Sonderdruck aus

Nicht im Handel

APPLIED MECHANICS

PROCEEDINGS OF THE ELEVENTH INTERNATIONAL CONGRESS OF APPLIED MECHANICS · MUNICH (GERMANY) 1964

Edited by HENRY GÖRTLER, Freiburg i. Br.

(Springer-Verlag, Berlin · Heidelberg New York)
(Printed in Germany)

Dynamic buckling of imperfection-sensitive structures

By

B. Budiansky and J. W. Hutchinson

Harvard University, Cambridge, Mass.

Introduction

Small geometrical imperfections in some structures can be responsible for large reductions in their static buckling strengths, As is well known, a thin shell is often very imperfection-sensitive in this sense, with a perfect specimen sometimes having a "classical" buckling strength several times higher than that of an imperfect one. Many analytical studies have sought to correlate reductions in buckling strength with assumed initial imperfections of various sizes and shapes. Such studies may eventually provide the quantitative information needed for the establishment of a statistical theory of buckling, which would relate the probability of buckling under a given static load to the spectrum of imperfections (see [*1*]). But at the present time, the design of shells leans heavily on experiment, and analysis has been mainly useful in identifying imperfection-sensitive structures and in establishing, in a qualitative way, the degree of this sensitivity.

Analyses have recently been made of the dynamic buckling of shells subjected to transient loading histories, wherein inertial forces must be considered [*2*, *3*, *4*, *5*]. There is not as yet a scientific consensus concerning an appropriate analytical definition of "dynamic buckling", or of the "dynamic buckling load", but regardless of this, imperfection-sensitivity can be expected to be as pertinent to dynamic buckling as to static buckling. It would appear, then, that in order to solve the problem of dynamic buckling theoretically, we might be faced with the necessity of having to analyze imperfection-sensitive structures for a wide variety of imperfections, for each different kind of transient loading history that is of interest.

The purpose of the present paper is to explore the possibility of bypassing such repetitious calculations by seeking to relate the dynamik buckling strength of a given imperfect structure directly to its static buckling strength. The viewpoint adopted as a working hypothesis is that the essential effects of imperfections reveal themselves in the extent to which they reduce static buckling strengths below their "classical" values, and that perhaps knowledge of the static reduction factors might therefore suffice for reasonably accurate predictions of dynamic buckling loads, without the need for further details concerning the imperfections themselves. The true static buckling loads needed in such a correlation, could, of course, be determined experimentally with much less difficulty than dynamic buckling loads.

In this paper, consideration will be restricted, for the most part, to elastic buckling under suddenly applied dead loads that are maintained at a constant magnitude. Following a discussion of criteria for dynamic buckling, the implications of some simple imperfection-sensitive models will be discussed. Next, on the basis of KOITER's theory of post-buckling behavior [*6*, *7*], general approximate theories of dynamic buckling will be formulated and their relations to the simple models will be studied. Finally, as particular examples, analyses will be made of the dynamic buckling under suddenly applied axial loads of circular cylindrical shells, stiffened by longitudinal stringers, as well as unstiffened.

Criteria for dynamic buckling

With respect to a given structure, consider the ensemble of loading histories $q(\vec{x}, t)$ generated by varying λ in the equation

$$q(\vec{x}, t) = \lambda q_0(\vec{x}, t) \quad (t \geq 0), \tag{1}$$

where $q_0(\vec{x}, t)$ is a particular function of position $\vec{x}$ and time t, and λ is a scalar parameter; we now propose to define a critical value of λ for dynamic buckling.

Let $R(\lambda, t)$ be a physically significant scalar measure of the response of the structure to $q(\vec{x}, t)$ (e.g. a stress, a deflection, an average deflection, etc.); further, define

$$R_{\max}(\lambda, T) = \max_{0 \leqq t \leqq T} [R(\lambda, t)], \tag{2}$$

where T is the largest value of t that is of interest. A typical plot of $R_{\max}(\lambda, T)$ vs. λ might then look like that shown in Fig. 1. If, as the hypothetical plot shows, there is indeed a narrow range in λ over which $R_{\max}$ rises very steeply, the critical value λ_D for dynamic buckling will be defined as the value of λ in the middle, more or less, of this range.

A sharper definition of λ_D, independent of T but appropriate for large T, may sometimes be possible on the basis of the variation of $R_{\max}(\lambda, \infty)$ with λ, which might display a finite discontinuity, as shown in Fig. 2a. [Such a discontinuity is generally not possible in $R_{\max}(\lambda, T)$ for finite T.] Indeed, for some idealized structures, $R_{\max}(\lambda, \infty)$ could actually be infinite for all λ greater than some λ_D (Fig. 2b).

The above definition of λ_D for finite T was introduced in [4] and also used in [5], wherein curves like that in Fig. 1 were found; the finite jump of Fig. 2a was used, essentially, in defining critical impulsive loads in [8]. The sharp definitions of λ_D implied by Fig. 2 are mathematically attractive, but the general conditions under which $R_{\max}(\lambda, \infty)$ could indeed be a discontinuous function of λ are not known.

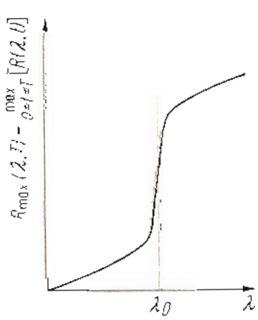

Fig. 1. Diagram for estimating dynamic buckling parameter λ_D

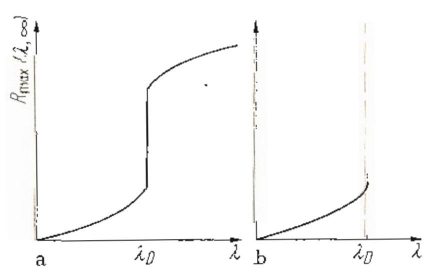

Fig. 2a and b. Hypothetical variations of $R_{\max}(\lambda,\infty)$ with λ

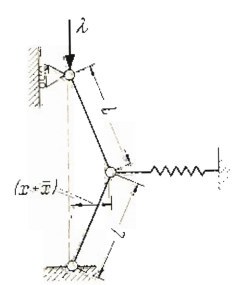

Fig. 3. Idealized column constrained by non-linear spring

Implications of some simple models

Imperfection-sensitivity is exhibited by the three-hinge, rigid-rod column shown in Fig. 3 when it is constrained laterally at its central hinge by a softening non-linear spring; a similar model was used by Kármán, Dunn and Tsien [9] in their pioneering elucidation of finite deformation effects in shell buckling.

Suppose the spring restoring force F is related to its shortening x by

$$F = KL(\xi - \alpha \xi^2), \tag{3}$$

where $\xi = x/L$, and $\alpha > 0$. If the unloaded structure has an initial displacement $\bar{x} = L\bar{\xi}$, then, assuming small rotations ($\xi, \bar{\xi} \ll 1$), *static* equilibrium relates the axial load λ to the *additional* displacement by

$$(1 - \lambda/\lambda_C)\, \xi - \alpha \xi^2 = (\lambda/\lambda_C)\, \bar{\xi}, \tag{4}$$

where $\lambda_C = KL/2$. Thus (Fig. 4a) the perfect column, with $\bar{\xi} = 0$, can buckle at the "classical" load λ_C, after which the load drops as ξ increases. With $\bar{\xi} \neq 0$ the structure deflects as soon as load is applied, and, for $\bar{\xi} > 0$, buckles statically at $\lambda = \lambda_S$ given by the maximum value attained by λ as it varies with ξ.

With $z = \xi/\bar{\xi}$, Eq. (4) gives

$$(1 - \lambda/\lambda_C)\, z - (\alpha \bar{\xi})\, z^2 = \lambda/\lambda_C \tag{5}$$

from which it is evident that λ_S/λ_C depends only on the parameter $(\alpha\bar{\xi})$, and, in fact, by setting $d\lambda/dz = 0$ we find that

$$(1 - \lambda_S/\lambda_C)^2 = 4(\alpha\bar{\xi})\,(\lambda_S/\lambda_C). \tag{6}$$

For small $(\alpha\bar{\xi})$, $(\lambda_S/\lambda_C) \approx 1 - 2\sqrt{\alpha\bar{\xi}}$; the larger α, the more imperfection-sensitive is the structure.

Next, consider dynamic equilibrium under a time dependent load $\lambda(t)$, assuming a mass M only at the central hinge; then Eq. (5) changes to

$$\ddot{z} + (1 - \lambda/\lambda_C)\, z - (\alpha\bar{\xi})\, z^2 = \lambda/\lambda_C, \tag{7}$$

where the dot denotes differentiation with respect to $t\sqrt{K/M}$. For the case of a step loading at $t = 0$ with initial conditions $z = \dot{z} = 0$, the first integral of (7) is

$$\dot{z}^2 + (1 - \lambda/\lambda_C)\, z^2 - \frac{2}{3}(\alpha\bar{\xi})\, z^3 = 2(\lambda/\lambda_C)\, z. \tag{8}$$

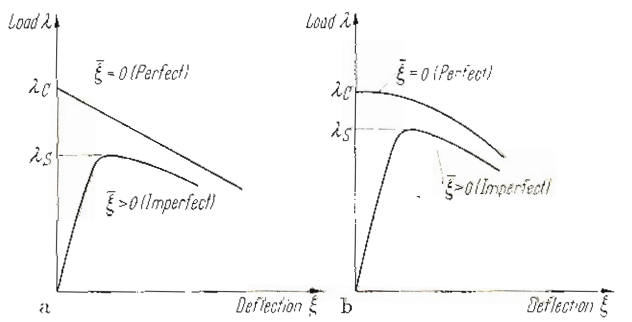

Fig. 4a and b.
Non-dimensional load-deflection curves. a) quadratic model; b) cubic model

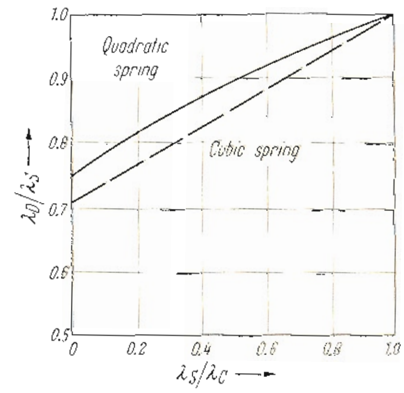

Fig. 5.
Dynamic buckling loads of simple models

For λ sufficiently small the motion is periodic, with a maximum amplitude $z_{\max}$ that satisfies

$$(1 - \lambda/\lambda_C)\, z_{\max}^2 - \frac{2}{3}(\alpha\bar{\xi})\, z_{\max}^3 = 2(\lambda/\lambda_C)\, z_{\max} \tag{9}$$

and gives a relation between λ and $z_{\max}$ like that shown in Fig. 2b between $R(\lambda, \infty)$ and λ. The critical value of λ, namely λ_D, for which the period becomes infinite and beyond which $z_{\max}$ is infinite is now determined by the condition $d\lambda/dz_{\max} = 0$; this, with (9), gives

$$(1 - \lambda_D/\lambda_C)^2 = \frac{16}{3}(\alpha\bar{\xi})\,(\lambda_D/\lambda_C). \tag{10}$$

And now, eliminating $\alpha\bar{\xi}$ between (6) and (10) provides the relation we have sought as

$$(\lambda_D/\lambda_S) = \frac{3}{4}\left(\frac{\lambda_C - \lambda_D}{\lambda_C - \lambda_S}\right)^2. \tag{11}$$

The variation of λ_D/λ_S with λ_S/λ_C is shown by the solid curve in Fig. 5.

Thus, the lower is λ_S/λ_C (and hence the more imperfect the structure) the smaller a fraction of the actual static buckling load is the dynamic buckling load; but this fraction is always at least 3/4. The most important feature of Eq. (11), which we hope can be generalized, is that the imperfection itself does not appear explicitly. Note also that as $\lambda_S \to \lambda_C$, $\lambda_D \to \lambda_C$; this result for the case of an imperfect structure *approaching* perfection is interesting, because dynamic axial loading of the perfect structure can never initiate any lateral motion.

A repetition of the above analysis for the case of a cubic spring having the characteristic

$$F = KL(\xi - \beta\xi^3) \quad (\beta > 0) \tag{12}$$

is easily executed. The analogue of Eq. (4) for static equilibrium is

$$(1 - \lambda/\lambda_C)\,\xi - \beta\xi^3 = (\lambda/\lambda_C)\,\bar{\xi} \tag{13}$$

and curves like those in Fig. 4b apply; note that now the curve for $\bar{\xi} = 0$ is symmetrical in ξ (with zero slope at $\xi = 0$) and static buckling of the imperfect structure is independent of the sign of $\bar{\xi}$. The counterpart of Eq. (7) for the dynamic case is

$$\ddot{z} + (1 - \lambda/\lambda_C)\,z - (\beta\bar{\xi}^2)\,z^3 = \lambda/\lambda_C \tag{14}$$

from which the results

$$(1 - \lambda_S/\lambda_C)^{3/2} = \frac{3\sqrt{3}}{2}\,(\sqrt{\beta}\,|\bar{\xi}|)\,(\lambda_S/\lambda_C) \tag{15}$$

and

$$(1 - \lambda_D/\lambda_C)^{3/2} = \frac{3\sqrt{6}}{2}\,(\sqrt{\beta}\,|\bar{\xi}|)\,(\lambda_D/\lambda_C) \tag{16}$$

are found. Finally,

$$(\lambda_D/\lambda_S) = \frac{\sqrt{2}}{2}\left(\frac{\lambda_C - \lambda_D}{\lambda_C - \lambda_S}\right)^{3/2} \tag{17}$$

which provides the dashed curve in Fig. 5. As seen, the results for (λ_D/λ_S) are not very different for the two models.

Our intention now is to try to determine whether formulas like (17) and (11) might be applicable to real structures. To this end, we shall exploit the general static buckling analysis of KOITER and extend it to dynamic conditions.

General analyses

Koiter's static theory

Field equations. A somewhat less general and slightly modified version of KOITER's theory for the static buckling of imperfect elastic structures will be presented briefly.

Generalized loads, stresses, strains, and displacements will be denoted simply by q, σ, ε, and u, respectively; depending on the structure and the theory used in its analysis, each of these symbols could stand for one or more functions of position. The functional notation $Q_i(u)$ will be used to denote a homogeneous functional of u of degree i; similarly, $Q_{ij}(u, v)$ will mean a homogeneous functional of degree i in u and j in v.

The *strain-displacement relation* will be written

$$\varepsilon = L_1(u) + \frac{1}{2} L_2(u) \tag{18}$$

and the notation

$$e = L_1(u)$$

will be used for the linear part of ε. The bilinear operator $L_{11}(u, v) = L_{11}(v, u)$ is then defined by the identity

$$L_2(u + v) = L_2(u) + 2L_{11}(u, v) + L_2(v)$$

and has the property $L_{11}(u, u) = L_2(u)$.

It will be assumed that for q and σ in equilibrium (in the presence of a displacement u) the principal of virtual work

$$I_{11}(\sigma, \delta\varepsilon) = E_{11}(q, \delta u) \tag{19}$$

holds for all δu, where I_{11} is internal virtual work, and E_{11} is external virtual work; here $\delta\varepsilon = \delta e + L_{11}(u, \delta u)$, where $\delta e = L_1(\delta u)$. Eq. (19) can be regarded as a variational *equation of equilibrium*. Finally, to complete the set of field equations, we postulate the linear *stress-strain relation*

$$\sigma = H_1(\varepsilon) \tag{20}$$

and also assume the reciprocal relation

$$I_{11}[H_1(\varepsilon_1), \varepsilon_2] = I_{11}[H_1(\varepsilon_2), \varepsilon_1]. \tag{21}$$

Trivial solution for the perfect structure. Now suppose the *prescribed* part of the external loading is λq_0, and that wherever loads are not prescribed, there are linear, homogeneous, prescribed geometrical conditions on u; then, for variations δu that are *admissible* (in the sense that they do not violate these geometrical conditions) the external virtual work is just

$$\lambda E_{11}(q_0, \delta u) \equiv \lambda B_1(\delta u).$$

With the use of the abbreviated notation $\{\ ,\ \}$ for $I_{11}(\ ,\ )$, the equation of equilibrium becomes

$$\{\sigma, \delta\varepsilon\} - \lambda B_1(\delta u) = 0 \tag{22}$$

for all admissible δu.

We now assume that the perfect structure has the "trivial" solution $\lambda\sigma_0$, $\lambda\varepsilon_0$, λu_0 for stress, strain and displacement, where u_0 has the property

$$L_{11}(u_0, \delta u) = 0 \tag{23}$$

for all δu; then it follows that $L_2(u_0) = 0$, $\varepsilon_0 = e_0$, $\sigma_0 = H_1(e_0)$, and the equilibrium equation is

$$\{\lambda\sigma_0, \delta e\} - \lambda B_1(\delta u) = 0. \tag{24}$$

Thus, the trivial solution is governed by a linear theory.[1]

Classical buckling of the perfect structure. To discover the eigenvalue λ_C for classical buckling we set

$$u = \lambda_C u_0 + u_C \tag{25}$$

in the field equations, retaining only linear terms in the buckling mode u_C. Then, by (23),

$$\varepsilon = \lambda_C e_0 + e_C,$$

where $e_C \equiv L_1(u_C)$, and, with $s_C \equiv H_1(e_C)$,

$$\sigma = \lambda_C\sigma_0 + s_C.$$

Also,

$$\delta\varepsilon = \delta e + L_{11}(u_C, \delta u)$$

and the equilibrium equation is

$$\{\lambda_C\sigma_0 + s_C, \delta e + L_{11}(u_C, \delta u)\} - \lambda B_1\ (\delta u) = 0.$$

But, by (24), and with further linearization, this gives

$$\lambda_C\{\sigma_0, L_{11}(u_C, \delta u)\} + \{s_C, \delta e\} = 0 \tag{26}$$

as the variational statement of the problem for the lowest eigenvalue λ_C (as well as for the higher ones). Note that (26) implies

$$\lambda_C\{\sigma_0, L_2(u_C)\} + \{s_C, e_C\} = 0 \tag{27}$$

and also that

$$\{\sigma_0, L_{11}(u^{(1)}, u^{(2)})\} = \{s^{(1)}, e^{(2)}\} = \{s^{(2)}, e^{(1)}\} = 0 \tag{28}$$

for any two buckling modes $u^{(1)}$, $u^{(2)}$ associated with distinct eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$. In what follows, we will first assume a single mode u_C (arbitrary, of course, to within a scalar factor) associated with λ_C; later multiple buckling modes will be considered.

---

[1] We are really assuming something about q_0, as well as a perfect structure, when we postulate (23); for certain loadings the hypothesis of linear behavior before buckling is not tenable.

Post-buckling behavior of the perfect structure. When λ reaches λ_C, the structure can begin to suffer deviations in the shape of u_C from its trivial configuration, and, simultaneously, λ will deviate from λ_C. The displacement of the structure in a slightly buckled state can always be written

$$u = \lambda u_0 + \xi u_C + w, \tag{29}$$

where the buckling mode u_C is now considered normalized in magnitude in a definite way, where w is orthogonal to u_C in the sense of Eqs. (28), and where ξ is a scalar. The stress is then

$$\sigma = \lambda\sigma_0 + \xi s_C + \frac{1}{2}\xi^2 H_1[L_2(u_C)] + H_1[L_1(w)] + \xi H_1[L_{11}(u_C, w)] + \frac{1}{2} H_1[L_2(w)]$$

and the equilibrium Eq. (22), simplified by use of (23) and (24), becomes

$$\begin{aligned} &\xi[\lambda\{\sigma_0, L_{11}(u_C, \delta u)\} + \{s_C, \delta e\}] \\ &+ \xi^2\left[\{s_C, L_{11}(u_C, \delta u)\} + \frac{1}{2}\{H_1(L_2(u_C)), \delta e\}\right] \\ &+ \frac{\xi^3}{2}\{H_1(L_2(u_C)), L_{11}(u_C, \delta u)\} \\ &+ \lambda\{\sigma_0, L_{11}(w, \delta u)\} + \{H_1(L_1(w)), \delta e\} \\ &+ \xi[\{s_C, L_{11}(w, \delta u)\} + \{H_1(L_{11}(u_C, w)), \delta e\} + \{H_1(L_1(w)), L_{11}(u_C, \delta u)\}] \\ &+ \cdots = 0, \end{aligned} \tag{30}$$

where the terms not written explicitly are all non-linear in w. It is useful to regard λ as a function of ξ, and to assume, tentatively, the asymptotic representation

$$w = \xi^2 u_2 + \xi^3 u_3 + \cdots$$

for w, hopefully valid for small ξ. The stress is then

$$\sigma = \lambda\sigma_0 + \xi s_C + \xi^2[s_2 + \frac{1}{2}H_1(L_2(u_C))] + \xi^3[s_3 + H_1(L_{11}(u_C, u_2))] + \cdots \tag{31}$$

and the equilibrium Eq. (30) becomes

$$\begin{aligned} &\xi[\lambda\{\sigma_0, L_{11}(u_C, \delta u)\} + \{s_C, \delta e\}] \\ &+ \xi^2[\lambda\{\sigma_0, L_{11}(u_2, \delta u)\} + \{s_2, \delta e\} + \{s_C, L_{11}(u_C, \delta u)\} + \frac{1}{2}\{H_1(L_2(u_C)), \delta e\}] \\ &+ \xi^3[\lambda\{\sigma_0, L_{11}(u_3, \delta u)\} + \{s_3, \delta e\} + \{s_C, L_{11}(u_2, \delta u)\} + \{H_1(L_{11}(u_C, u_2)), \delta e\} \\ &+ \{s_2, L_{11}(u_C, \delta u)\} + \frac{1}{2}\{H_1(L_2(u_C)), L_{11}(u_C, \delta u)\}] + \cdots = 0, \end{aligned} \tag{32}$$

where $e_2 = L_1(u_2)$, $s_2 = H_1(e_2)$, etc., and where the omitted terms are of degree ξ^4 and higher.

The variation $\delta u = u_C\,\delta\xi$ gives the scalar equation relating λ to ξ

$$\begin{aligned} &-\xi(\lambda_C - \lambda)\{\sigma_0, L_2(u_C)\} + \frac{3\xi^2}{2}\{s_C, L_2(u_C)\} \\ &+ \xi^3\left[2\{s_C, L_{11}(u_C, u_2)\} + \{s_2, L_2(u_C)\} + \frac{1}{2}\{H_1(L_2(u_C)), L_2(u_C)\}\right] + \cdots = 0, \end{aligned} \tag{33}$$

where the orthogonality to u_C of u_2 and u_3 has been used, as have Eqs. (21) and (27).

Next, set $\delta u = \delta\tilde{u}$, where $\delta\tilde{u}$ is orthogonal to u_C, in (32), to get

$$\begin{aligned} &\xi^2\left[\lambda\{\sigma_0, L_{11}(u_2, \delta\tilde{u})\} + \{s_2, \delta\tilde{e}\} + \{s_C, L_{11}(u_C, \delta\tilde{u})\} + \frac{1}{2}\{H_1(L_2(u_C)), \delta\tilde{e}\}\right] \\ &+ \xi^3[\lambda\{\sigma_0, L_{11}(u_3, \delta\tilde{u})\} + \cdots] + \cdots = 0. \end{aligned} \tag{34}$$

Then, since $\lim_{\xi\to 0}\lambda = \lambda_C$, cancelling out ξ^2 and letting ξ vanish gives

$$\lambda_C\{\sigma_0, L_{11}(u_2, \delta\tilde{u})\} + \{s_2, \delta\tilde{e}\} = -\{s_C, L_{11}(u_C, \delta\tilde{u})\} - \frac{1}{2}\{H_1(L_2(u_C)), \delta\tilde{e}\} \tag{35}$$

the solution of which for u_2 (orthogonal to u_C) can then be used to evaluate the coefficient of ξ^3 in (33). To obtain more terms in (33) would require solving equations analogous to (35) for u_3, u_4, and so on.

Behavior of the imperfect structure. Now suppose that the structure has a small initial, stress-free, displacement $\bar{u} = \bar{\xi} u_C$ in the shape of the classical buckling mode, and then undergoes an *additional* displacement u when the external loading is applied. The strain-displacement relation (18) must be changed to

$$\varepsilon = L_1(u + \bar{u}) + \frac{1}{2} L_2(u + \bar{u}) - L_1(\bar{u}) - \frac{1}{2} L_2(\bar{u})$$

$$= L_1(u) + \frac{1}{2} L_2(u) + \bar{\xi} L_{11}(u, u_C). \tag{36}$$

The variational equation of equilibrium (22) still holds, but now

$$\delta\varepsilon = \delta e + L_{11}(u, \delta u) + \bar{\xi} L_{11}(u_C, \delta u). \tag{37}$$

We can still represent u in the form (29), and then regard λ as a function of ξ and $\bar{\xi}$: an appropriate representation for w is now

$$\begin{aligned} w = {} & \xi^2 u_2 + \xi^3 u_3 + \cdots \\ & + \bar{\xi}[\xi u_{11} + \xi^2 u_{21} + \cdots] \\ & + \bar{\xi}^2 [\xi u_{12} + \xi^2 u_{22} + \cdots] \\ & + \cdots . \end{aligned} \tag{38}$$

The expression (31) for σ is augmented by terms involving products of $\bar{\xi}$ and ξ, of order $\bar{\xi}\xi$ and higher; similarly, the term

$$\bar{\xi}\{\lambda\sigma_0, L_{11}(u_C, \delta u)\}$$

together with others of order $\bar{\xi}\xi$ and higher are added to the variational Eq. (32). Then the extra terms in (33) are

$$\bar{\xi}\lambda\{\sigma_0, L_2(u_C)\} + 0(\bar{\xi}\xi), \tag{39}$$

while those added to (34) are just $0(\bar{\xi}\xi)$. Note that $\lim_{\xi \to 0} \lambda(\xi, \bar{\xi}) = 0$ for all $\bar{\xi} \neq 0$, but that $\lim_{\xi \to 0} \lim_{\bar{\xi} \to 0} \lambda(\xi, \bar{\xi}) = \lambda_C$. Hence, letting $\bar{\xi} \to 0$ in the modified version of (34), cancelling out ξ^2, and then letting $\xi \to 0$ reveals that u_2 is the same as in the case of the perfect structure.

Following KOITER, we now limit ourselves to a first approximation for the influence of $\bar{\xi}$ on λ by neglecting all terms $0(\bar{\xi}\xi)$, adding only (39) to the equilibrium equation (33). The result then implied for $\lambda(\xi, \bar{\xi})$ still displays the above mentioned non-uniform limiting behavior for vanishing $\bar{\xi}$ and ξ, and will therefore constitute a *uniformly* valid approximation for small $\bar{\xi}$ and ξ.

If, in (33), $\{s_C, L_2(u_C)\} \neq 0$, it follows [with the use of (27)] that for sufficiently small ξ,

$$\xi(1 - \lambda/\lambda_C) + \frac{3\xi^2}{2} \frac{\{s_C, L_2(u_C)\}}{\{s_C, e_C\}} = \frac{\lambda\bar{\xi}}{\lambda_C} \tag{40}$$

which is entirely analogous to Eq. (4) for the simple model with a quadratic spring. On the other hand, if $\{s_C, L_2(u_C)\} = 0$, the equation

$$\xi(1 - \lambda/\lambda_C) - \frac{\xi^3\left[2\{s_C, L_{11}(u_C, u_2)\} - \{s_2, L_2(u_C)\} - \frac{1}{2}\{H_1(L_2(u_C)), L_2(u_C)\}\right]}{\{s_C, e_C\}} = \frac{\lambda\bar{\xi}}{\lambda_C} \tag{41}$$

which is essentially Eq. (13) for the cubic spring model, is found to hold for small enough ξ. The structure represented by Eq. (40) is always imperfection-sensitive (for one sign or the other of $\bar{\xi}$); the "cubic" structure of Eq. (41) is imperfection-sensitive only if the coefficient of ξ^3 is negative.[1]

[1] Note that $\{s_C, e_C\}$ is twice the strain energy of linear elasticity theory — and hence positive; it follows from Eq. (27) that $\{-\sigma_0, L_2(u_C)\}$ is positive when $\lambda_C > 0$.

Multiple buckling modes. If corresponding to λ_C there are several simultaneous, linearly independent buckling modes $u_C^{(1)}$, $u_C^{(2)}, \ldots,$ the displacement during loading is written $u = \lambda u_0 + \Sigma\, \xi_n u_C^{(n)} + w$ where, for convenience, the modes are made orthogonal to each other, and w is orthogonal to all of them. With the initial imperfection

$$\bar{u} = \sum \bar{\xi}_n u_C^{(n)}$$

and with the retention onlyof terms up to order $\xi_m \xi_n$, simultaneous equations for the ξ_n's analogous to Eq. (40) are readily found to be

$$\xi_n (1 - \lambda/\lambda_C)\, \{s_C^{(n)}, e_C^{(n)}\} + \{\sum \xi_m s_C^{(m)}, L_{11}(\sum \xi_m u_C^{(m)}, u_C^{(n)})\} + \frac{1}{2} \{s_C^{(n)}, L_2(\sum \xi_m u_C^{(m)})\} = \frac{\lambda}{\lambda_C} \bar{\xi}_n \{s_C^{(n)}, e_C^{(n)}\}. \tag{42}$$

This quadratic approximation, independent of w, will be adequate for the example of the circular cylinder under axial compression, but it can, of course, be improved; indeed, if all the quadratic terms should vanish, a better approximation becomes essential.

The outline of some of KOITER'S results has now been completed. It may be mentioned that KOITER'S derivations lean on the principle of stationary potential energy; we have preferred to write variational equations directly by way of the principal of virtual work. We turn next to the introduction of inertial loads.

Dynamic theory

Inertial loads. In order to incorporate dynamic effects into the general static field equations heretofore considered it suffices to include inertia loading q_I in the external virtual work E_{11} of Eq. (19). This loading can be written as the linear functional of acceleration

$$q_I = -\, M_1\left(\frac{\partial^2 u}{\partial t^2}\right) \tag{43}$$

and then the variational equation of equilibrium (22) becomes

$$\{\sigma, \delta\varepsilon\} + E_{11}\left[M_1\left(\frac{\partial^2 u}{\partial t^2}\right), \delta u\right] - \lambda B_1(\delta u) = 0, \tag{44}$$

where σ, u, and λ are now time-dependent. The operators E_{11} and M_1 are assumed to obey the reciprocal relation

$$E_{11}[M_1(a), b] = E_{11}[M_1(b), a]. \tag{45}$$

In the present analysis we will set

$$M_1(u_0) = 0 \tag{46}$$

or, in words, the inertia loads associated with the "trivial" displacements will be neglected. (We are therefore explicitly ignoring the kind of dynamic effects studied by GOODIER and MCIVOR [*10*] wherein breathing oscillations of a laterally compressed ring transfer their energy to bending motion.) Turning, now, immediately to consideration of the *imperfect* structure, with initial displacement $\bar{\xi} u_C$ as before, we can still assert that the additional displacement is given by (29) in the case of a unique classical buckling mode u_C. We consider next the "quadratic" and "cubic" structures separately.

Quadratic structure. We note that the static analysis of the imperfect quadratic structure was tantamount to letting w vanish in the equilibrium Eq. (30), dropping the term in ξ^3, adding the imperfection term (39), and then letting $\delta u = u_C\, \delta\xi$; doing this again, but including the extra term

$$\left(\frac{d^2\xi}{dt^2}\right) E_{11}\{M_1(u_C), \delta u\} \tag{47}$$

in the equilibrium equation[1] gives

$$\frac{d^2\xi}{dt^2} E_{11}\{M_1(u_C), u_C\} + \xi(1 - \lambda/\lambda_C)\, \{s_C, e_C\} + \frac{3\xi^2}{2}\, \{s_C, L_2(u_C)\} = \frac{\lambda}{\lambda_C} \bar{\xi}\, \{s_C, e_C\} \tag{48}$$

[1] Note that the term $\frac{d^2\lambda}{dt^2} E_{11}[M_1(u_0), \delta u]$ is dropped on the basis of the assumption (46).

which, by appropriate changes of variable, is exactly reducible to Eq. (7) for the simple "quadratic" model. Consequently, the implications of the study of the simple model for suddenly applied loads, in particular Eq. (11) relating λ_D, λ_S, and λ_C, appear applicable to structures for which $\{s_C, L_2(u_C)\} \neq 0$. This is subject, of course, to the qualification implied by the assumption (46) that the characteristic time associated with the establishment of the displacement λu_0 is very small compared with that required for the growth of the additional contribution ξu_C to the total displacement u.

Cubic structure. The result (41) for the cubic structure would follow from the variational Eq. (30), augmented by the imperfection term (39), by execution of the steps (a) let $w = \gamma u_2$, where u_2 is the solution of (34), dropping all terms of order higher than γ and $\gamma\xi$; (b) take $\delta u = u_2\,\delta\gamma$, letting $\lambda = \lambda_C$ in the resulting equation, and dropping terms of order higher than γ and ξ^2; (c) finally, take $\delta u = u_C\,\delta\xi$. Now we add

$$E_{11}\left[\frac{d^2\xi}{dt^2} M_1(u_C) + \frac{d^2\gamma}{dt^2} M_1(u_2), \delta u\right]$$

to (30) and repeat these steps, getting

$$\left(\frac{d^2\gamma}{dt^2}\right)E_{11}[M_1(u_2), u_2] + \left(\frac{d^2\xi}{dt^2}\right)E_{11}[M_1(u_C), u_2] + (\gamma - \xi^2)\,[\lambda_C\{\sigma_0, L_2(u_2)\} + \{s_2, e_2\}] = 0, \tag{49}$$

$$\left(\frac{d^2\xi}{dt^2}\right)E_{11}[M_1(u_C), u_C] + \left(\frac{d^2\gamma}{dt^2}\right)E_{11}[M_1(u_2), u_C] + \xi\left(1 - \frac{\lambda}{\lambda_C}\right)\{s_C, e_C\} + \frac{\xi^3}{2}\{H_1(L_2(u_C)), L_2(u_C)\} + \gamma\xi\,[2\{s_C, L_{11}(u_2, u_C)\} + \{s_2, L_2(u_C)\}] = \frac{\bar{\xi}\lambda}{\lambda_C}\{s_C, e_C\}. \tag{50}$$

To facilitate study of these equations, we will assume *that the buckling mode u_C is also a natural vibration mode of the unloaded structure.* The variational equation for a natural vibration mode of frequency ω is

$$\{s, \delta e\} = \omega^2 E_{11}\{M_1(u), \delta u\}$$

for admissible δu. Consequently, since $\{s_C, e_2\} = 0$, it follows that $E_{11}[M_1(u_C), u_2] = E_{11}[M_1(u_2), u_C] = 0$.

Noting that the frequency ω_C of the mode u_C is given by

$$\omega_C^2 = \frac{\{s_C, e_C\}}{E_{11}[M_1(u_C), u_C]} \tag{51}$$

let $\tau = \omega_C t$ in (49) and (50), and let $z_1 = \xi/\bar{\xi}$, $z_2 = \gamma/\bar{\xi}^2$, to get

$$\ddot{z}_2 + \eta(z_2 - z_1^2) = 0, \tag{52}$$

$$\ddot{z}_1 + (1 - \lambda/\lambda_C)\,z_1 - \bar{\xi}^2 k\,[z_1 z_2 - r z_1^3] = \lambda/\lambda_C, \tag{53}$$

where

$$k = -\frac{2\{s_C, L_{11}(u_2, u_C)\} + \{s_2, L_2(u_C)\}}{\{s_C, e_C\}}$$

$$r = -\frac{\frac{1}{2}\{H_1(L_2(u_C)), L_2(u_C)\}}{2\{s_C, L_{11}(u_2, u_C)\} + \{s_2, L_2(u_C)\}}$$

$$\eta = \frac{\lambda_C\{\sigma_0, L_2(u_2)\} + \{s_2, e_2\}}{\omega_C^2 E_{11}[M_1(u_2), u_2]}$$

and $(\dot{\ }) = \frac{d}{d\tau}(\ )$.

Note that the static result (41) is recovered from (52) and (53) when the time dependent terms are dropped, and that the structure is imperfection-sensitive only if $k > 0$ and $r < 1$. The parameter η can be written

$$\eta = (1 - \lambda_C/\bar{\lambda}) \left(\frac{\bar{\omega}}{\omega_C}\right)^2 \qquad (54)$$

where $\bar{\omega}^2$ is the Rayleigh quotient for free vibrations obtained from u_2

$$\bar{\omega}^2 = \frac{\{s_2, e_2\}}{E_{11}[M_1(u_2), u_2]} \qquad (55)$$

and $\bar{\lambda}$ is the Rayleigh quotient for classical buckling

$$\bar{\lambda} = -\frac{\{s_2, e_2\}}{\{\sigma_0, L_2(u_2)\}}. \qquad (56)$$

It is certain that $\bar{\lambda} > \lambda_C$ by Rayleigh's principle[1], but there is no general rule for the ordering of ω_C and $\bar{\omega}$, because ω_C is not necessarily the *lowest* natural frequency.

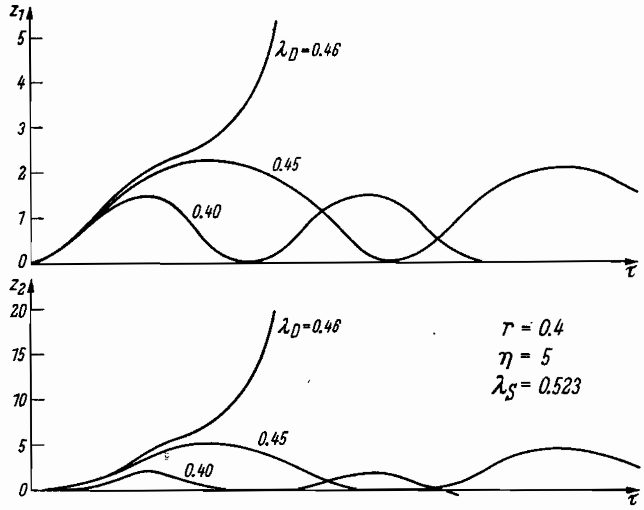

Fig. 6. Typical responses for cubic structure (two degrees of freedom)

For η very large, (52) implies $z_2 = z_1^2$, the use of which in (53) gives Eq. (14) for the simple model, with β identified as $k(1 - r)$, and then the result (17) and the dashed curve of Fig. 5 apply. In order to study the implications of (52) and (53) for finite η these equations were solved numerically for various combinations of η, r and k for the case of a suddenly applied load with initial conditions $z_1 = z_2 = \dot{z}_1 = \dot{z}_2 = 0$. Typical responses found for z_1 and z_2 are shown in Fig. 6, which shows how closely the dynamic buckling parameter (evidently associated with a

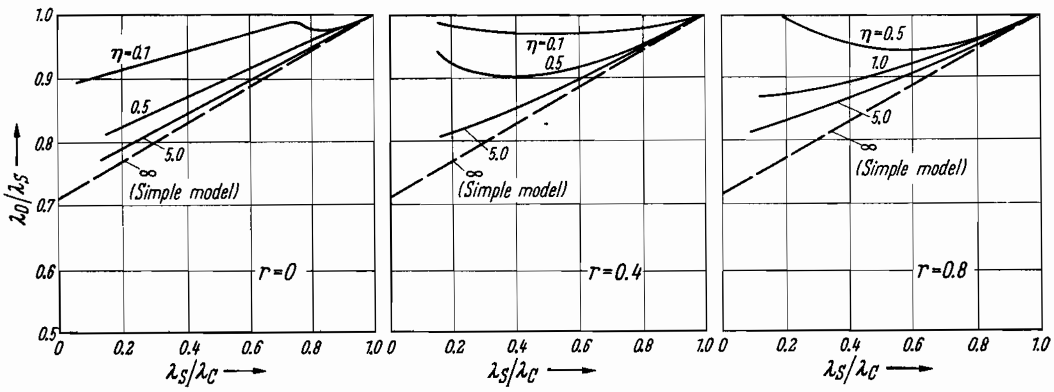

Fig. 7. Dynamic buckling loads of cubic structure

response pattern like that of Fig. 2b) can be estimated. Fig. 7 shows how λ_D/λ_S thus found varies with λ_S/λ_C [as given by Eq. (16)] for several values of η and r. The lower r, the greater is the imperfection sensitivity, and the closer do the results tend to approximate those for the simple model ($\eta = \infty$). Note that the predictions of the simple model are, in all cases, conservative.

Multiple buckling modes. With the simplification

$$E_{11}[M_1(u_C^{(m)}), u_C^{(n)}] = 0 \; (m \neq n), \qquad (57)$$

[1] In fact, since u_2 is orthogonal to u_C in the sense of Eq. (27), $\bar{\lambda}$ is larger than the *second* eigenvalue of the classical buckling problem.

which holds rigorously if *each* classical buckling mode $u_C^{(n)}$ is also a vibration mode, the coupled dynamical equations that correspond to the static Eqs. (42) are

$$\left(\frac{1}{\omega^{(n)}}\right)^2 \frac{d^2\xi_n}{dt^2} + (1 - \lambda/\lambda_C)\,\xi_n + \frac{\left[\left\{\Sigma\,\xi_m s_C^{(m)},\, L_{11}(\Sigma\,\xi_m u_C^{(m)},\, u_C^{(n)})\right\} + \frac{1}{2}\left\{s_C^{(n)}\, L_2(\Sigma\,\xi_m u_C^{(m)})\right\}\right]}{\{s_C^{(n)},\, e_C^{(n)}\}} = (\lambda/\lambda_C)\,\bar{\xi} \quad (58)$$

where

$$\omega^{(n)} = \frac{E_{11}[M_1(u_C^{(n)}),\, u_C^{(n)}]}{\{s_C^{(n)},\, e_C^{(n)}\}}\,.$$

Simultaneous solutions of these equations for various $\bar{\xi}_n$ would lead to λ_D, which could then be compared with the λ_S that is implied by the solution of the static Eqs. (42).

Examples: Cylindrical shells under axial compression

Narrow panels between longitudinal stiffeners

On the basis of his general theory KOITER has studied [*11*] the influence of initial imperfections on the static buckling under axial compression of a long, thin cylindrical cylinder (Fig. 8a) subdivided into narrow panels by stiffeners that remain straight but offer no resist-

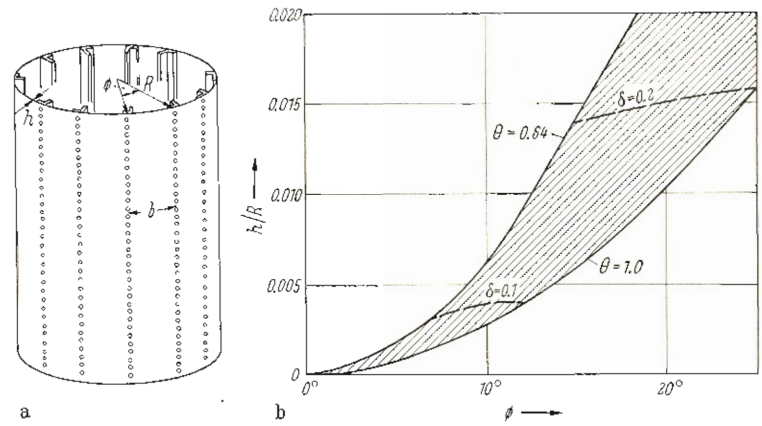

Fig. 8a and b. Stiffened cylinder. a) geometrical parameters; b) domain of imperfection sensitivity

ance to twisting. Let us identify the load parameter λ with the average *compressive* stress; the classical buckling stress is associated with a repeating pattern of square buckles between the stiffeners, and in terms of the "narrowness" parameter

$$\theta = \frac{1}{2\pi}\,[12(1 - \nu^2)]^{1/4}\,\frac{b}{(Rh)^{1/2}} \quad (59)$$

is given by

$$\lambda_C = \frac{4\pi^2 D}{b^2 h}\,(1 + \theta^4) \quad (60)$$

where $D = Eh^3/12(1 - \nu^2)$, E is Young's modulus and ν is Poisson's ratio. The parameter θ must not exceed unity for the panel to be considered narrow; for all $\theta > 1$, λ_C remains equal to the critical value for an unstiffened cylinder. KOITER shows that the narrow panels constitute a structure of the cubic type, and that there is imperfection-sensitivity only for $\theta > 0.64$. Combinations of h/R and opening angles $\phi = b/R$ between stiffeners for which the panel is both "narrow" and imperfection-sensitive are shown by the shaded region in Fig. 8b.

We wish now to check some of the simplifying assumptions made in the last section in the course of establishing the probable conservatism of the predictions for dynamic buckling obtained from the simple cubic model. We note first that, as assumed, the classical buckling mode is indeed a natural mode of vibration if inertial loads in the longitudinal and circumferential directions are neglected; in shallow shell theory this simplification is justified for sufficiently low frequencies. Next, to estimate how quickly the "trivial" stress state would be established relative to the time it would take for the buckling deformations to develop, we introduce the parameter δ defined as the ratio of the time for an axial stress wave to traverse a buckle (of length b) to the quarter-period of the natural mode just mentioned. Assuming the one-dimensional wave speed $\sqrt{E/\varrho}$, this ratio is found to be

$$\delta = \frac{[2(\theta^2 + \theta^{-2})\, h/R]^{1/2}}{[3(1-\nu^2)]^{1/4}} . \tag{61}$$

Loci of constant values of δ are given by the dotted lines in Fig. 8b; the small values that occur tend to discount the possibility of serious error incurred by the assumption $\delta = 0$ that was, effectively, made in the general dynamic analysis.

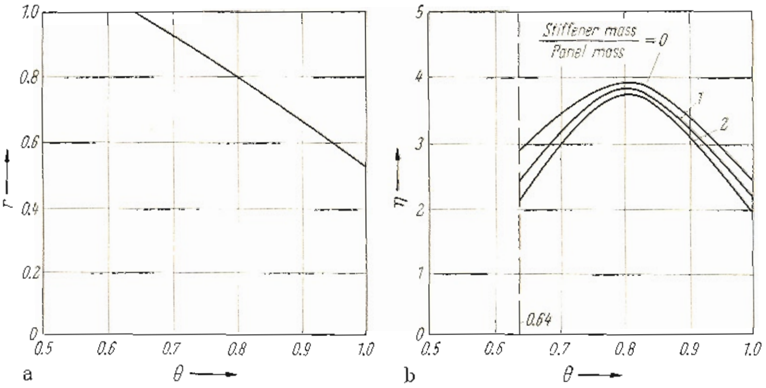

Fig. 9a and b. Parameters for the dynamic analysis of stiffened cylinders

It may be of interest to estimate, next, how conservative might be the dynamic buckling loads given by Eq. (17) for the simple model. On the basis of KOITER's detailed calculations, we find that the parameters r and η [see Eqs. (52) and (53)] vary with θ as shown in Figs. 9a and 9b; the curves for η are for various values of the ratio of stiffener mass to skin mass. Although heavier stiffeners tend to lower η and hence (see Fig. 7) tend to raise the dynamic buckling strength, the results are not very sensitive to stiffener mass. All told, the curves of Fig. 7 indicate that, for most of the ranges $0.5 < r < 1$ and $2 < \eta < 4$ that appear appropriate for imperfection-sensitive narrow panels, the predictions of the simple model ($\eta = \infty$) are not unduly conservative.

Unstiffened cylinder

The unstiffened cylinder under axial compression has a multiplicity of buckling modes associated with the classical critical compressive stress

$$\lambda_C = \frac{2E}{p_0^2} \tag{62}$$

where

$$p_0^4 = 12(1-\nu^2)\left(\frac{R}{h}\right)^2 \tag{63}$$

and the influence (notoriously great) of imperfections on the buckling strength has been studied by KOITER [6, 7] on the basis of his general theory. A self-contained analysis, aimed at the evaluation and study of the dynamical Eqs. (58), will be given here.

In a Donnell-type non-linear theory for circular cylindrical shells

$$D\nabla^4 W = hS[F, W] - \frac{h}{R}\frac{\partial^2 F}{\partial x^2} \tag{64}$$

$$\nabla^4 F = \frac{-E}{2} S[W, W] + \frac{E}{R}\frac{\partial^2 W}{\partial x^2} \tag{65}$$

where $S[a, b] \equiv \frac{\partial^2 a}{\partial x^2}\frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 a}{\partial y^2}\frac{\partial^2 b}{\partial x^2} - 2\frac{\partial^2 a}{\partial x \partial y}\frac{\partial^2 b}{\partial x \partial y}$ and where W is the radial displacement, F is the Airy stress function, x and y are axial and circumferential coordinates, respectively. With reference to the operator L_2 of the general theory [see Eq. (18)], note that the non-linear contributions to the normal and shear membrane strains ε_x, ε_y and γ_{xy} are $\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2$, $\frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2$, and $\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}$ respectively. The classical buckling equations are obtained by letting

$$F = -\frac{\lambda_C y^2}{2} + f$$

and then linearizing Eqs. (64) and (65) with respect to f and w to get

$$D\nabla^4 W + h\lambda_C \frac{\partial^2 W}{\partial x^2} = -\frac{h}{R}\frac{\partial^2 f}{\partial x^2}$$

$$\nabla^4 f = \frac{E}{R}\frac{\partial^2 W}{\partial x^2}$$

It is then easily found that the lowest eigenvalue λ_C given by Eq. (62),[1] corresponds to the modes

$$W = e^{\pm i\frac{(px \pm ny)}{R}} \tag{66}$$

$$f = -\left[\frac{Dp_0^2}{Rh}\right] e^{\pm i\frac{(px \pm ny)}{R}} \tag{67}$$

where p and n are related by

$$p^2 - p_0 p + n^2 = 0. \tag{68}$$

For DONNELL's theory to be applicable, the wave number n should be large, but the special case $n = 0$, $p = p_0$ is also acceptable. We will limit ourselves to consideration only of the simultaneous occurrence of this axisymmetric mode and the one with square buckles corresponding to $n = p_0/2$, $p = p_0/2$; thus we can use ξ_1 and ξ_2 in Eqs. (58) as the coefficients of the modes

$$W_C^{(1)} = h\cos\frac{p_0 x}{R} \qquad \text{and} \qquad W_C^{(2)} = h\sin\frac{p_0 x}{2R}\cos\frac{p_0 y}{2R}$$

$$f_C^{(1)} = -\frac{Dp_0^2}{R}\cos\frac{p_0 x}{R} \qquad\qquad f_C^{(2)} = -\frac{Dp_0^2}{R}\sin\frac{p_0 x}{2R}\cos\frac{p_0 y}{2R}$$

respectively, and then Eqs. (58) may be written

$$\begin{aligned}\frac{1}{[\omega^{(1)}]^2}\frac{d^2\xi_1}{dt^2} + (1 - \lambda/\lambda_C)\,\xi_1 + \frac{1}{\{s_C^{(1)}, e_C^{(1)}\}}\Big[\frac{3}{2}\xi_1^2\{s_C^{(1)}, L_2(u_C^{(1)})\} \\ + \xi_1\xi_2\left(\{2s_C^{(1)}, L_{11}(u_C^{(1)}, u_C^{(2)})\} + \{s_C^{(2)}, L_2(u_C^{(1)})\}\right) \\ + \xi_2^2\left(\{s_C^{(2)}, L_{11}(u_C^{(1)}, u_C^{(2)})\} + \frac{1}{2}\{s_C^{(1)}, L_2(u_C^{(2)})\}\right)\Big] = (\lambda/\lambda_C)\bar{\xi}_1\end{aligned} \tag{69}$$

$$\begin{aligned}\frac{1}{[\omega^{(2)}]^2}\frac{d^2\xi_2}{dt^2} + (1 - \lambda/\lambda_C)\,\xi_2 + \frac{1}{\{s_C^{(2)}, e_C^{(2)}\}}\Big[\frac{3}{2}\xi_2^2\{s_C^{(2)}, L_2(u_C^{(2)})\} \\ + \xi_1\xi_2\left(\{2s_C^{(2)}, L_{11}(u_C^{(1)}, u_C^{(2)})\} + \{s_C^{(1)}, L_2(u_C^{(2)})\}\right) \\ + \xi_1^2\left(\{s_C^{(1)}, L_{11}(u_C^{(1)}, u_C^{(2)})\} + \frac{1}{2}\{s_C^{(2)}, L_2(u_C^{(1)})\}\right)\Big] = (\lambda/\lambda_C)\bar{\xi}_2.\end{aligned} \tag{70}$$

Now, by Eq. (27)

$$\{s_C^{(1)}, e_C^{(1)}\} = -\lambda_C\{\sigma_0, L_2(u_C^{(1)})\}.$$

where σ_0 is determined by $F = -\frac{y^2}{2}$; so

$$\{s_C^{(1)}, e_C^{(1)}\} = h\lambda_C\int_0^{2\pi R} dy\int_0^l dx\left[\frac{\partial W_C^{(1)}}{\partial x}\right]^2 = \frac{2\pi E l h^3}{R}$$

[1] This assumes that the cylinder is long and that there are no degrading effects associated with the ends.

where $l \gg R$ is the cylinder length. Similarly

$$\{s_C^{(2)}, e_C^{(2)}\} = \frac{\pi E l h^3}{4R}.$$

Further

$$\{s_C^{(2)}, L_{11}(u_C^{(1)}, u_C^{(2)})\} = h \int_0^{2\pi R} dy \int_0^l dx \left[\frac{\partial^2 f_C^{(2)}}{\partial y^2} \frac{\partial W_C^{(1)}}{\partial x} \frac{\partial W_C^{(2)}}{\partial x} - \frac{\partial^2 f_C^{(2)}}{\partial x \partial y} \frac{\partial W_C^{(1)}}{\partial x} \frac{\partial W_C^{(2)}}{\partial y}\right] = -\frac{\pi}{8} \sqrt{3(1-\nu^2)} \frac{E l h^3}{R}$$

and

$$\{s_C^{(1)}, L_2(u_C^{(2)})\} = h \int_0^{2\pi R} \int_0^l dx \left[\frac{\partial^2 f_C^{(1)}}{\partial x^2} \left(\frac{\partial W_C^{(2)}}{\partial y}\right)^2\right] = -\frac{\pi}{8} \sqrt{3(1-\nu^2)} \frac{E l h^3}{R}$$

but

$$\{s_C^{(1)}, L_2(u_C^{(1)})\} = \{s_C^{(2)}, L_2(u_C^{(1)})\} = \{s_C^{(2)}, L_2(u_C^{(2)})\} = \{s_C^{(1)}, L_{11}(u_C^{(1)}, u_C^{(2)})\} = 0.$$

Finally, with circumferential and axial inertias neglected, the use of Eq. (51) gives the vibration frequencies

$$\omega^{(1)} = \frac{\sqrt{2}}{R} \sqrt{\frac{E}{\varrho}}$$

and

$$\omega^{(2)} = \frac{\sqrt{2}}{2R} \sqrt{\frac{E}{\varrho}}$$

associated with the two buckling modes. And so, with the variable change $\tau = \omega^{(2)} t$, Eqs. (69) and (70) become

$$(\ddot{\xi}_1/4) + (1 - \lambda/\lambda_C)\,\xi_1 - \left(\frac{3c}{32}\right) \xi_2^2 = (\lambda/\lambda_C)\,\bar{\xi}_1 \tag{71}$$

$$\ddot{\xi}_2 + [1 - \lambda/\lambda_C - \left(\frac{3c}{2}\right) \xi_1]\,\xi_2 = (\lambda/\lambda_C)\,\bar{\xi}_2 \tag{72}$$

where $c = \sqrt{3(1-\nu^2)}$.

If $\bar{\xi}_2 = 0$, $\bar{\xi}_1 \neq 0$ (i.e. only axisymmetric imperfections assumed) *static* buckling of the cylinder occurs by a bifurcation process in which ξ_2 remains zero while ξ_1 varies with λ as

$$\xi_1 = \frac{\lambda/\lambda_C \bar{\xi}_1}{1 - \lambda/\lambda_C}$$

until the coefficient of ξ_2 in Eq. (72) — which is homogeneous in ξ_2 — vanishes; this gives for the static compressive buckling stress the equation

$$(1 - \lambda/\lambda_C)^2 = \left[\frac{3c}{2} \bar{\xi}_1\right] (\lambda_S/\lambda_C), \tag{73}$$

which was found by KOITER in [7][1]. On the other hand, if $\bar{\xi}_1 = 0$, $\bar{\xi}_2 \neq 0$, simultaneous solution of (71) and (72) gives, statically,

$$(1 - \lambda/\lambda_C)^2\,\xi_2 - \frac{9}{64} c^2 \xi_2^3 = (\lambda/\lambda_C)(1 - \lambda/\lambda_C)\,\bar{\xi}_2,$$

which implies that $\lambda_{max} \equiv \lambda_S$ must satisfy[2]

$$(1 - \lambda_S/\lambda_C)^2 = \left[\frac{9\sqrt{3}\,c}{16} |\bar{\xi}_2|\right] (\lambda_S/\lambda_C). \tag{74}$$

In the presence of both $\bar{\xi}_1$ and $\bar{\xi}_2$, maximization of λ leads to

$$\frac{\left[(1 - \lambda_S/\lambda_C)^2 - \left(\frac{3c\bar{\xi}_1}{2}\right)(\lambda_S/\lambda_C)\right]^{\frac{3}{2}}}{1 - \lambda_S/\lambda_C} = \left(\frac{9\sqrt{3}\,c\,|\bar{\xi}_2|}{16}\right)(\lambda_S/\lambda_C). \tag{75}$$

[1] In [7], this result is shown to hold, with $\bar{\xi}_1$ replaced by $|\bar{\xi}_1|$, under much less restrictive assumptions on the choice of non-axisymmetrical buckling modes.

[2] An analogous, but different, result was given by KOITER in [12] for another type of non-axisymmetric imperfection which led to a bifurcation type of buckling instead of the attainment of a "smooth" maximum in the variation of λ with ξ as found here.

In order to see how the dynamic buckling load varies with λ_S/λ_C the dynamical Eqs. (71), (72) were solved numerically for various imperfection ratios $\bar{\xi}_1/\bar{\xi}_2$ and the results for λ_D/λ_C together with the corresponding roots (λ_S/λ_C) of Eq. (75) were used to plot λ_D/λ_S against λ_S/λ_C in Fig. 10. The curves indicate that λ_D/λ_S is lowered more by the non-axisymmetric component of the imperfection than by the axisymmetric one (although the reverse is true for λ_S/λ_C) and the lowest values of λ_D/λ_S are found for $\bar{\xi}_1 = 0$. A good approximation to this case can be found analytically by dropping $\ddot{\xi}_1/4$ in Eq. (71) to get

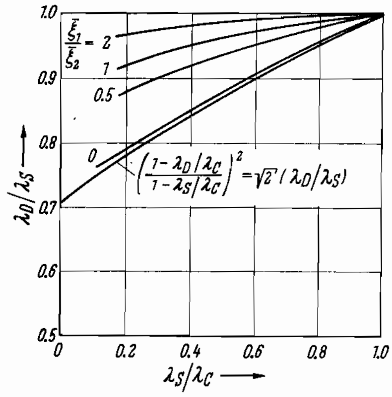

Fig. 10. Dynamic buckling stress of axially loaded cylindrical shell

$$\ddot{\xi}_2 + [1 - \lambda/\lambda_C]\,\xi_2 - \left[\frac{9c^2}{64(1-\lambda/\lambda_C)}\right]\xi_2^3 = (\lambda/\lambda_C)\,\bar{\xi}_2,$$

the solution of which provides

$$(1 - \lambda_D/\lambda_C)^2 = \left[\frac{9\sqrt{3}\,c}{8\sqrt{2}}\,|\bar{\xi}_2|\right](\lambda_D/\lambda_C). \tag{76}$$

Combining (74) and (76) gives

$$\left(\frac{1-\lambda_D/\lambda_C}{1-\lambda_S/\lambda_C}\right)^2 = \sqrt{2}\,(\lambda_D/\lambda_S), \tag{77}$$

from which the lowest curve in Fig. 10 is obtained.

It may be mentioned, finally, that the ratio of the time it takes an axial compressive stress wave to travel a half-wave-length of a buckle to its stress-free quarter-vibration-period has the small value

$$\delta = \frac{2}{[3(1-\nu^2)]^{1/4}}\sqrt{\frac{h}{R}}$$

for *all* of the buckling modes.

Concluding remarks

The three simple Eqs. (11), (17) and (77) that have been uncovered in this paper are all closely approximated, conservatively, by the formula

$$\lambda_D/\lambda_S = \frac{7 + 3(\lambda_S/\lambda_C)}{10} \tag{78}$$

for the ratio of dynamic to static buckling strength in the case of a suddenly applied load that is then maintained at a constant value. In the absence of a more detailed analysis, this formula — or the still more comfortable one $\lambda_D = 0.7\,\lambda_S$ — is suggested as a basis for design.

It is very tempting, next, to study the implication of the simple models for other types of dynamic loading histories. Thus, for example, the simple quadratic model will buckle under an impulsive loading $\lambda = I\,\delta(t)$ at the value

$$I = \frac{4}{\sqrt{3}}\left(\frac{\lambda_s}{\omega}\right)\left(\frac{1}{1-\lambda_s/\lambda_c}\right)^2$$

where ω is the stress-free natural frequency. Can formulas like this be applied reliably to structures in general? *Which* natural frequency ω should be used? The situation appears more uncertain than in the case of step-function loading, but seems well worth exploration.

References

[1] BOLOTIN, V. V.: Statistical Methods in Structural Mechanics (in Russian), Moscow 1961.

[2] AGAMIROV, V. L., and A. S. VOL'MIR: Behavior of Cylindrical Shells Under Dynamic Loading by Hydrostatic Pressure or by Axial Compression. J. A.R.S. 31 (1961) No. 1, p. 98 (English translation from Bull. Acad. Sci. USSR, Div. Tech. Sci., Mechanics and Machine Construction, 1959, No. 3, pp. 78—83).

[3] KADASHEVICH, V. I., and A. K. PERTSEV: Loss of Stability of a Cylindrical Shell under Dynamic Loads. J. A.R.S. 32 (1962) No. 1, p. 140 (English translation from Bull. Acad. Sci. USSR, Div. Tech. Sci., Mechanics and Machine Construction, 1960, No. 3, pp. 30—33).

[4] BUDIANSKY, B., and R. S. ROTH: Axisymmetric Dynamic Buckling of Clamped Shallow Spherical Shells. In Collected Papers on

Instability of Shell Structures, NASA TN D-1510, 1962.
[5] ROTH, R. S., and J. M. KLOSNER: Nonlinear Response of Cylindrical Shells Subject to Dynamic Axial Loads. AIAA, 2, October 1964.
[6] KOITER, W. T.: On the Stability of Elastic Equilibrium (in Dutch), Thesis, Delft, Amsterdam: H. J. Paris 1945.
[7] KOITER, W. T.: Elastic Stability and Post Buckling Behavior. In Non Linear Problems, edited by R. E. Langer. University of Wisconsin Press 1963.
[8] HUMPHREYS, J., and S. BODNER: Dynamic Buckling of Shallow Shells under Impulsive Loads. J. Eng. Mech. Div. Proc. ASCE, Vol. 88, No. EM 2 (April 1962).
[9] VON KARMAN, TH., L. G. DUNN and H. S. TSIEN: The Influence of Curvature on the Buckling Characteristic of Structures. J. Aero. Sci. 7 (1940) No. 7.
[10] GOODIER, J. N., and I. K. MCIVOR: The Elastic Cylindrical Shells under Nearly Uniform Radial Impulse. J. Appl. Mech., Vol. 31, No. 2 (June 1964).
[11] KOITER, W. T.: Buckling and Postbuckling Behavior of a Cylindrical Panel under Axial Compression, Report S. 476, National Aero. Research Inst., Amsterdam 1956.
[12] KOITER, W. T.: The Effect of Axisymmetric Imperfections on the Buckling of Cylindrical Shells under Axial Compression, Koninkl. Nederl. Akademie van Wetenschappen, Amsterdam, Ser. B, 66 (1963) No. 5.