Buckling of Imperfect Cylindrical Shells under Axial Compression and External Pressure

JOHN HUTCHINSON*
Harvard University, Cambridge, Mass.

Introduction

THIS note presents the results of a brief investigation of the effect of initial imperfections on the buckling of cylindrical shells under combinations of axial compression and external pressure. Although consideration is restricted to shells with axisymmetric imperfections, the important features of previously reported experimental findings are reproduced.

Analysis of Cylindrical Shell with Axisymmetric Imperfections

In this note we consider a cylindrical shell with an initial imperfection in the form of the axisymmetric buckling mode of the perfect shell under axial compression. That is

$$w_0 = \bar{\xi} h \cos q_0(x/R) \quad (1)$$

where w_0 is the initial radial displacement, $\bar{\xi}$ is the magnitude of the imperfection relative to the shell thickness h, x is the axial distance, R is the shell radius, and

$$q_0^4 = 12(1 - \nu^2)(R/h)^2$$

The length of the shell L is assumed to be sufficiently long to insure that the end conditions (assumed sufficiently strong) do not have an appreciable effect on the buckling load of the cylinder when loaded under axial compression alone. Theoretical analyses[1] seem to indicate that this requirement is met for thin shells of length greater than or about equal to the radius.

Koiter[2] has obtained an upper bound to the buckling load of the unpressurized shell described previously for the case of pure axial loading. Imperfections of the order of magnitude of the shell thickness (i.e., $\bar{\xi}$ order unity) reduce the buckling load to a small fraction of the axial buckling load of the unpressurized perfect shell

$$P_0 = 2\pi[3(1 - \nu^2)]^{-1/2} E h^2$$

Koiter's analysis has been extended to obtain an upper bound to the axial buckling load of pressurized cylinders.[3] The upper bound expression in Ref. 3 is also valid for buckling under combined axial compression and external pressure. Here, these results will be exploited without repeating the details of the analysis other than to give a very brief description of the method employed.

Received June 23, 1965; revision received August 2, 1965. This work was supported in part by NASA under Grant NsG-559 and by the Division of Engineering and Applied Physics, Harvard University.

* Assistant Professor of Structural Mechanics.

Reprinted from AIAA JOURNAL

The large-deflection Donnell equations for an imperfect cylindrical shell yield a very simple prebuckling solution for the loading combinations considered here and for an initial imperfection of the form given by Eq. (1). Bifurcation from the prebuckling solution occurs at a certain value of the axial load (for a given external pressure) in the form of a non-axisymmetric buckling mode. The load-deflection curve falls subsequent to bifurcation, and, thus, the bifurcation value is the buckling load.

The eigenvalue equations for the bifurcation load are solved approximately in a manner which insures that the approximate eigenvalue expression yields an upper bound to the exact bifurcation load (again, for fixed values of external pressure and initial imperfection). To effect this approximate solution, it is necessary to assume a form for the nonaxisymmetric buckling deflection. The form assumed was

$$w = \sin\tfrac{1}{2}q_0(x/R)\,\sin(\gamma/2)q_0(y/R) \qquad (2)$$

where y is the circumferential distance, and γ is a free parameter to be chosen to minimize the upper bound value.

The eigenvalue equation, obtained in Ref. 3, is

$$\left(1 - \frac{P}{P_0}\right)^2\left[\frac{(1+\gamma^2)^2}{4} + \frac{4}{(1+\gamma^2)^2} - 2\bar{p}\gamma^2 - \frac{2P}{P_0}\right] - c\gamma^2\bar{\xi}\left(1 - \frac{P}{P_0}\right)\left[\frac{P}{P_0} + \frac{8}{(1-\gamma^2)^2}\right] + 4(c\gamma^2\bar{\xi})^2\left[\frac{1}{(1+\gamma^2)^2} + \frac{1}{(9+\gamma^2)^2}\right] = 0 \qquad (3)$$

where $c^2 = 3(1 - \nu^2)$, and the pressure parameter is

$$\bar{p} = cpR^2/Eh^2$$

Here P is the total axial load (the load resulting from external pressure on the capped ends plus the additional applied axial load).†

To find the upper bound estimate of the axial buckling load P/P_0 for a given value of $\bar{p}$ and initial imperfection $\bar{\xi}$, it is necessary to solve Eq. (3) for P/P_0 in terms of γ and then find the value of γ such that P/P_0 is minimized. An equivalent, but considerably easier, procedure is to solve Eq. (3) for $\bar{p}$ in terms of P/P_0, $\bar{\xi}$, and γ and then to minimize $\bar{p}$ with respect to γ. This procedure leads to the upper bound curves plotted as solid lines in Fig. 1. These curves correspond to three values of $\bar{\xi}$ chosen such that shell buckles at $P/P_0 = 0.3$, 0.5, and 0.7 when subject to axial compression with no external pressure. The upper bound character can be construed in either of the following two ways: 1) for a given value of external pressure $\bar{p}$ the associated value of axial load is an upper bound to the actual axial buckling load or 2) for a given value of total axial load P/P_0 the associated value of $\bar{p}$ is an upper bound to the actual buckling pressure.

Included in Fig. 1 are buckling load curves (dashed) for combined axial compression and external pressure for an initially perfect shell. These curves are straight-line approximations to those obtained, for example, in Ref. 1 on the basis of linear buckling equations for a perfect cylinder. The several curves shown correspond to different values of the frequently defined length parameter

$$z = (1 - \nu^2)^{1/2}(R/h)(L/R)^2$$

where L is the length of the shell. Although the upper bound results are independent of shell length and end conditions, the dashed buckling curves for the perfect cylinders are dependent on both. The curves in Fig. 2 correspond to a shell, which is simply supported with respect to the radial displacement, free with respect to additional axial stresses, and clamped with respect to the circumferential displacement. Completely clamped end conditions can raise the buckling load under pure

† With $\bar{p} = 0$, Koiter used this expression to obtain the results for the unpressurized cylinder.

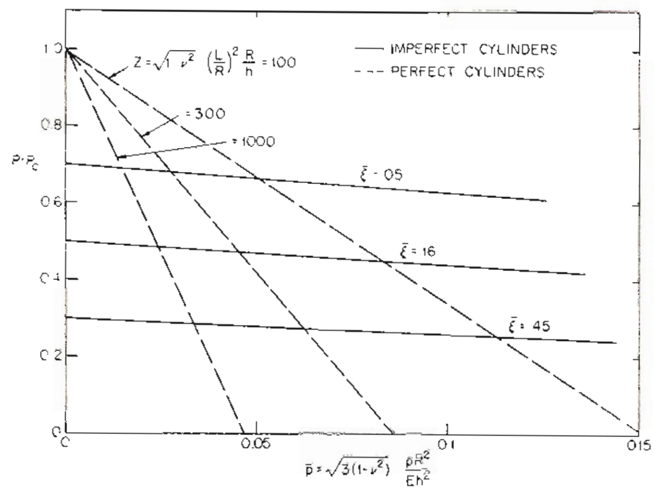

Fig. 1 Effect of external pressure on buckling load of initially imperfect cylindrical shell.

external pressure as much as 40%.[1] For other end conditions, completely clamped for example, the curves for the perfect cylinder would also be essentially straight line curves. However, they would intersect the $\bar{p}$ axis at different values than shown. The effect of end conditions on the axial buckling load can be considered negligible for the purposes of this note as long as they are sufficiently strong.

Reduction of the load carrying capacity resulting from the presence of the imperfection of the assumed axisymmetric form occurs only if the shell buckles in a mode, which is, more or less, of the form given by Eq. (2). Under pure external pressure, for example, the shell buckles in a mode with only one-half wavelength over the entire length of the cylinder. The assumed imperfection has essentially no effect on the buckling load and, consequently, the predictions of the linear theory are valid. We expect the upper bound predictions to be appropriate for combinations of axial load and external pressure such that the upper bound predictions fall below those for the linear buckling theory. On the other hand, the linear theory will be valid, both with respect to buckling load and mode form, for combinations such that results of the linear theory fall below the upper bound predictions. Thus, the curve of critical load combinations (sometimes referred to as the interaction curve) consists of the following two branches: 1) the upper bound branch on which the shell buckles in a mode of the form of Eq. (2) and 2) the linear buckling theory branch.

For a given value of external pressure the axial buckling load will be either the upper bound value (actually less than

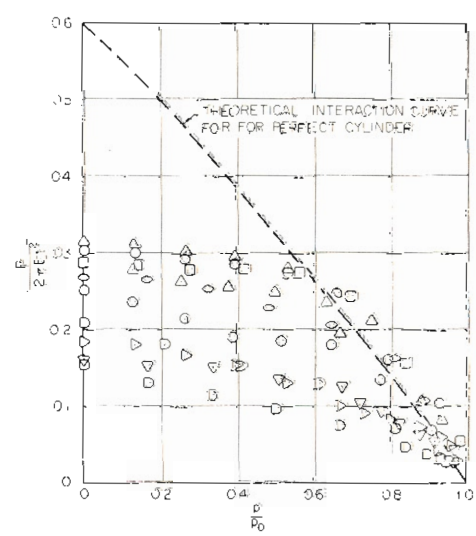

Fig. 2 Experiment data, Weingarten and Seide.

or equal to this value) or the value predicted for the perfect shell, whichever is less.

Discussion of Results and Comparison with Experiment

Recently published experimental data by Weingarten and Seide[4] are reproduced from their paper in Fig. 2. A typical specimen of the nine plotted was buckled repeatedly for loads ranging from axial compression with no external pressure to pure external pressure. In effect, the buckling load data for each specimen represents a complete interaction curve for a cylindrical shell with a given initial imperfection. It is therefore meaningful to make a direct comparison with the interaction curves of Fig. 1. Of course, it should be remembered that the theoretical interaction curves are strictly applicable only to shells with the assumed axisymmetric imperfection.

A typical experimental point in Fig. 2 represents a critical combination of total axial load P and external pressure p. The total axial load has been normalized with respect to $2\pi Eh^2$, and, thus, the value of the ordinate for $p = 0$ reflects the extent to which the axial buckling load falls below that for a perfect shell $P/(2\pi Eh^2) = 0.6$. In addition, the abscissa value is p/p_0 where p_0 is the experimentally determined pressure for buckling under external pressure alone. The dotted line labeled interaction curve is the curve of critical load combinations as predicted by the linear buckling theory for a perfect cylinder.

The two main features of the theoretical results are 1) the two branches of the interaction curve and 2) the lack of strong dependence of the axial buckling load on pressure on the upper bound branch. Both these features are characteristic, to a certain extent, of almost all of the specimens. About half of the experimental interaction curves can be reproduced quite accurately by the theoretical curves, if appropriate choices are made for the imperfection magnitude $\bar{\xi}$, and if the pressure coordinate is normalized with respect to p_0. Apparently, however, some specimens exhibit a greater dependence on pressure than is predicted for a shell with axisymmetric imperfections.

At least two possible explanations can be suggested to account for the discrepancy between the theory, as presented here, and experiment. Firstly, the predictions on upper bound branch are, as has been emphasized, of an upper bound nature and, most likely, are overestimates. Probably more important is the failure to account for other than axisymmetric imperfections. In this connection, it is noted that a significant distinction was found between the roles of axisymmetric and asymmetric imperfections in reducing the axial buckling load of pressurized cylinders.[3] Although asymmetric imperfections are ironed out by internal pressure, axisymmetric ones are not. An ironing in effect caused by external pressure (that is, the effect of the external pressure in pushing in the initial asymmetric imperfection) is expected to give a greater pressure dependence than is predicted for a shell with only axisymmetric imperfections.

A multiplicity of buckling modes is associated with the critical load of a perfect cylindrical shell under axial compression. Imperfections in the form of any of the buckling modes are particularly degrading. On the basis of linear shell theory, it is a simple matter to show that the smaller the number of circumferential wavelengths associated with one of these imperfections the less internal or external pressure will be necessary to iron it out or in. Thus, a cylinder whose buckling behavior is determined by an asymmetric imperfection with relatively few circumferential wavelengths should be expected to show a definite pressure dependence over the entire interaction curve. This effect, however, will be less noticeable in the case of externally pressurized cylinders than for those internally pressurized. This follows because the external pressure on a thin shell with z greater than 50, say, can never exceed $\bar{p} = 0.2$, although the ironing out effect only becomes significant at internal pressures of this order or larger.

References

[1] Almroth, B. O., "Influence of edge conditions on the stability of axially compressed cylindrical shells," NASA Contact Rept. 161 (1965).

[2] Koiter, W. T., "The effect of axisymmetric imperfections on the buckling of cylindrical shells under axial compression," Koninkl. Ned. Akad. Wetenschap, Proc., Ser. B **66**, 265–279 (1963).

[3] Hutchinson, J. W., "Axial buckling of pressurized imperfect cylindrical shells," AIAA J. **3**, 1461–1466 (1965).

[4] Weingarten, V. I. and Seide, P., "Elastic stability of thin-walled cylindrical and conical shells under combined external pressure and axial compression," AIAA J. **3**, 913–920 (1965).