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Imperfection Sensitivity of Externally Pressurized Spherical Shells[1]

The initial postbuckling behavior of a shallow section of a spherical shell subject to external pressure is studied within the context of Koiter's general theory of postbuckling behavior. Imperfections in the shell geometry are shown to have the same severe effect on the buckling strengths of spherical shells as has been demonstrated for axially compressed cylindrical shells. Large reductions in the buckling pressure result from small deviations, relative to the shell thickness, of the shell middle surface from the perfect configuration.

Introduction

In this paper, the initial postbuckling behavior of a spherical shell subject to external pressure loading is determined on the basis of Koiter's general theory of postbuckling behavior [1].[2] As might well be expected, the most important features of this problem show a striking similarity to aspects present in the behavior of cylindrical shells under axial compression. Imperfections in the shell geometry are found to have the same severe effect on spherical shells as has been demonstrated for axially compressed cylinders [1, 2].

Perhaps the main feature which distinguishes this investigation from previous work is that here consideration has not been restricted to rotationally symmetric buckling deformations. In fact, it is clearly demonstrated that the initial postbuckling behavior is decidedly not rotationally symmetric but is analogous to the cylindrical shell behavior in which a number of modes combine to give rise to the highly imperfection-sensitive character of the structure.

Thompson [3] has also employed the Koiter theory to study the initial postbuckling behavior of the complete sphere. His approach, however, is fundamentally different than that taken here because of the restriction to rotationally symmetric deformations. This work will be discussed further in the body of the present paper. Other investigators [4, 5, 6] have determined the large-deflection behavior with the aid of various methods but in each case under the aforementioned assumption of rotational symmetry. The large-deflection, rotationally symmetric equilibrium configurations appear to be in reasonable agreement with experimental observations in the same large-deflection range.

To obtain a clear understanding of the effects of imperfections on the buckling strength of this structure, it is necessary to study its initial postbuckling behavior. It is this study which forms the substance of the present paper.

Shallow-Shell Equations

Nonlinear shallow-shell equations will be employed in this analysis. The consistency of applying this representation to the complete sphere will be discussed as the analysis proceeds. In anticipation, however, we remark that the adequacy of this description follows from the fact that the characteristic buckle wavelengths are small compared to the shell radius. Thus it is possible to choose a shallow section of the shell surface in which the buckle pattern is duplicated may times.[3] For essentially the same reason, the initial postbuckling behavior of an axially compressed cylinder can also be obtained within the context of shallow-shell theory (which, for cylinders, is identical to nonlinear Donnell theory).

A shallow section S_0 of the sphere is imagined to be isolated as shown in Fig. 1. Cartesian coordinates x and y are chosen in the base plane of the shallow section, and z is normal to this plane. The stress-strain relations and the bending strain-displacement relations of shallow-shell theory are linear while the membrane strain-displacement relations are nonlinear. Listed here are these nonlinear relations which, along with the other shallow shell equations, are given, for example, by Sanders [7]. The membrane strains ϵ_x, ϵ_y, and ϵ_{xy} are given in terms of the tangential displacements U and V and the normal displacement W by

$$\begin{aligned} \epsilon_x &= U_{,x} + W/R + \frac{1}{2} W_{,x}{}^2 \\ \epsilon_y &= V_{,y} + W/R + \frac{1}{2} W_{,y}{}^2 \qquad (1) \\ \epsilon_{xy} &= \frac{1}{2}(U_{,y} + V_{,x}) + \frac{1}{2} W_{,x}W_{,y} \end{aligned}$$

where R is the radius of curvature of the spherical section.

The three equilibrium equations of nonlinear shallow shell theory can be replaced by one equilibrium equation and one compatibility equation written in terms of W and a stress function F. These two equations are

[3] Here, shallow is taken in the sense that the slopes of the surface measured from the section base are small and, thus, the shallow shell approximations are valid.

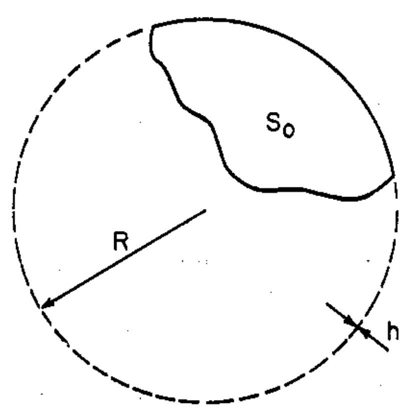

Fig. 1 Shallow section of complete sphere

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[2] Numbers in brackets designate References at end of paper.

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$$D\nabla^4W + \frac{1}{R}\nabla^2F - F_{,xx}W_{,yy} - F_{,yy}W_{,xx} + 2F_{,xy}W_{,xy} = -p \quad (2)$$

$$\frac{1}{Eh}\nabla^4F - \frac{1}{R}\nabla^2W + W_{,xx}W_{,yy} - W_{,xy}^2 = 0 \quad (3)$$

where $D = Eh^3/12(1-\nu^2)$; E and ν are Young's modulus and Poisson's ratio, respectively; h is the shell thickness, ∇^4 and ∇^2 are the two-dimensional biharmonic and Laplacian operators; and p is the external pressure. The resultant membrane stresses are given in terms of the stress function by

$$N_x = F_{,yy}, \quad N_y = F_{,xx} \quad \text{and} \quad N_{xy} = -F_{,xy}$$

Classical Buckling Analysis

Prior to buckling the perfect spherical shell is in a uniform membrane state of stress ($N_x^0 = N_y^0 = -\frac{1}{2}pR$) with an associated inward radial displacement $W_0 = -(1-\nu)pR^2/2Eh$. With

$$F = -\tfrac{1}{4}(x^2 + y^2)pR + f$$

and

$$W = -(1-\nu)pR^2/2Eh + w$$

f and w are zero prior to buckling. The critical pressure p_c, often called the classical buckling pressure, at which bifurcation from the prebuckling state of stress occurs, is predicted by the linear buckling analysis. The linear buckling equations are obtained by substituting for F and W into equations (2) and (3) and then linearizing with respect to f and w. One finds

$$D\nabla^4w + \frac{1}{R}\nabla^2f + \frac{1}{2}pR\nabla^2w = 0 \quad (4)$$

and

$$\frac{1}{Eh}\nabla^4f - \frac{1}{R}\nabla^2w = 0 \quad (5)$$

Periodic solutions to these homogeneous eigenvalue equations are sought in the form of products of sinusoidal functions such as

$$w = \cos\left(k_x\frac{x}{R}\right)\cos\left(k_y\frac{y}{R}\right)$$
$$f = B\cos\left(k_x\frac{x}{R}\right)\cos\left(k_y\frac{y}{R}\right) \quad (6)$$

The eigenvalue associated with this choice is

$$p = \frac{2Eh}{R}\left[(k_x^2 + k_y^2)^{-1} + q_0^{-4}(k_x^2 + k_y^2)\right] \quad (7)$$

with $B = -EhR(k_x^2 + k_y^2)^{-1}$, and where

$$q_0^4 = 12(1-\nu^2)\left(\frac{R}{h}\right)^2$$

The classical buckling pressure is found by minimizing p as given by equation (7) with respect to k_x and k_y. This critical pressure is

$$p_c = 4Eh/Rq_0^2 = \frac{2E}{[3(1-\nu^2)]^{1/2}}\left(\frac{h}{R}\right)^2 \quad (8)$$

and is associated with any combination of wave numbers k_x and k_y satisfying

$$k_x^2 + k_y^2 = q_0^2 \quad (9)$$

This critical pressure, obtained on the basis of shallow shell theory, is exactly that predicted by equations for a full sphere (see, for example, Flügge [8]).

The shallow shell representation of the portion S_0 of the complete sphere can only be valid if the wavelengths of the buckle pattern are small compared to the radius of the shell or—what is the same—if the wave numbers k_x and k_y are both large compared to unity. Associated with the critical buckling pressure is a multiplicity of buckling modes and, as seen from equation (9), combinations of k_x and k_y are possible such that both are of order q_0 and, therefore, sufficiently large. An exception to the requirement of large wave numbers occurs if either k_x or k_y is identically zero, that is, if the buckling deformation is independent of either x or y. In such cases, as well, the shallow-shell description is accurate for shallow sections of a complete sphere. This is analogous to the situation for axial buckling of cylinders for which the shallow-shell equations are accurate for the axisymmetric mode but not, for example, for the Euler column mode in which only one wavelength, in effect, spans the shell circumference.

The inital postbuckling behavior of the spherical shell is investigated in the remainder of this paper. It will be seen that sets of either two or three of the buckling modes associated with the classical buckling pressure couple give rise to a load-deflection behavior which falls sharply in the initial postbuckling regime. As previously indicated, the analysis will be carried out within the framework of Koiter's general theory of postbuckling behavior. This theory is outlined in the next section.

Koiter Theory for Multimode Buckling

The procedure which is sketched in this section is an application of the variational principle of potential energy to obtain equations characterizing equilibrium in the prebuckling and initial postbuckling regimes of a structure with a multiplicity of buckling modes associated with the critical buckling load. These equations are in the form of simultaneous nonlinear, algebraic equations relating the magnitude of the externally applied load to the deflections in the various buckling modes. The magnitudes of assumed geometrical imperfections also appear. The notation and development of Koiter's general theory displayed here is taken for the most part from reference [9]. The reader is referred to this reference or Koiter's [1] own work for certain arguments and points of rigor which there is no need to reestablish here.

Generalized stress, strain, and displacement fields are denoted by σ, ϵ, and u, respectively. The magnitude of the applied load system is taken to be directly proportional to the load parameter λ.

The potential energy expression for the structure is conveniently written in the compact form

$$PE = \frac{1}{2}\{\sigma, \epsilon\} - \lambda B_1(u) \quad (10)$$

where, of course, the stresses and strains are calculated from the kinematically admissible displacement field u. Here, in general, $\{\sigma', \epsilon''\}$ denotes the internal virtual work of the stress field σ' through the strain field ϵ''; and $\lambda B_1(u)$ is the work of the applied force field of intensity λ through a displacement u of the structure.

We consider only structures which can be adequately described by nonlinear strain-displacement relations of the form

$$\epsilon = L_1(u) + \frac{1}{2}L_2(u) \quad (11)$$

where L_1 and L_2 are homogeneous functionals which are linear and quadratic, respectively, in u. Furthermore, the stress-strain relations are assumed to be linear and are written symbolically as

$$\sigma = H_1(\epsilon) \quad (12)$$

where H_1 is a linear, homogeneous functional of the strain components. The set of nonlinear shallow shell equations is of this form. With this notation, for example, L_2 is zero in calculating the bending strain while L_2 is $W_{,x}{}^2$ in calculating ϵ_x.

An initial deviation $\bar{u}$ of the unloaded structure from the perfect form is called the initial imperfection. In the presence of an initial imperfection, the strain arising from an additional displacement u is

$$\epsilon = L_1(u) + \frac{1}{2} L_2(u) + L_{11}(u, \bar{u}) \tag{13}$$

where $L_{11}(u, \bar{u}) = L_{11}(\bar{u}, u)$ is the bilinear, homogeneous functional of u and $\bar{u}$ which appears in the identity

$$L_2(u + \bar{u}) = L_2(u) + 2L_{11}(u, \bar{u}) + L_2(\bar{u})$$

(As an illustration, if the initial deviation of the shell middle surface from a spherical shape is denoted by $\bar{W}$, then, by shallow-shell theory, ϵ_x is, using (1) and (13), $U_{,x} + W/R + \frac{1}{2} W_{,x}{}^2 + W_{,x}\bar{W}_{,x}$, where W is the additional radial displacement.)

It is assumed that there are several linearly independent buckling modes $u_c{}^{(1)}$, $u_c{}^{(2)}$, ..., associated with the critical value of the load parameter λ_c. The complete displacement of the structure is written quite generally as

$$u = \lambda u_0 + \sum_n \xi_n u_c{}^{(n)} + \bar{u} \tag{14}$$

where λu_0 is the prebuckling displacement of the perfect structure subject to the external load intensity corresponding to λ. For the spherical shell under uniform pressure, this is just a uniform radial displacement. Each of the modes $u_c{}^{(i)}$ is taken orthogonal to one another and each is orthogonal to $\bar{u}$. The orthogonality condition is

$$\{\sigma_0, L_{11}(u_c{}^{(i)}, u_c{}^{(j)})\} = 0 \qquad i \neq j$$

where $\sigma_0 = H_1[L_1(u_0)]$. Imperfections in the form of the buckling modes may result in significant reductions of the buckling load; thus we take

$$\bar{u} = \sum_n \bar{\xi}_n u_c{}^{(n)} \tag{15}$$

Now, the potential energy is evaluated using the expressions for u and $\bar{u}$ with equations (12)–(14) and the orthogonality conditions. The result is:

$$\begin{aligned} PE = \text{const} &+ \frac{1}{2} (\lambda - \lambda_c)\Sigma\xi_i{}^2\{\sigma_0, L_2(u_c{}^{(i)})\} \\ &+ \frac{1}{2} \{\Sigma\xi_i s_c{}^{(i)}, L_2(\Sigma\xi_i u_c{}^{(i)})\} + \Sigma\xi_i\bar{\xi}_i\lambda\{\sigma_0, L_2(u_c{}^{(i)})\} \\ &+ \text{terms of order } \xi^4, \xi\bar{\xi}^2, \ldots \end{aligned} \tag{16}$$

where $s_c{}^{(i)} = H_1[L_1(u_c{}^{(i)})]$. Only terms up to and including third powers of the ξ and imperfection terms such as $\xi\bar{\xi}$ in the potential energy are displayed. This expression for a "quadratic structure" is in the form given by Koiter [(1b), equation (28)]. It is noted that $\bar{u}$ does not appear in potential energy since it contributes to quartic but not cubic terms in the ξ. The potential energy expression in the truncated form given here can provide an accurate description of the structure only so long as the ξ_i and imperfections $\bar{\xi}_i$ are sufficiently small to insure that the terms neglected are small compared to those retained.

Equilibrium equations relating the ξ_i to the load parameter λ are obtained from the requirement that the first variations of the potential energy with respect to the ξ_i vanish. These equations are

$$\begin{aligned} \xi_i(1 - \lambda/\lambda_c) + [\{\Sigma\xi_n s_c{}^{(n)}, L_{11}(\Sigma\xi_n u_c{}^{(n)}, u_c{}^{(i)})\} \\ + \frac{1}{2} \{s_c{}^{(i)}, L_2(\Sigma\xi_n u_c{}^{(n)})\}]/(-\lambda_c\{\sigma_0, L_2(u_c{}^{(i)})\}) \\ = (\lambda/\lambda_c)\bar{\xi}_i \qquad i = 1, 2, \ldots \end{aligned} \tag{17}$$

Finally, we give the generalized load-deflection relation for a perfect, multimode structure, which can be obtained from the general theory; namely,

$$\frac{B_1(u)}{B_1(u_{0c})} = \frac{\lambda}{\lambda_c} - \frac{1}{2} \sum_n \xi_n{}^2 \frac{\lambda_c\{\sigma_0, L_2(u_c{}^{(n)})\}}{\lambda_c{}^2\{\sigma_0, L_1(u_0)\}} \tag{18}$$

where, from equation (10), $B_1(u)$ represents the generalized displacement through which the external loading system acts, and where $u_{0c} = \lambda_c u_0$.

Postbuckling of Spherical Shell: Two Operative Modes

In forming the nonlinear equations of equilibrium for the spherical shell, it is necessary to take into account all the buckling modes associated with the critical buckling pressure; that is, all modes whose wave numbers satisfy

$$k_x{}^2 + k_y{}^2 = q_0{}^2 \tag{9}$$

It will be shown that the nonlinear equations for the ξ_i decouple into separate sets of equations corresponding to interaction between either two or three of these critical modes. Two modes will interact when one of the operative modes has a zero wave number associated with either the x or y-coordinate. This case is considered first, and the three-mode situation is discussed in the next section.

Translating from the general notation to the shallow-shell notation, we take as $u_c{}^{(1)}$

$$w_c{}^{(1)} = h \cos\left(q_0 \frac{x}{R}\right) \tag{19}$$

with the associated stress function

$$f_c{}^{(1)} = -ERh^2q_0{}^{-2} \cos\left(q_0 \frac{x}{R}\right)$$

Of all the modes satisfying (9), only a mode with $k_x = q_0/2$, and thus $k_y = \sqrt{3}\, q_0/2$, will interact with $w_c{}^{(1)}$, as will be shown. For $u_c{}^{(2)}$, take

$$\begin{aligned} w_c{}^{(2)} &= h \sin\left(\frac{1}{2} q_0 \frac{x}{R}\right) \sin\left(\frac{\sqrt{3}}{2} q_0 \frac{y}{R}\right) \\ f_c{}^{(2)} &= -ERh^2q_0{}^{-2} \sin\left(\frac{1}{2} q_0 \frac{x}{R}\right) \sin\left(\frac{\sqrt{3}}{2} q_0 \frac{y}{R}\right) \end{aligned} \tag{20}$$

The coefficients of the various terms in the equilibrium equations (17) are easily calculated. Some of the details are shown as follows: First,

$$\begin{aligned} \lambda_c\{\sigma_0, L_2(u_c{}^{(1)})\} &= -\frac{1}{2} p_c R \int_{S_0} (w_{c,x}{}^{(1)})^2 dS \\ &= -\frac{2Eh^3}{R^2} \int_{S_0} \left(\sin q_0 \frac{x}{R}\right)^2 dS \\ &= -\frac{Eh^3}{R^2} S_0 \end{aligned}$$

where S_0 is the area of the shallow section. Consistent with the fact that the buckle wavelength is short compared to the characteristic length of the section S_0, only the constant part of $\sin^2 q_0 \frac{x}{R} = \frac{1}{2}\left(1\text{-}\cos 2q_0 \frac{x}{R}\right)$ is evaluated in arriving at the foregoing expression. Similarly,

$$\lambda_c\{\sigma_0, L_2(u_c{}^{(2)})\} = -\frac{Eh^3}{2R^2} S_0$$

and

$$\{s_c^{(1)}, L_2(u_c^{(2)})\} = \int_{S_0} f_{c,xx}^{(1)}(w_{c,y}^{(2)})^2 dS$$

$$= -\frac{3C}{16}\frac{Eh^3}{R^2} S_0$$

where $C = [3(1 - \nu^2)]^{1/2}$. This last coefficient, which gives rise to coupling between $w_c^{(1)}$ and $w_c^{(2)}$, would vanish if the k_x associated with $w_c^{(2)}$ were not $\frac{1}{2}q_0$. The other nonzero coupling coefficient is

$$\{s_c^{(2)}, L_{11}(u_c^{(1)}, u_c^{(2)})\} = \int_{S_0} [f_{c,yy}^{(2)}w_{c,x}^{(1)}w_{c,x}^{(2)} - f_{c,xy}^{(2)}w_{c,x}^{(1)}w_{c,y}^{(2)}]dS$$

$$= -\frac{3C}{16}\frac{Eh^3}{R^2} S_0$$

while

$$\{s_c^{(1)}, L_{11}(u_c^{(1)}, u_c^{(2)})\} = \{s_c^{(1)}, L_2(u_c^{(1)})\} = \{s_c^{(2)}, L_2(u_c^{(2)})\}$$
$$= \{s_c^{(2)}, L_2(u_c^{(1)})\} = 0$$

Lastly,

$$\lambda_c^2\{\sigma_0, L_1(u_0)\} = \frac{2}{3(1+\nu)}\frac{Eh^3}{R^2} S_0$$

With these coefficients, the two equilibrium equations for ξ_1 and ξ_2 in terms of the external pressure p become

$$\left(1 - \frac{p}{p_c}\right)\xi_1 - \frac{9C}{32}\xi_2^2 = \frac{p}{p_c}\bar{\xi}_1 \quad (21)$$

$$\left(1 - \frac{p}{p_c}\right)\xi_2 - \frac{9C}{8}\xi_1\xi_2 = \frac{p}{p_c}\bar{\xi}_2 \quad (22)$$

and it is important to note that the deflections in the modes, ξ_1 and ξ_2, as well as the imperfection magnitudes, $\bar{\xi}_1$ and $\bar{\xi}_2$, are measured relative to the shell thickness. This follows from the choice of $w_c^{(1)}$ and $w_c^{(2)}$ in equations (19) and (20).[4]

For the *perfect shell* ($\bar{\xi}_1 = \bar{\xi}_2 = 0$), equations (21) and (22) admit only the trivial solution when the prebuckling pressure is less than the critical value p_c. When p attains p_c, bifurcation from the membrane state of stress occurs, and the equilibrium equations are easily solved for ξ_1 and ξ_2:

$$\xi_1 = \frac{8}{9C}\left(1 - \frac{p}{p_c}\right)$$

$$\xi_2 = \pm\frac{16}{9C}\left(1 - \frac{p}{p_c}\right)$$

This behavior, sketched in Fig. 2, is characteristic of a "quadratic-type" structure and has been discussed by Koiter [1] in some detail for the general case. The equilibrium pressure in the postbuckling regime is greatly reduced even where the buckling deflections are only a small fraction of the shell thickness; i.e., ξ_1 and ξ_2 a small fraction of unity.

The generalized load-deflection relation valid in the initial postbuckling region is easily calculated using equation (18). The generalized displacement $B_1(u)$ in this case corresponds to the average normal displacement, w_{avg}, of the shallow section. One finds

$$\frac{w_{avg}}{w_c^0} = \frac{p}{p_c} + \frac{16}{27(1-\nu)}\left(1 - \frac{p}{p_c}\right)^2 \quad (23)$$

where

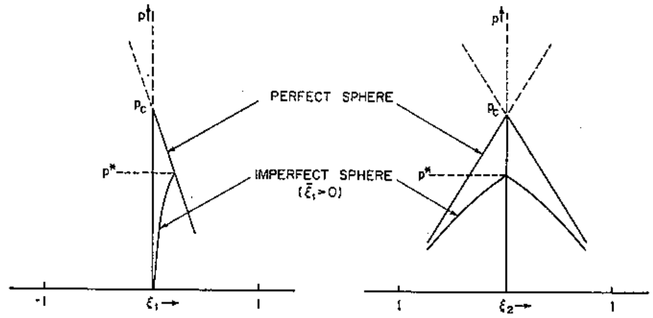

Fig. 2 Pressure-mode deflection behavior

$$w_c^0 = -\left(\frac{1-\nu}{3(1+\nu)}\right)^{1/2} h$$

is the prebuckling normal displacement at the bifurcation pressure.

An *imperfect shell* suffers deflections in the buckling modes with the first application of external pressure. The behavior for the case $\bar{\xi}_2 = 0$, $\bar{\xi}_1 > 0$ is also depicted in Fig. 2. Prior to buckling, the load increases with deflection in the ξ_1-mode with

$$\xi_1 = \frac{\bar{\xi}_1 p/p_c}{1 - p/p_c} \quad (24)$$

until the coefficient of ξ_2 in equation (22) vanishes. At this point, bifurcation occurs. Following bifurcation, the equilibrium pressure falls with deflections occurring in both modes; thus the maximum (buckling) pressure, denoted by p^*, is the bifurcation pressure which satisfies

$$\left(1 - \frac{p^*}{p_c}\right)^2 = \frac{9C}{8}\bar{\xi}_1\frac{p^*}{p_c} \quad (25)^5$$

Small imperfections (relative to the shell thickness) result in large reductions of the buckling pressure as shown in the plot of equation (25) in Fig. 3.

If $\bar{\xi}_1 = 0$ but $\bar{\xi}_2 \neq 0$, the maximum value of p is obtained by substituting for ξ_1 in terms of ξ_2 from equation (21) into equation (22) and determining then the value of p such that $dp/d\xi_2 = 0$. One finds

$$\left(1 - \frac{p^*}{p_c}\right)^2 = \frac{27\sqrt{3}C}{32}|\bar{\xi}_2|\frac{p^*}{p_c} \quad (26)$$

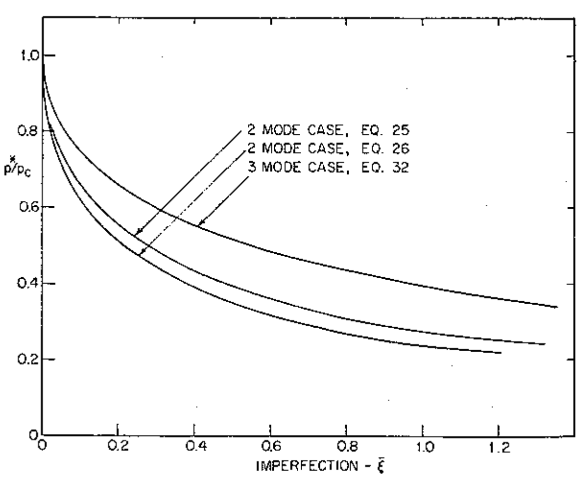

Fig. 3 Buckling pressure of imperfect spherical shells (ν = 1/3)

[4] Note that equations (21) and (22), as well as the predictions to be obtained from them, do not depend on the area of the shallow section S_0.

[5] If we had chosen $w_c^{(2)} = h\cos(q_0x/2R)\sin(\sqrt{3}q_0y/2R)$ instead of (20), equation (25) would be $(1 - p^*/p_c)^2 = -9C\bar{\xi}_1p^*/8p_c$ and thus valid for $\bar{\xi}_1 < 0$.

This formula is also plotted in Fig. 3, and it is seen that an imperfection in the form of the ξ_2-mode causes a greater reduction than an equal imperfection in the ξ_1-mode.

Three Operative Buckling Modes

Interaction between two modes occurs only if one mode has a zero wave number; otherwise, the modes will interact in sets of three. To illustrate such a situation, we take as $u_c^{(1)}$, $u_c^{(2)}$, and $u_c^{(3)}$

$$\begin{bmatrix} w_c^{(1)} \\ f_c^{(1)} \end{bmatrix} = \begin{bmatrix} h \\ -ERh^2q_0^{-2} \end{bmatrix} \cos\left(\alpha_1 q_0 \frac{x}{R}\right) \cos\left(\beta_1 q_0 \frac{y}{R}\right)$$

$$\begin{bmatrix} w_c^{(2)} \\ f_c^{(2)} \end{bmatrix} = \begin{bmatrix} h \\ -ERh^2q_0^{-2} \end{bmatrix} \sin\left(\alpha_2 q_0 \frac{x}{R}\right) \sin\left(\beta_2 q_0 \frac{y}{R}\right) \qquad (27)$$

and

$$\begin{bmatrix} w_c^{(3)} \\ f_c^{(3)} \end{bmatrix} = \begin{bmatrix} h \\ -ERh^2q_0^{-2} \end{bmatrix} \sin\left(\alpha_3 q_0 \frac{x}{R}\right) \sin\left(\beta_3 q_0 \frac{y}{R}\right)$$

where, by equation (9), $\alpha_i^2 + \beta_i^2 = 1$ $(i = 1, 2, 3)$.

The coefficients in the algebraic equilibrium equations are evaluated in much the same manner as in the last section. Coupling between the three modes will occur only if coefficients such as $\{s_c^{(1)}, L_{11}(u_c^{(2)}, u_c^{(3)})\}$ do not vanish. Evaluating this term, we find

$$\{s_c^{(1)}, L_{11}(u_c^{(2)}, u_c^{(3)})\} = \int_{S_0} [f_{c,yy}^{(1)} w_{c,x}^{(2)} w_{c,x}^{(3)}$$

$$+ f_{c,xx}^{(1)} w_{c,y}^{(2)} w_{c,y}^{(3)} - f_{c,xy}^{(1)}(w_{c,x}^{(2)} w_{c,y}^{(3)} + w_{c,y}^{(2)} w_{c,x}^{(3)})]dS$$

$$= -\frac{C}{8}\frac{Eh^3}{R^2} S_0(-\beta_1^2\alpha_2\alpha_3 + \alpha_1^2\beta_2\beta_3 - \alpha_1\beta_1\alpha_3\beta_2 + \alpha_1\beta_1\alpha_2\beta_3)$$

if $\alpha_2 + \alpha_3 = \alpha_1$ and $\beta_1 + \beta_2 = \beta_3$ but is zero otherwise.[6] These two equations, along with the conditions $\alpha_i^2 + \beta_i^2 = 1$, uniquely determine the magnitudes of any five of the α and β in terms of the remaining one. From this follows the statement made earlier that the equilibrium equations decouple into sets made up of three interacting modes unless, of course, one mode wave number is zero. Then the three-mode case degenerates to the case discussed in the last section.

The remaining nonlinear, coupling coefficients are nonvanishing only under the same conditions and are easily found to be

$$\{s_c^{(2)}, L_{11}(u_c^{(1)}, u_c^{(3)})\} = -\frac{C}{8}\frac{Eh^3}{R^2} S_0(\beta_2^2\alpha_1\alpha_3 + \alpha_2^2\beta_1\beta_3 + \alpha_2\beta_2\alpha_1\beta_3 + \alpha_2\beta_2\beta_1\alpha_3)$$

and

$$\{s_c^{(3)}, L_{11}(u_c^{(1)}, u_c^{(2)})\} = -\frac{C}{8}\frac{Eh^3}{R^2} S_0(\beta_3^2\alpha_1\alpha_2 - \alpha_3^2\beta_1\beta_2 + \alpha_3\beta_3\alpha_1\beta_2 - \alpha_3\beta_3\beta_1\alpha_2)$$

with all others zero. Finally,

$$\lambda_c\{\sigma_0, L_2(u_c^{(i)})\} = -\frac{1}{2}\frac{Eh^3}{R^2} S_0 \qquad i = 1, 2, 3$$

The coefficient of the nonlinear term in each of the three equilibrium equations is

$$\{s_c^{(1)}, L_{11}(u_c^{(2)}, u_c^{(3)})\} + \{s_c^{(2)}, L_{11}(u_c^{(1)}, u_c^{(3)})\} + \{s_c^{(3)}, L_{11}(u_c^{(1)}, u_c^{(2)})\}$$

[6] Actually, there are other combinations for which this coefficient will not vanish; for example, $\alpha_2 + \alpha_3 = -\alpha_1$ and $\beta_1 + \beta_2 = -\beta_3$, but these are, in effect, included within the combination listed earlier.

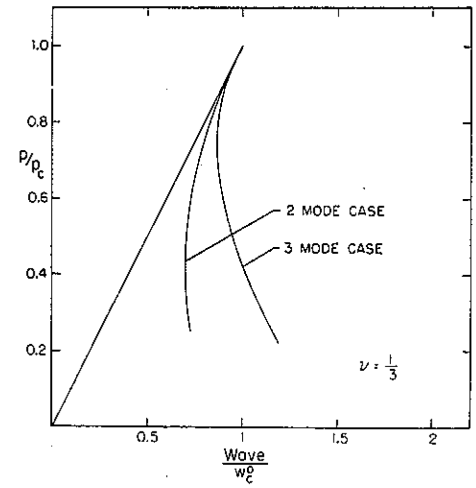

Fig. 4 Generalized load-deflection curves for shallow section of perfect sphere

This coefficient reduces to a constant value,

$$-\frac{9C}{32}\frac{Eh^3}{R^2} S_0$$

independent of which three-mode set is under consideration. Thus the three equilibrium equations for a given decoupled set are

$$\left(1 - \frac{p}{p_c}\right)\xi_1 - \frac{9C}{16}\xi_2\xi_3 = \frac{p}{p_c}\bar{\xi}_1 \qquad (28)$$

$$\left(1 - \frac{p}{p_c}\right)\xi_2 - \frac{9C}{16}\xi_1\xi_3 = \frac{p}{p_c}\bar{\xi}_2 \qquad (29)$$

$$\left(1 - \frac{p}{p_c}\right)\xi_3 - \frac{9C}{16}\xi_1\xi_2 = \frac{p}{p_c}\bar{\xi}_3 \qquad (30)$$

If a three-mode set is operative in the initial postbuckling deformation of the *perfect sphere,* then

$$\xi_1 = \xi_2 = \xi_3 = \frac{16}{9C}\left(1 - \frac{p}{p_c}\right)$$

and the generalized load-deflection relation, again using equation (18), is

$$\frac{w_{avg}}{w_c^0} = \frac{p}{p_c} + \frac{32}{27(1-\nu)}\left(1 - \frac{p}{p_c}\right)^2 \qquad (31)$$

One notes that the load-deflection relation for the three-mode case differs from that for the two-mode situation. The initial postbuckling load-deflection curves for both cases are plotted in Fig. 4.

If the shell is *imperfect* with an imperfection in only one of the modes—$\bar{\xi}_3 > 0$, say—the prebuckling deformation is in the ξ_3-mode

$$\xi_3 = \frac{\bar{\xi}_3 p/p_c}{1 - p/p_c}$$

Bifurcation from this solution will occur at the value of p when the determinant of the coefficients of ξ_1 and ξ_2 in equations (28) and (29) vanishes. This gives the buckling pressure of the imperfect shell, which is found to be

$$\left(1 - \frac{p^*}{p_c}\right)^2 = \frac{9C}{16}\bar{\xi}_3\frac{p^*}{p_c} \qquad (32)$$

This relation is plotted, along with the results for two-mode situation, in Fig. 3. A single imperfection in the three-mode case is not as degrading as a single imperfection when two modes are operative.

Discussion of Results

The equilibrium equations of the general theory decouple into sets of either two or three interacting modes. The set, or combination of sets, which is actually operative in the initial postbuckling regime of the *perfect sphere* is indeterminate within the context of the general theory. This indeterminacy parallels the initial postbuckling situation for axially compressed cylindrical shells which Koiter studied via his general theory. Like the spherical shell, the cylindrical shell has a multiplicity of buckling modes associated with the classical buckling load; but unlike the sphere, *all* the modes can couple through the nonlinear terms in the equilibrium equations. The relative magnitudes of these modes, however, remain undetermined by the general theory. On the other hand, the load-end shortening relation is uniquely determined by the general theory. The generalized load-deflection relation (i.e., p versus w_{avg}) for the sphere is not uniquely determined by the general theory but depends on whether a two or three-mode set is operative. Quite likely, the indeterminacy in both problems would be removed if higher-order terms were retained in the potential energy expression.

An imperfection in the form of any given mode has the effect of determining which set of modes will be operative. The imperfection which appears to cause the greatest reduction in the maximum support pressure is that in the form of the ξ_7-mode of the two-mode case [see equation (26)]. For axially compressed cylindrical shells, imperfections in the form of the axisymmetric buckling mode are most critical [2]. The relation of buckling load to imperfection for cylinders with this imperfection is

$$\left(1 - \frac{\lambda^*}{\lambda_c}\right)^2 = \frac{3C}{2}\bar{\xi}\frac{\lambda^*}{\lambda_c}$$

where λ^*/λ_c is the ratio of the buckling load to the classical buckling load. Here, the imperfection magnitude is measured relative to the shell thickness in the same manner as in the analogous formulas in this paper. The effect of the most critical imperfection on each structure is almost identical [compare this equation with equation (26)].

With the restriction that the deformations be rotationally symmetric, Thompson [3] has shown, using equations for a complete sphere, and on the basis of Koiter's general theory, that the slope of the pressure-deflection curve is negative in the initial postbuckling regime according to

$$\frac{\partial(p/p_c)}{\partial(A/h)} = -\frac{3\sqrt{3}}{4\pi}[12(1-\nu^2)]^{1/4}\left(\frac{h}{R}\right)^{1/2}$$

where A is the buckling deflection at the pole of symmetry. For small values of h/R, this slope will be quite small compared to the corresponding values predicted by the present analysis, $9[3(1 - \nu^2)]^{1/2}/16$ and $9[3(1 - \nu^2)]^{1/2}/8$.

The comparison between the two results is more meaningful, however, if deflections of the complete sphere away from the pole are considered, since the relatively large deflections of the rotationally symmetric buckling mode only occur near the poles.[7] Buckling deflections away from the poles are on the order of $(h/R)^{1/4}$ of A and, thus, referred to deflections of this magnitude, the slope of the pressure-deflection curve will be on the order of $(h/R)^{1/4}$. It follows, then, that the rotationally symmetric analysis indicates that, for comparable reductions in the buckling pressure, the imperfection magnitudes must be on the order of $(R/h)^{1/4}$ times the values predicted by the present analysis. The rotational-symmetry restriction precludes the possibility of the strong coupling between critical modes which has been demonstrated here.

Limitations of General Theory

As previously emphasized, one can have confidence in the general theory only when the terms dropped from the potential energy expression are sufficiently small compared to the terms retained. In particular, the larger the imperfections, the more one should question the buckling load predictions.

When the imperfection is in the form of a buckling mode with one zero wave number, it is possible to obtain an independent estimate of the buckling pressure of the spherical shell. If we take an imperfection in the form

$$\bar{w} = \bar{\xi}h \cos q_0(x/R)$$

the nonlinear shallow shell equations for an initially imperfect spherical shell admit an exact solution for the prebuckling deformation of the shell, and the deformation is independent of the y-coordinate. At a certain value of the external pressure, bifurcation from this y-independent deformation occurs. The approximate calculation for determining the relation of the bifurcation pressure to the imperfection is sketched briefly in the Appendix. The analogous calculation for the effect of axisymmetric imperfections on the buckling load of cylindrical shells has been given a careful treatment by Koiter [2]. The method of calculation insures that the estimate of the bifurcation pressure, although approximate, is an upper bound to the actual bifurcation pressure. The results of this calculation are shown in Fig. 5, where the upper-bound results can be compared with the general-theory predictions for the same imperfection, equation (25). The agreement, as in the case for the cylindrical shell, is surprisingly good even for imperfections which reduce the buckling load to as little as 30 percent of the classical value. For small imperfections, the upper-bound estimate and the general-theory prediction approach each other asymptotically.

Application to Spherical Caps

The conclusions reached with regard to the effects of imperfections on shallow sections of complete spherical shells obviously apply to spherical caps if the buckling wavelengths are small compared to the base dimension of the cap. The shortest buckle wavelength is

$$2\pi[12(1-\nu^2)]^{-1/4}(Rh)^{1/2}$$

To obtain a rough estimate of range in which the results of the present analysis should be at least partially applicable, we will demand, quite arbitrarily, that the foregoing wavelength be less than one third the base diameter of the cap. With this constraint, it is easily shown that the shell rise H to the thickness h must satisfy

$$\left(\frac{H}{h}\right)^{1/2} > (3\pi/2)[3(1-\nu^2)]^{-1/4}$$

In terms of the frequently defined shallowness parameter $\lambda = 2[3(1-\nu^2)]^{1/4}(H/h)^{1/2}$, this implies that $\lambda > 3\pi$.

It is interesting to note that Huang [10] has calculated the buckling pressure for initially perfect spherical caps which are clamped on the base edge. For sufficiently large rise to height—$\lambda > 3\pi$, say—the buckling pressure is about 85 percent of the pressure necessary to buckle a complete perfect sphere with the same radius of curvature and thickness. Experimental data in this range show considerable scatter, with buckling pressures in

[7] The author is indebted to Professor Koiter for a helpful discussion of this point. Professor Koiter [11] has obtained some results for multimode buckling of externally pressurized spherical shells, based on equations for the complete sphere. Although there is some difficulty in comparing the two approaches, the two sets of results are at least in qualitative agreement. The main difference appears to be that, while the results obtained from the shallow shell approach are independent of R/h, those based on the complete sphere equation are not.

many instances less than 30 percent of the value predicted for the perfect cap. It seems clear that initial imperfections account for the discrepancy between the experimental data and the results for the initially perfect cap.

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APPENDIX

Upper-Bound Calculation

We consider a shallow spherical section with an imperfection in the form

$$\bar{w} = \bar{\xi} h \cos q_0 (x/R) \quad (33)$$

The nonlinear shallow shell equations for an initially imperfect spherical shell admit a relatively simple, exact prebuckling solution for an imperfection of the form of equation (33).[8] This solution is

$$W = \left\{ -(1-\nu)pR^2/Eh2 + \bar{\xi} h \frac{p/p_c}{1-p/p_c} \cos q_0 \frac{x}{R} \right\} + w \quad (34)$$

$$F = \left\{ -\frac{1}{4}(x^2+y^2)pR - EhRq_0^{-2}\bar{\xi} h \frac{p/p_c}{1-p/p_c} \cos q_0 \frac{x}{R} \right\} + f \quad (35)$$

[8] For a y-independent initial imperfection, the shallow-shell equations are altered by appending the term $-F_{,yy}\bar{w}_{,xx}$ to the left-hand side of equation (2) and $+W_{,yy}\bar{w}_{xx}$ to the left-hand side of equation (3).

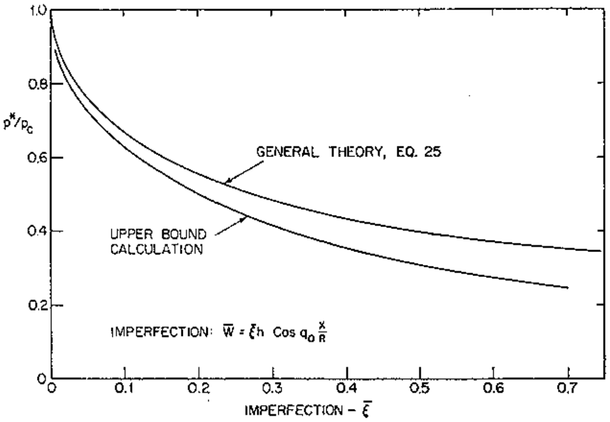

Fig. 5 Comparison between general theory and an independent upper-bound calculation ($\nu = 1/3$)

where w and f are zero prior to buckling. Buckling occurs with bifurcation from this y-independent prebuckling solution. Thus we look for the value of the pressure p at which the nonlinear shallow shell equations admit nonzero solutions for w and f. Substituting (34) and (35) into the full nonlinear equations for an initially imperfect spherical shell and then linearizing with respect to w and f, we obtain two homogeneous, nonconstant coefficient equations in w and f for determining the eigenvalue p^*. In the interests of brevity, these equations will not be listed here. The approximate method of solution of this eigenvalue problem, only described subsequently, is the subject of a paper by Koiter [2] for the analogous problem of a cylinder with axisymmetric imperfections.

One of the two linear eigenvalue equations is a compatibility equation which is solved exactly for f in terms of an assumed w

$$w = \sin\left(\frac{1}{2} q_0 \frac{x}{R}\right) \sin\left(\gamma q_0 \frac{y}{R}\right)$$

where γ is a free parameter. Then, f and w are used in conjunction with the Rayleigh-Ritz method to solve the second equation, an equilibrium equation, approximately and to obtain an equation for the eigenvalue. The eigenvalue equation relating p^* and $\bar{\xi}$ is

$$(1 - p^*/p_c)^2(Q^2 + 1 - 2Qp^*/p_c) - C\gamma^2\bar{\xi}(1 - p^*/p_c)(p^*/p_c + 2/Q) + (C\gamma^2\bar{\xi})^2(Q^{-2} + (9/4 + \gamma^2)^{-2}) = 0 \quad (36)$$

where

$$Q = 1/4 + \gamma^2$$

This approximate method of solution insures that the estimate of the buckling pressure p^* for a given imperfection $\bar{\xi}$ is an upper bound to the exact bifurcation pressure. In calculating p^* for a given value of $\bar{\xi}$ the free parameter γ is chosen so that the prediction of p^* based on (36) is a minimum. The results of these calculations are plotted in Fig. 5, where they can be compared with the predictions of the general theory.

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