plane, the plastic strain resulting from a shear strain γ on this slip system is

$$\epsilon^p{}_{ij} = \tfrac{1}{2}\gamma\,(m_i\,n_j + m_j\,n_i) = \gamma\alpha_{ij}.$$

The total plastic strain of the grain is the sum of the contributions of the slips on all systems, so that

$$\epsilon^p{}_{ij} = \sum_n \gamma^{(n)}\,\alpha_{ij}{}^{(n)}. \tag{6}$$

With symbols topped with dots denoting the increments of these quantities, the incremental form of the above equations is

$$\dot{\epsilon}^p{}_{ij} = \sum_n \dot{\gamma}^{(n)}\,\alpha_{ij}{}^{(n)}. \tag{7}$$

Slip occurs on the nth slip system when the magnitude of the resolved shear stress on that system, $\left|\tau^{(n)}\right|$, reaches the yield stress of that system, τ_y, where

$\tau^{(n)} = \sigma_{ij} \, \alpha_{ij}^{(n)} \equiv s_{ij} \, \alpha_{ij}^{(n)}$. Corresponding to a positive or negative shear stress, there must be a positive or negative shear strain. The following convention is introduced for notational simplicity. Instead of allowing both positive and negative slip on each slip system, only positive slip resulting from a positive shear stress is permitted. To account for negative slip resulting from a negative shear stress on any physical system, a twin system is introduced such that the α_{ij} defining the second slip system is negative the α_{ij} of its twin. With this convention the number of slip systems is doubled and the yield stress of each system is positive as is the shear strain.

The strain hardening rule chosen to represent the single crystal behaviour was suggested by Taylor, namely the yield stress on each system is equal to τ_y where

$$\tau_y = F \left[\sum_{(n)} \gamma^{(n)} \right]. \tag{8}$$

The yield surface of the single crystal as viewed in stress space undergoes a uniform expansion. Certainly this is in contradiction to the Bauschinger effect of the single crystals and this rule would not be valid for reversible stress histories.

With the relations between the stress and plastic strain of the embedded crystal given by (5) and the constitutive relations specified, the entire stress–strain history of the embedded crystal for the case of either simple tension or simple shear can be calculated. The stress in the crystal is σ_{ij}. A change in $\mathring{R}^0{}_{ij}$, $\mathring{R}^0{}_{ij}$, results in a change of the stress in the crystal, $\dot\sigma_{ij}$. If for the nth slip system

$$\sigma_{ij} \, \alpha_{ij}^{(n)} \equiv s_{ij} \, \alpha_{ij}^{(n)} = \tau_y, \tag{9}$$

then this system is said to be potentially active. Either this system remains active, in which case the resolved shear stress on this system remains equal to the yield stress

$$\dot s_{ij} \, \alpha_{ij}^{(n)} = \dot\tau_y \quad \text{and} \quad \dot\gamma^{(n)} \geqslant 0; \tag{10}$$

or this system unloads and becomes inactive. For an unloading system

$$\dot s_{ij} \, \alpha_{ij}^{(n)} < \dot\tau_y \quad \text{and} \quad \dot\gamma^{(n)} = 0. \tag{11}$$

If a system is not potentially active it is said to be inactive and

$$s_{ij} \, \alpha_{ij}^{(n)} < \tau_y \quad \text{and} \quad \dot\gamma^{(n)} = 0. \tag{12}$$

With the incremental form of (5) and (7) and a linear version of (8) substituted into (10) and (11), a set of linear algebraic equations for the unknown incremental $\dot\gamma^{(n)}$'s along with auxiliary conditions is obtained. For example, (10) for all potentially active systems which do not unload becomes

$$\mathring{R}^0{}_{ij} \, \alpha_{ij}^{(n)} - \frac{2\,(7 - 5\nu)}{15\,(1 - \nu)} \, G \sum_m \dot\gamma^{(m)} \, \alpha_{ij}^{(m)} \, \alpha_{ij}^{(n)} = b \sum_m \dot\gamma^{(m)} \tag{13}$$

where b is the linear Taylor hardening parameter. The equations for potentially active systems which unload are similar.

A solution to the above equations consists of a set of $\dot\gamma$'s which satisfies all the above conditions. From this set all the other physically significant quantities can be determined. At any stage in the calculation the potentially active systems will be known. From this set of n systems a trial set of m active systems is chosen. The m $\dot\gamma$'s associated with this choice are determined from

the set (13), m equations and m unknowns. The remaining n-m $\dot{\gamma}$'s are supposed zero. If the m $\dot{\gamma}$'s are not negative, and if the shear stress increment on each of the remaining n-m slip systems is either equal to or less than but not greater than that on the active systems, then a solution has been obtained. If one of the above criteria is not satisfied then a different trial set must be chosen. It can be shown that if the set of m alphas are linearly dependent then the corresponding $m \times m$ determinant of the set (13) is zero. As no more than five slip systems can be independent, the set of equations to be inverted will never be greater than five by five. The set of $\dot{\gamma}$'s obtained is not necessarily unique, but both $\dot{\epsilon}^p_{ij}$ and $\dot{\tau}_y$ as well as the set of unloading slip systems can be shown to be unique. Once the $\dot{\gamma}$'s have been determined they are linearly dependent upon $\dot{R}^o_{ij}$ until one of the inactive slip systems becomes potentially active. At this point the procedure must again be repeated.

The ideal plasticity solution, i.e. with $b = 0$, is of special interest. This solution for the f.c.c. structure was obtained by BUDIANSKY and WU (1962). It can be seen from (13) that once five independent slip systems become active, $\dot{s}_{ij} = 0$ and no further slip systems become active or unload. Such a situation is referred to by saying the stress has reached a corner of the yield surface. The macroscopic stress of the aggregate is the average of the stress in the individual crystals. As $\dot{R}^o_{ij}$ increases, the state is approached in which the stresses in all the crystals reach corners; and thus, since the macroscopic stress is just the average of these corner stresses, it also approaches a limit value. By the incremental form of (5) it is seen that, once $\dot{s}_{ij} = 0$ in a crystal, $\dot{\epsilon}^p_{ij} = 15 (1 - \nu)/2 (7 - 5\nu) G \dot{R}^o_{ij}$. Thus, as $\dot{R}^o_{ij}$ increases, the state is asymptotically approached in which the plastic strain increments in all the crystals are equal. This was the assumption Taylor used in his polycrystalline deformation analysis for large strains. With $b \neq 0$ it will be seen that five independent slip systems do not necessarily ever become active, and the plastic strain increments in the crystals do not become equal.

3. THE TAYLOR HARDENING SOLUTION

The deformation of the spherical crystal obeying Taylor's strain hardening rule can be given in terms of the solution for the ideally plastic grain. The following solution, produced under the assumption of linear Taylor hardening, can be extended to other functional dependencies on the total slip. The equations for the single crystal deforming according to Taylor's rule, (9) through (13), are the equations governing the deformation of an ideally plastic single crystal if b is taken to be zero. The Taylor hardening solution is given in terms of the ideal plasticity solution for a grain of the same orientation by

$$R^o_{ij} = \lambda \bar{R}^o_{ij}, \quad s_{ij} = \lambda \bar{s}_{ij}, \quad \epsilon^p_{ij} = \lambda \bar{\epsilon}^p_{ij}, \quad \gamma^{(n)} = \lambda \bar{\gamma}^{(n)}, \tag{14}$$

where bars denote ideal plasticity quantities and

$$\lambda = \frac{1}{1 - \dfrac{b}{\tau^o_y} \displaystyle\sum_n \bar{\gamma}^{(n)}}. \tag{15}$$

λ has been obtained from the equation

$$\lambda \tau^o_y = \tau^o_y + b \sum_n \gamma^{(n)} = \tau^o_y + b\lambda \sum_n \bar{\gamma}^{(n)}$$

and thus

$$s_{ij} \alpha_{ij}^{(n)} = \lambda \bar{s}_{ij} \alpha_{ij}^{(n)} = \lambda \tau^o_y = \tau^o_y + b \sum_m \gamma^{(m)}$$

for an active system, and

$$s_{ij}\,\alpha_{ij}^{(n)} = \lambda\bar{s}_{ij}\,\alpha_{ij}^{(n)} < \lambda\tau^o{}_y = \tau^o{}_y + b\sum_m \gamma^{(m)}$$

for an inactive system. Clearly, the auxiliary conditions are also satisfied.

The sets of systems which become active and those which unload in the Taylor hardening solution are the same as those for the ideal plasticity solution; but it is interesting to note that, if $b\sum_n \bar{\gamma}^{(n)} = \tau^o{}_y$ prior to stage where the last set of slip systems is activated in the perfect plasticity solution, then this set will not be activated in the Taylor hardening solution. It is to be concluded in such a case that five independent slip systems are never activated; and, as mentioned previously, the plastic strain does not approach a uniform value in each grain.

4. NUMERICAL CALCULATIONS

The procedure for calculating the stress–strain history of a grain of any orientation has been described. The averaging integral defined by (2) must be replaced by a numerical integration. The symmetry of the f.c.c. crystal is such as to divide the hemisphere of Fig. 1 into twenty-four spherical triangles. For simple tension and simple shear the integration can be replaced by an integration over any one of these spherical triangles. In addition, in simple tension $\epsilon^p{}_{11}$ is independent of ϕ and thus the integral can be replaced by a numerical integration over one of the spherical triangles. To evaluate the integral the β variation (the hemisphere was chosen so that $0 \leqslant \beta \leqslant \frac{1}{2}\pi$) was divided into twelve equal divisions. Then the η variations corresponding to each of the thirteen fixed values of β were divided into equal divisions ranging from one integration station at $\beta = 0$ to thirteen stations at $\beta = \frac{1}{2}\pi$. In all, ninety-one orientation stations were used for the numerical evaluation.
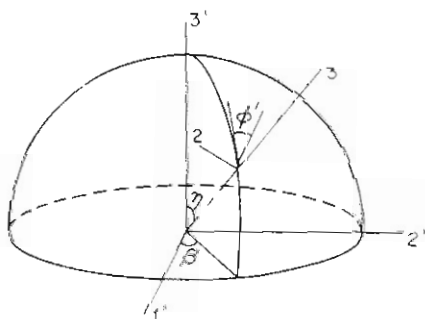

FIG. 1.

For each of these ninety-one orientations the history of $\epsilon^p{}_{11}$ was determined as a function of $R^o{}_{11}$. Then, for fixed values of $R^o{}_{11}$, $\epsilon^p{}_{11}$ was averaged over all orientations in accordance with (2). Once $E^p{}_{11}$ had been obtained, the corresponding value of the tensile stress, σ, was obtained from (4) as

$$\sigma = \tfrac{3}{2}S^o{}_{11} = \tfrac{3}{2}\left[R^o{}_{11} - \frac{2\,(7 - 5\nu)}{15\,(1 - \nu)}\,GE^p{}_{ij}\right].$$

These calculations were executed by a 7090 IBM computer with Poisson's ratio set equal to $\frac{1}{3}$. To obtain an estimate of the accuracy, a calculation identical to the one described above was made with twenty-eight instead of ninety-one orientation stations. The divergence between the two predictions was nowhere greater than 0·4 per cent. It would be expected that this is the order of magnitude of the error of the ninety-one station calculation.

The calculation of both the Bauschinger effect and the cyclic tensile loading were performed with twenty-eight orientation stations. The reduction in the number of orientation stations resulted in a proportional saving in calculation time and yet did not sacrifice much accuracy. While ϵ^P_{11} is independent of ϕ in the tensile calculation, ϵ^P_{12} is dependent on this angle in simple shear. The variation in ϕ, 0 to $\frac{1}{2}\pi$, was divided into ten equal divisions. For each fixed value of ϕ the η and β integration was replaced by a numerical integration over twenty-eight stations as described above. Thus 280 orientation stations were used.

5. THEORETICAL PREDICTIONS

(a) Tensile stress–strain relations

The tensile stress–strain curve for a polycrystal of ideally plastic single crystals was obtained by BUDIANSKY and WU (1962). The relationship between the stress and strain in a grain of the Taylor hardening solution and similarly oriented grain of the perfect plasticity solution gives rise, as will be seen in the next subsection, to a very good approximate formulae for the stress–strain curve of the Taylor hardening polycrystal in terms of the corresponding curve for the ideally plastic polycrystal. Exact calculations were made for several values of the strain hardening parameter, b. The results of these calculations are given in Table 1 and shown in Fig. 2 along with the perfect plasticity curve. As the tensile strain increases, the tensile stress of the polycrystal of perfectly plastic single crystals becomes asymptotic to a limit value. while the slope of the stress–strain curve of the Taylor hardening polycrystal approaches a limiting value. Both of these properties have significance in terms of work done previously by Taylor.
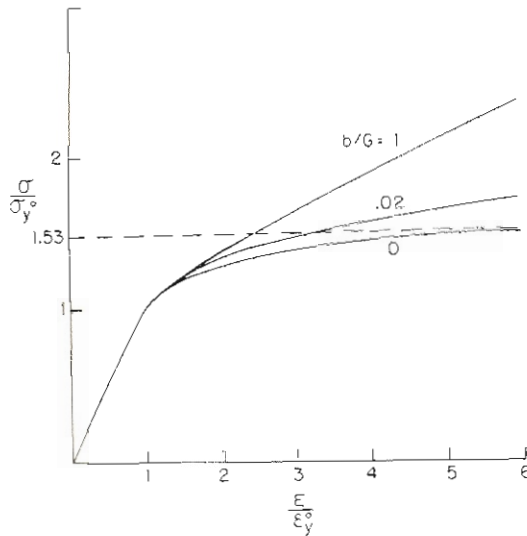

FIG. 2. Tensile stress–strain curves for various values of the hardening parameter, b/G. σ^0_y and ϵ^0_y are the stress and strain associated with the initial elastic limit. $\nu = \frac{1}{3}$.

At this point it is appropriate to mention one of the inadequacies of the Taylor formula (8). Experiments with single crystals of aluminium, such as those by DAVIS et al. (1957), valid for strains larger than those of direct interest in this analysis, have shown that the parameter b of the linear strain hardening rule is

dependent on the particular set of slip systems which is active. Such a parameter dependence could be incorporated into the present calculations without difficulty. Unfortunately, no data valid for small strains exist which would permit such a calculation. Furthermore, it seems that it would be very difficult to obtain single crystal multi-slip data for such small strains since this would require very accurate orientation of the single crystals relative to the tensile axis.

TABLE 1

$b/G = 0$		$b/G = 0.02$		$b/G = 0.10$	
σ/σ^0_y	E^p/ϵ^0_y	σ/σ^0_y	E^p/ϵ^0_y	σ/σ^0_y	E^p/ϵ^0_y
1·125	0·125	1·128	0·120	1·138	0·104
1·195	0·341	1·204	0·326	1·232	0·280
1·252	0·580	1·268	0·554	1·317	0·471
1·363	1·228	1·327	0·789	1·400	0·667
1·447	2·422	1·424	1·293	1·630	1·283
1·520	9·30	1·561	2·398	1·984	2·360
1·581	20·95	2·131	9·948	4·556	10·91
1·536	∞	2·85	20·42	7·54	20·10

(b) Taylor's formula for large plastic strains

If the plastic strain is large TAYLOR (1938) assumed the strain in each grain is homogeneous and equal to the macroscopic strain of the polycrystal. Since the strain is assumed large, the elastic strain is ignored. To achieve the prescribed uniform strain in general slip must occur on five independent slip systems in each crystal. Taylor assumed each of the slip systems hardened uniformly in accordance with (8). Of all the possible combinations of slips which give the prescribed plastic strain, the combination for which the sum of the magnitudes of the slips, $\sum_n \gamma^{(n)}$,

is a minimum is chosen. The relation between the plastic strain and stress of the polycrystal in simple tension is

$$\sigma = \left[F\left(m\, E^p \right) m \right]_{ave.} \tag{16}$$

Here, as in the present model, the average is over all the possible orientations of the grain axes relative to the tensile axis and m is the quantity $\sum_n \gamma^{(n)}/E^p$,

associated with each orientation. The above formula may be approximated, as Taylor seems to have done, by assuming for any given plastic strain that each grain has hardened equally, and

$$\sigma = F\left(m_{ave}\, E^p \right) m_{ave.} \tag{17}$$

For linear hardening

$$\sigma = (m_{ave})^2\, b\, E^p. \tag{18}$$

The value for m_{ave} which Taylor obtained for the f.c.c. tensile calculation was 3·06. Taylor's procedure was extended and verified by BISHOP and HILL (1951) who made additional calculations using a simplified procedure. The final

slope of the stress–strain curve as predicted by (18) is compared with the prediction of the present theory in Fig. 3. Also included in this figure is the final slope as predicted by Lin's extension of Taylor's model. These results were obtained by making use of a correspondence between the Lin model and the Budiansky–Wu model demonstrated in Section 6. For $b/G = 0.1$ the final slope, $\dot{\sigma}/\dot{E}^p$, predicted by (18) is about 15 per cent higher than the corresponding value of the present analysis. As was noted previously the slip systems activated in the latter stages of plastic deformation in the perfect plasticity solution remain dormant with Taylor hardening. Thus, for larger plastic strains, instead of five slip systems contributing to the plastic strain as assumed by Taylor, in most of the grains only three or four slip systems contribute. The strain in each grain does not become uniform, and divergence from Taylor's formula should be expected. This effect is more pronounced the larger the strain hardening parameter. With bars denoting the perfect plasticity quantities, equations (14) and (15) can be averaged directly under the assumption of uniform hardening to give for simple tension

$$\sigma = \frac{1}{1 - \dfrac{b}{\tau^0_y} \, m_{ave} \, \overline{E}^p} \, \overline{\sigma}, \qquad E^p = \frac{1}{1 - \dfrac{b}{\tau^0_y} \, m_{ave} \, \overline{E}^p} \, \overline{E}^p.$$

Although the number, m_{ave}, is only appropriate when five slip systems are active and not in the initial stages of slip, this formula nevertheless gives very close agreement with the exact calculations for small values of the strain hardening parameter. For large values the formula displays about the same error as was noted in the discussion of the Taylor formula (18).
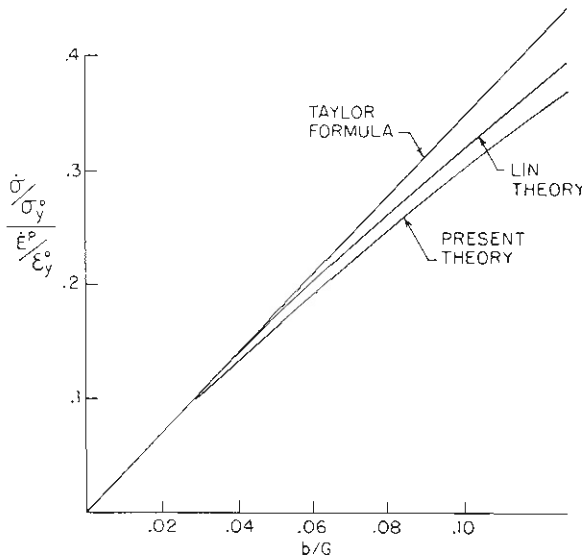

FIG. 3.

The limit yield stress of the f.c.c. polycrystal comprized of perfectly plastic single crystals is $\sigma/\sigma^0_y = 1.536$ in agreement with values obtained by both Taylor and BISHOP-HILL, as required theoretically.

(c) *The stress–strain curve in simple shear*

Again this curve for a polycrystal comprised of perfectly plastic single crystals was obtained by BUDIANSKY and WU (1962). The limit yield stress of this solution, $\tau_{lim}/\tau^0{}_y$, is 1·656 in agreement with the value obtained by Bishop and Hill. When the stress–strain curve in simple tension is known, it is possible to calculate the stress–strain curve in simple shear by assuming either J_2 theory or the τ_{max} vs. γ^p_{max} theory. The affinity relationships of these two phenomenological theories are

$$J_2 \text{ theory}: \quad \frac{\tau}{\tau^0{}_y} = \frac{2}{\sqrt{(3)}}\frac{\sigma}{o^0{}_y}, \quad \frac{\gamma^p}{\gamma^0{}_y} = \frac{\sqrt{(3)}}{1+\nu}\frac{E^p}{\epsilon^0{}_y};$$

$$\tau_{max} \text{ vs. } \gamma^p_{max} \text{ theory}: \quad \frac{\tau}{\tau^0{}_y} = \frac{\sigma}{o^0{}_y}, \quad \frac{\gamma^p}{\gamma^0{}_y} = \frac{3}{2(1+\nu)}\frac{E^p}{\epsilon^0{}_y}.$$

BATDORF (1950) interpreted some biaxial experiments with aluminium polycrystals by OSGOOD (1947) in terms of J_2 and τ_{max} vs. γ^p_{max} theories. Using the experimental tensile stress–strain curve in conjunction with the two phenomeno-
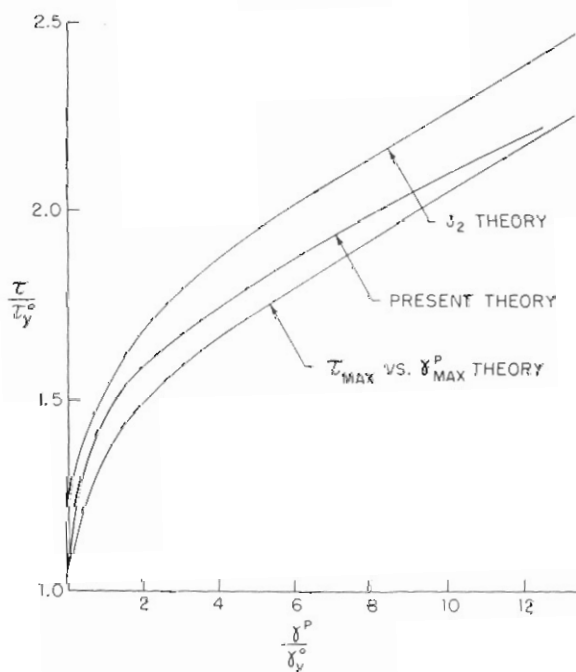

FIG. 4. Comparison with phenomenological theories.

logical theories, he could predict the pure shear curves and then compare these with the experimental shear curve. TAYLOR (1931) made a similar comparison with copper polycrystals. He employed, however, Mises (J_2) theory and Tresca's hypothesis. If his results are reinterpreted in terms of J_2 theory and τ_{max} vs. γ^p_{max} theory, they are seen to be very similar to the findings of Batdorf. The experimental shear curve lies more or less half way between the two phenomenological shear curves approaching the τ_{max} vs. γ^p_{max} curve as the plastic strain increases. A comparison in the same spirit was made with the tensile and shear

curves predicted by the present model; b/G was chosen to be 0.02 as a representative value of hardening. The relationship between the three curves, the two phenomenological curves as determined from the theoretical tensile curve and the theoretical shear curve, is shown in Fig. 4 and is similar to that described for the Batdorf and Taylor plots.

(d) The polycrystalline Bauschinger effect

As is well known, the tensile elastic limit of a polycrystal is increased by plastic extension and the magnitude of the compressive elastic limit is diminished. This phenomenon is commonly called the Bauschinger effect. The experimental data on tension-compression stress–strain relations of single crystals are not complete enough to warrant a calculation of the polycrystalline Bauschinger effect as
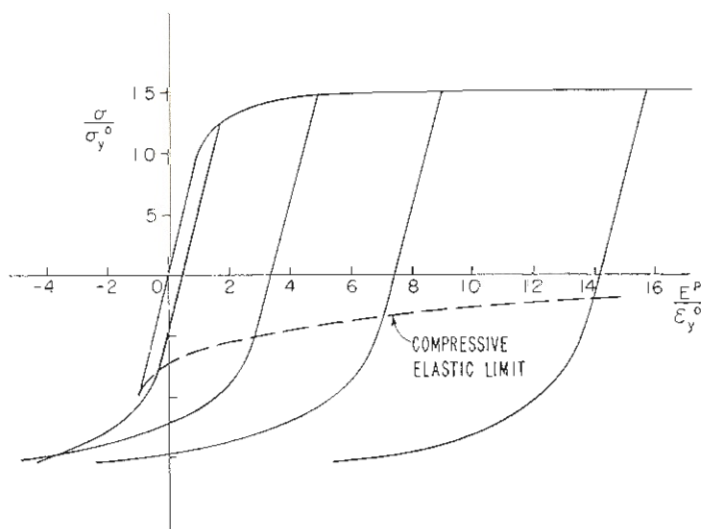

FIG. 5. Polycrystalline Bauschinger effect.

dependent on those of the single crystals.   On the other hand, an analysis based on perfectly plastic single crystals (the yield stresses being constant and equal on all systems) predicts a polycrystalline Bauschinger effect. A similar result was first obtained by CZYZAK et al. (1961) using Lin's extension to Taylor's model.   As shown in Section 6 there is reason to expect that the calculations of Czyzak et al. are in error.   The correspondence between the two theories enables the corrected results to be obtained along with the predictions of the present theory.

A series of stress–strain curves of f.c.c. polycrystals under tension and subsequent compression have been calculated for different values of initial plastic extensions. These results are shown in Fig. 5. When the tensile load on the polycrystal is diminished it would seem that the strain increments in all the crystals would become pure elastic. But according to this model, as well as that of Lin, this is not the case. Upon decreasing the applied tensile stress after a large initial plastic extension a small amount of plastic strain occurs in almost one quarter of the crystals. The plastic strains in these crystals combine to contribute a macroscopic plastic strain totalling only about $3/2$ per cent of the elastic strain.

The amount of plastic strain in this nominal elastic region decreases for diminishing values of the initial plastic extension, a fact which can easily be explained. The slip systems that continue to remain active after the tensile stress has been diminished are those which are activated in the latter stages of slip. Thus, if the stress on a crystal is such that the last stages of slip are not activated, the crystal will deform elastically instead of plastically when the tensile stress is diminished. Since there is not a truly elastic range under reversed loading, the compressive elastic limit was established by assuming the nominal elastic range included the region in which the plastic strain contribution was less than 2 per cent of the elastic strain contribution.

It is noted that following a very large plastic extension the compressive elastic limit is $\sigma/\sigma^0_y = -0.17$. The theoretical results also show (see Fig. 5) that the width of the nominal elastic region shrinks as the initial plastic extension increases. This phenomenon has been reported for aluminium polycrystals by IVEY (1961). Any completely successful prediction of the polycrystalline Bauschinger effect must necessarily include these same single crystal effects, but the present analysis indicates that the grain interaction contributes to this effect in a significant way.

(e) Cyclic tensile loading

The perculiar prediction of small plastic strain contributions upon reversal of the applied load, noted in conjunction with the Bauschinger effect, would lead one to suspect that the model might predict an accumulation of plastic tensile strain under cyclic tensile loading. The following calculation was made. The f.c.c. specimen of perfectly plastic single crystals was given an initial tensile stress which deformed the specimen plastically; then the tensile stress was reduced to zero. Next, the stress was increased again to its initial value and the resulting increment in tensile plastic strain, $\Delta E^p/\epsilon^0_y$, over this cycle was calculated. This procedure was repeated for several cycles until either the plastic strain increments

TABLE 2

cycle	$E^p_0/\epsilon^0_y = 14.0$ $\Delta E^p/\epsilon^0_y$	cycle	$E^p_0/\epsilon^0_y = 7.6$ $\Delta E^p/\epsilon^0_y$	cycle	$E^p_0/\epsilon^0_y = 4.0$ $\Delta E^p/\epsilon^0_y$
1	1.33	1	0.39	1	0.15
2	0.50	2	0.06	2	elastic
3	0.45	3	elastic		
4	0.37				

became small or the specimen started to behave elastically throughout the entire cycle. In a typical cycle the specimen deforms with a small plastic strain as the load is diminished from the initial stress to zero stress. As the load then increases from zero the specimen deforms elastically until a stress, lower than the initial stress, is reached at which the specimen begins to deform plastically. It is in this stage that the major contribution to the cyclic plastic strain increment is made. If, in any cycle, the specimen deforms elastically upon diminishing the stress from the initial tensile value, the behaviour will be purely elastic in all the subsequent cycles. The results of this calculation for several values of the initial plastic extension, E^p_0/ϵ^0_y, are given in Table 2.

There do not seem to be any data which would verify either the prediction of small amounts of plastic strain upon reversal of the tensile loading or the prediction of cyclic plastic strain increments.   Indeed, high accuracy of measurement would be required and these phenomena would be obscured by such phenomena as creep and the elastic after effect. Further insight can be gained by considering the effect of strain hardening. With all systems hardening equally according to Taylor's rule it was seen that, the higher the strain hardening, the more the tendency that the final slip systems to be activated in the perfectly plastic crystal would remain dormant in the Taylor hardening crystal. Since these are the slip systems which contribute the plastic strain upon reversal of the tensile load, it is to be expected that the phenomena which we are considering will not be observed in specimens which display high strain hardening. The assumption of equal hardening on all slip systems is certainly only an approximation.   If the latent slip systems tend to harden slower than the active slip systems than these phenomena should be expected to be amplified and vice-versa.

6.   Correspondence between Lin's Model and the Present Model

Each crystal of the Lin model (1957) is assumed to have the same total strain as the polycrystalline aggregate as in Taylor's model.   Lin included the elastic strain and thus did not restrict his analysis to large plastic strains.   As in the present theory, the plastic strain is obtained for each orientation; and then the aggregate stress and plastic strain are obtained as an average over all the orientations.   The total strain deviator, e_{ij}, is imposed on each grain and is the sum of the plastic strain, ϵ^p_{ij}, and the elastic strain deviator, e^E_{ij}.   The constitutive equation for each deforming crystal are the same as those of the Budiansky-Wu model, equations (8) to (12), but the relation between stress and plastic strain in each crystal is not given by (5) as in the present theory but by

$$s_{ij} = 2\, G\, e^E_{ij}$$

where $$e^E_{ij} = e_{ij} - \epsilon^p_{ij} \quad\text{and}\quad \epsilon^p_{ij} = \sum_n \gamma^{(n)}\, \alpha_{ij}^{(n)}.$$

Thus, for example, for a potentially active system not unloading,

$$2G\left[\dot{e}_{ij}\, \alpha_{ij}^{(n)} - \sum_m \dot{\gamma}^{(m)}\, \alpha_{ij}^{(m)}\, \alpha_{ij}^{(n)}\right] = b \sum_m \dot{\gamma}^{(m)} \quad\text{and}\quad \dot{\gamma}^{(n)} \geqslant 0.$$

Recalling that R^o_{ij} is the independent variable in the present calculations while e_{ij} is prescribed in Lin's calculation, we set (with the circumflex accent denoting Lin model quantities)

$$R^o_{ij} = \beta\, \hat{e}_{ij}.$$

Reference to the equations of the Budiansky-Wu model, (9)–(13), and to those above reveals that a solution to the Lin equations is a solution to the Budiansky-Wu equations if

$$\beta = \frac{1}{1+\nu}\,, \quad \hat{\epsilon}^p_{ij} = \frac{7-5\nu}{15\,(1-\nu)}\, \epsilon^p_{ij}, \quad \hat{b} = \frac{15\,(1-\nu)}{7-5\nu}\, b.$$

The polycrystalline stress and plastic strain of the Lin model, just the average

of these quantities over all orientations, are given in terms of the quantities of the present analysis by

$$(\hat{s}_{ij})_{ave} = S^o{}_{ij} \quad \text{and} \quad \hat{E}^p{}_{ij} = \frac{7 - 5\nu}{15\,(1 - \nu)}\,E^p{}_{ij}.$$

The relationship between the two models is now clear, For perfect plasticity, $b = \dot{b} = 0$, if corresponding to a stress, σ (simple tension), the present theory predicts a plastic strain, E^p, then the Lin theory predicts a plastic strain of $\left[7 - 5\nu/15\,(1 - \nu)\right]E^p$. With $\nu = 1/3$ the latter strain is $8/15$ of the value predicted by the Budiansky–Wu model. By the equation for the polycrystalline stress the limit yield stress predicted by both theories is the same. Czyzak *et al.* have made several calculations on the basis of Lin's model, including the tensile stress–strain curve of a polycrystal comprized of ideally plastic f.c.c. crystals. Their calculation provided the value $\sigma_{\text{lim}}/\sigma^o{}_y$, and this is in agreement with the value obtained. A figure comparing the tensile curves of the two theories is given by Budiansky and Wu but is inaccurately plotted. A similar correspondence can be made with non zero strain hardening, but account must be taken of the relation $\dot{b} = \left[15\,(1 - \nu)\,(7 - 5\nu)\right]b$. The final slope of the stress–strain curve of the Lin model is compared with that of the Budiansky–Wu model and also the Taylor formula (23) in Fig. 3.

The Bauschinger curve according to the present theory has been calculated. A similar calculation for f.c.c. polycrystals based on the Lin model has been made by Czyzak *et al.* When these two calculations are brought into comparison through the correspondence noted above, considerable disagreement is apparent. It is immediately seen that limit yield stress in compression according to the present calculation is exactly the negative of the value obtained for tension. This can be shown to be a theoretical requirement of both theories. Reference to the curve on p. 65 of Czyzak *et al.* (1961) indicates that their calculation is not in agreement with this theoretical requirement, and thus it is suggested that the authors have made an error at some point in their calculation.

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