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SINGULAR BEHAVIOUR AT THE END OF A TENSILE CRACK IN A HARDENING MATERIAL*

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Summary

Distributions of stress occurring at the tip of a crack in a tension field are presented for both plane stress and plane strain. A total deformation theory of plasticity, in conjunction with two hardening stress–strain relations, is used. For applied stress sufficiently low such that the plastic zone is very small relative to the crack length, the dominant singularity can be completely determined with the aid of a path-independent line integral recently given by Rice (1967). The amplitude of the tensile stress singularity ahead of the crack is found to be larger in plane strain than in plane stress.

1. Introduction

The background for the present study has been surveyed comprehensively by McClintock and Irwin (1965) and more recently by Rice (1967a). Our investigation is concerned with the plastic deformation at the tip of a crack in a two-dimensional stress field which is uniaxial tension in a direction perpendicular to the crack far from it. We have limited consideration to only the zone in the immediate neighbourhood of the crack tip in which the plastic strains are large compared to the elastic strains. Nonlinearity is introduced in this study only through the stress–strain relations which are chosen to model certain aspects of the elastic–plastic behaviour of common metals. Equilibrium conditions and the strain-displacement equations are taken to be linear. Results for plane stress and plane strain are contrasted.

2. Deformation Theory Equations for a Crack in a Strain Hardening Material

The first representation of nonlinear elastic–plastic behaviour we will consider is of the type suggested by Ramberg and Osgood (1943) to model the tensile stress–strain relations of certain metals, namely

$$\epsilon = \sigma + \alpha\sigma^n. \tag{1}$$

Throughout this paper, unless otherwise stated, stress quantities if unbarred will be non-dimensionalized by a yield stress $\bar{\sigma}_y$ and unbarred strain quantities will be normalized with respect to the corresponding yield strain $\bar{\epsilon}_y = \bar{\sigma}_y/E$ where E is the initial

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slope of the stress–strain curve. Tensile curves with $\alpha = 0{\cdot}02$ corresponding to a usual engineering definition of yield are shown in Fig. 1 for several values of the strain hardening coefficient n. Applicability of (1) is restricted to monotonically increasing stress or, as more commonly stated in plasticity theory, under the condition of no unloading.

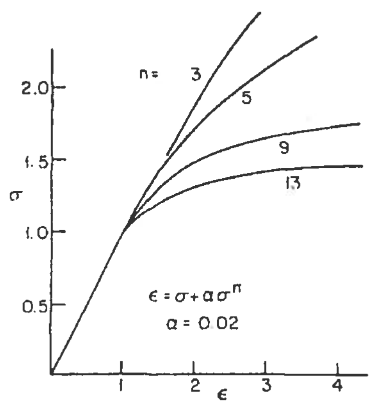

FIG. 1. Ramberg–Osgood-type stress–strain relation.

Only the simplest total deformation theory will be employed in the present study. Plastic deformation is assumed to be independent of the hydrostatic component of the stress, $\frac{1}{3}\sigma_{kk}$, and, further, is assumed to be completely determined by the first invariant of the stress deviator :

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\,\delta_{ij}. \tag{2}$$

It is convenient to introduce the invariant in the form of the 'effective stress' σ_e defined by

$$\sigma_e^2 = \tfrac{3}{2}\,s_{ij}\,s_{ij}. \tag{3}$$

Thus, in simple tension $\sigma_e = \sigma$; and therefore the Mises yield condition auxiliary to the theory is $\sigma_e = 1$ or $\bar{\sigma}_e = \bar{\sigma}_y$ (of course a smooth stress–strain representation such as (1) admits only an effective, or approximate, yield condition).

The generalized stress–strain relation which reduces to (1) in simple tension is

$$\epsilon_{ij} = (1+\nu)\,s_{ij} + \frac{1-2\nu}{3}\,\sigma_{pp}\,\delta_{ij} + \tfrac{3}{2}\,\alpha\sigma_e^{\,n-1}\,s_{ij}$$

where ν is Poisson's ratio.

Since unloading must be excluded for this relation to retain any validity, any non-steady solution must be checked in retrospect to insure that σ_e at every point is non-decreasing. Solutions must also be checked to determine the extent to which the deformation history at every point is proportional—that is, the extent to which the stress components remain in a fixed proportion as the deformation proceeds. It is well known that deformation theory is inadequate where there are large departures from proportionality; but when this does not occur this theory can be considered no more objectionable than the corresponding incremental flow theory (BUDIANSKY 1959). The application of this theory to the crack problem will be discussed at a later point.

The complementary potential energy functional, appropriate to a stress boundary value problem and to be specialized to either plane stress or plane strain, is

$$\int_A \left\{ \tfrac{1}{3}(1+\nu)\,\sigma_e^2 + \frac{1-2\nu}{6}\,\sigma_{kk}^2 + \frac{\alpha}{n+1}\,\sigma_e^{n+1} \right\} dA. \tag{4}$$

In either generalized plane stress or plane strain, equilibrium is ensured for all stresses derived from a stress function by

$$\left.\begin{aligned} \sigma_r &= r^{-1}\,\phi' + r^{-2}\,\ddot{\phi}, \\ \sigma_\theta &= \phi'', \\ \sigma_{r\theta} &= -\,(r^{-1}\,\dot{\phi})'. \end{aligned}\right\} \tag{5}$$

where cylindrical coordinates r and $\theta\,[(\ )' = \partial/\partial r$ and $(\ )^{\cdot} = \partial/\partial\theta]$ are centred at the right end of the crack as shown in Fig. 2. The non-dimensional stress function ϕ and coordinate r are given in terms of the dimensional quantities (barred) by

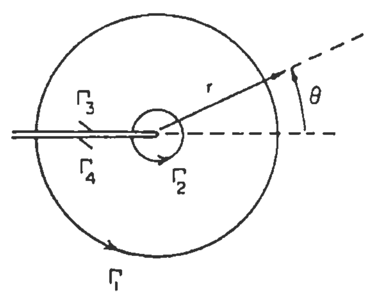

FIG. 2. Conventions at crack tip.

$$\phi = \frac{1}{\sigma_y\,L^2}\,\bar{\phi}, \quad r = \frac{1}{L}\,\bar{r},$$

where L is the half length of the crack.

In *plane stress*, dealt with first, the stress components acting on the plane parallel to the plate are neglected. The effective stress is

$$\sigma_e^2 = \sigma_r^2 + \sigma_\theta^2 - \sigma_r\,\sigma_\theta + 3\sigma_{r\theta}^2 \tag{6}$$

while the strains are given by

$$\left.\begin{aligned} \epsilon_r &= \sigma_r - \nu\sigma_\theta + \alpha\sigma_e^{n-1}\,(\sigma_r - \tfrac{1}{2}\,\sigma_\theta), \\ \epsilon_\theta &= \sigma_\theta - \nu\sigma_r + \alpha\sigma_e^{n-1}\,(\sigma_\theta - \tfrac{1}{2}\,\sigma_r), \\ \epsilon_{r\theta} &= (1+\nu)\,\sigma_{r\theta} + \tfrac{3}{2}\,\alpha\sigma_e^{n-1}\,\sigma_{r\theta}. \end{aligned}\right\} \tag{7}$$

The partial differential equation governing the stress function (under the restriction of no unloading) can be obtained by eliminating the strains from the compatibility equation

$$r^{-1}\,(r\epsilon_\theta)'' + r^{-2}\,\ddot{\epsilon}_r - r^{-1}\,\epsilon_r' - 2r^{-2}\,(\dot{\epsilon}_{r\theta}\,r)' = 0$$

or, alternatively, can be obtained as the Euler equation associated with the first variation of the complementary potential energy.* The result for plane stress is†

$$\nabla^4 \phi + \frac{\alpha}{2} \{ r^{-1} [\sigma_e^{n-1} (2r\phi'' - \phi' - r^{-1} \ddot{\phi})]'' + 6r^{-2} [\sigma_e^{n-1} r (r^{-1} \dot{\phi})']^{\cdot\prime} + r^{-1} [\sigma_e^{n-1} (-2r^{-1} \phi' - 2r^{-2} \ddot{\phi} + \phi'')]' + r^{-2} [\sigma_e^{n-1} (-\phi'' + 2r^{-1} \phi' + 2r^{-2} \ddot{\phi})]^{\cdot\cdot} \} = 0. \quad (8)$$

For a stress-free crack the boundary conditions on the crack can be taken as

$$\phi = \dot{\phi} = 0. \quad (9)$$

3. Singular Behaviour at the Crack Tip

A crack in a far field which is tensile in a direction perpendicular to it is considered. A solution to (8) is sought in the immediate vicinity of the crack tip where the stresses are large, and, on the basis of the theory which has been laid down, are unbounded as the crack tip is approached. An asymptotic expansion of the solution is attempted in the form

$$\phi = r^s \tilde{\phi}_1(\theta) + r^t \tilde{\phi}_2(\theta) + \ldots \quad (10)$$

where, if the first term is to be singled out as the dominant one, $s < t$, etc. Our search will be restricted to only the dominant term in such an expansion :

$$\phi = K r^s \tilde{\phi}(\theta) \quad (11)$$

where explicit display of the amplitude K will permit us to adjust the amplitude of $\tilde{\phi}$ in some arbitrary, yet appropriate, manner.

Except for its amplitude, the dominant term is determined entirely by the nonlinear terms in the governing equation (8). Formally, this follows from a substitution of the expansion (10) into (8), or by a comparison of the elastic (quadratic) and plastic ($n + 1^{\text{th}}$ power) contributions to the complementary energy functional (4). For any $s < 2$, one can choose a sufficiently small neighbourhood of the crack tip such that the elastic energy is an arbitrarily small fraction of the plastic energy. On physical grounds the energy in any finite zone containing the crack tip must be finite. This imposes the restriction, easily obtained using (4) and (11), that

$$s > \frac{2n}{n+1}. \quad (12)$$

With the biharmonic term omitted, (8) is homogeneous in both ϕ and r (and derivatives with respect to r). For this reason, an exact separation of the nonlinear terms in (8) is accomplished by the term assumed (11). The resulting equation, which is homogeneous in $\tilde{\phi}$ and is associated with homogeneous boundary conditions, is in the form of an eigenvalue equation for s :

$$\left[n(s-2) - \frac{\partial^2}{\partial \theta^2} \right] \cdot [\tilde{\sigma}_e^{n-1} \{ s(s-3) \tilde{\phi} - 2\ddot{\tilde{\phi}} \}] + \{ n(s-2) + 1 \} \{ n(s-2) \} \tilde{\sigma}_e^{n-1} \{ s(2s-3) \tilde{\phi} - \ddot{\tilde{\phi}} \} + 6 \{ n(s-2) + 1 \} (s-1) (\tilde{\sigma}_e^{n-1} \dot{\tilde{\phi}})^{\cdot} = 0 \quad (13)$$

*Strictly speaking, (4) can only be employed for finite regions, but for the purposes of discussion in this paper it is sufficient. A modification similar to that introduced by BUDIANSKY and VIDENSEK (1955) for application to the entire plane could be used.

†Thinning of the plate is a nonlinear effect which has been neglected in the derivation of this equation. Equations for perfect plasticity which take thinning into account have been given by HILL (1950).

$$\left.\begin{aligned} \text{where}\quad \sigma_e &= K r^{s-2}\,\tilde{\sigma}_e(\theta) = K r^{s-2}\,(\tilde{\sigma}_r^2 + \tilde{\sigma}_\theta^2 - \tilde{\sigma}_r\,\tilde{\sigma}_\theta + 3\tilde{\sigma}_{r\theta}^2)^{\frac{1}{2}} \\ \text{and}\quad \sigma_r &= K r^{s-2}\,\tilde{\sigma}_r(\theta) = K r^{s-2}\,(s\tilde{\phi} + \ddot{\tilde{\phi}}) \\ \sigma_\theta &= K r^{s-2}\,\tilde{\sigma}_\theta(\theta) = K r^{s-2}\,s\,(s-1)\,\tilde{\phi} \\ \sigma_{r\theta} &= K r^{s-2}\,\tilde{\sigma}_{r\theta}(\theta) = K r^{s-2}\,(1-s)\,\dot{\tilde{\phi}}. \end{aligned}\right\} \qquad (14)$$

The stress field surrounding the crack will be symmetric about $\theta = 0$ and thus $\tilde{\phi}(\theta)$ will be also. Stress-free boundary conditions (9) require

$$\tilde{\phi}(\pm\,\pi) = \dot{\tilde{\phi}}(\pm\,\pi) = 0$$

or, with the imposed symmetry, $\dddot{\tilde{\phi}}(0) = \dot{\tilde{\phi}}(0) = 0$.

The nonlinear eigenvalue equation (13) and its counterpart for plane strain, equation (31), were solved numerically using an iteration scheme described in the Appendix. Rather remarkably the calculated values of s for various integer values of n for both plane stress and plane strain were found to be given to within one tenth of one percent by the simple formula

$$s = \frac{2n+1}{n+1}. \qquad (15)$$

As a consequence of this relation, the energy density varies exactly as $1/r$ as the crack tip is approached. A direct proof of the validity of this result can be obtained with a path-independent integral recently given by RICE (1967). This demonstration is given in the following section.

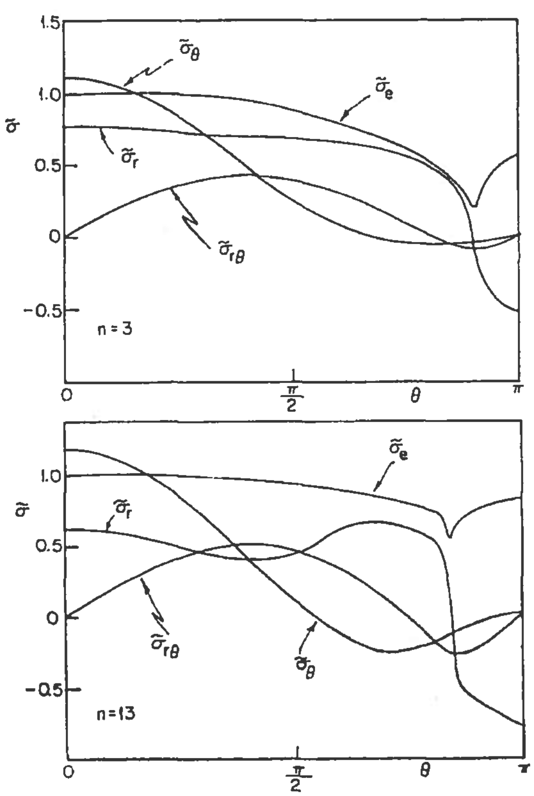

FIG. 3 (a) and (b). θ-variation of stresses at crack tip for plane stress.

The θ-variations of the three stress components and the effective stress associated with the dominant singularity are shown in Figs. 3a and b for $n = 3$ and 13. Typically, then, the effective stress is given by

$$\sigma_e = Kr^{-\frac{1}{n+1}} \tilde{\sigma}_e(\theta) \tag{16}$$

where, as seen from the plots, the maximum amplitude of $\tilde{\phi}$ is fixed so that $\tilde{\sigma}_e$ attains the maximum value of unity. Note that $\tilde{\sigma}_e$ does not vanish for any value of θ; and therefore, neglect of the elastic energy density is justified for all θ as well.

The amplitude K is undetermined and must be related to the applied tensile stress σ^∞. In general, this would appear to be a rather formidable task entailing a matching procedure to connect the far field to the solution in the regime in which the elastic and plastic strains are of comparable magnitude and then in turn to the solution at the crack tip. It might be feasible, under certain circumstances, to make use of finite difference methods in much the same way as SWEDLOW, WILLIAMS and YANG (1966) have done but to incorporate directly the form of the singularity. It happens that it is not necessary to resort to these lengths when the plastic zone is very small in comparison with the crack length. In this case, application of Rice's path integral yields the amplitude, K, with recourse to only a relatively simple calculation which completely by-passes the necessity of considering the range of comparable elastic and plastic strains.

4. AMPLITUDE OF THE DOMINANT SINGULAR TERM FOR SMALL SCALE YIELDING : PLANE STRESS*

As a preliminary to the calculations in this section, we now demonstrate that when yielding occurs on a small scale (that is, the applied stress is sufficiently low to ensure that the plastic zone is very small compared to the crack length) the deformation is governed by a set of similar solutions. Conditions for no unloading and near-proportional loading, on which the applicability of deformation theory rests, are discussed.

The leading singular term in the purely elastic solution for a cracked plate subject to a tension $\bar{\sigma}^\infty = \bar{\sigma}_y \sigma^\infty$ is

$$\phi = \frac{\sigma^\infty}{\sqrt{2}} r^{\frac{3}{2}} (\cos \tfrac{1}{2}\theta + \tfrac{1}{3}\cos \tfrac{3}{2}\theta). \tag{17}$$

For plane stress the effective stress is easily found to be

$$\sigma_e = \frac{\sigma^\infty}{\sqrt{2}} r^{-\frac{1}{2}} (\cos^2 \tfrac{1}{2}\theta + \tfrac{3}{4}\sin^2 \theta)^{\frac{1}{2}}. \tag{18}$$

The boundary value problem for small-scale yielding is posed in the following manner. It is assumed that the applied stress is sufficiently low to ensure that the plastic zone is not only very small compared to the crack length but is also embedded in an elastic field which is dominated by the singularity of the elastic solution. For small-scale yielding, then, ϕ must satisfy the full nonlinear equation (8) and approach the function (17) of the elastic solution as r becomes large.

*The term " small-scale yielding " has been introduced by RICE (1967). Parts of the present discussion are similar to his and that given by MCCLINTOCK and IRWIN (1965).

The following non-dimensionalization collapses the set of solutions corresponding to different values of $\bar{\sigma}^{\infty}$ to one set of similar solutions :

$$\phi_0 = \left(\frac{\bar{\sigma}_y}{\bar{\sigma}^{\infty}}\right)^4 \cdot \frac{\bar{\phi}}{\bar{\sigma}_y L^2}, \quad r_0 = \left(\frac{\bar{\sigma}_y}{\bar{\sigma}^{\infty}}\right)^2 \cdot \frac{\bar{r}}{L}, \quad \sigma_0 = \frac{1}{\bar{\sigma}_y}\bar{\sigma}.$$

The equation governing ϕ_0 is again (8) with ϕ, σ_e and r replaced by ϕ_0, σ_{e0} and r_0, respectively. Far from the crack tip

$$\phi_0 \sim \frac{1}{\sqrt{2}} r_0^{\frac{3}{2}} (\cos \tfrac{1}{2}\theta + \tfrac{1}{3}\cos \tfrac{3}{2}\theta).$$

In terms of the similarity solution the dimensional quantities are given by, typically,

$$\bar{\sigma}_e(\bar{r}, \theta) = \bar{\sigma}_y \, \sigma_{e0}(r_0, \theta) = \bar{\sigma}_y \, \sigma_{e0}\left[\left(\frac{\bar{\sigma}_y}{\bar{\sigma}^{\infty}}\right)^2 \frac{\bar{r}}{L}, \theta\right].$$

If no unloading is to occur with monotonically increasing $\bar{\sigma}^{\infty}$ it is necessary that $\sigma_{e0}(r_0, \theta)$ be a non-decreasing function of r_0 as r_0 approaches zero from infinity for each value of θ. While a complete check of this condition must await an inspection of the full solution, one might expect with some confidence that it will be satisfied. In much the same manner, proportionality or lack thereof at any point depends on the extent of proportionality on the corresponding radial ray of the similarity solution. We return to this discussion at the end of the paper.

The boundary value problem posed for small-scale yielding is, in effect, the first term in an asymptotic expansion of the full solution about the zero applied stress level. It is not clear how large the applied stress may be such that the one term in the expansion remains a good approximation. Some insight into the range of validity can be obtained from an examination of the solution of the simpler problem of an anti-plane shear crack which has been studied in detail by McClintock (1958), Neuber (1961) and Koskinen (1963). Rice (1966), in particular, has looked closely at this solution and has suggested that the small-scale yielding assumption remains valid for overall applied stress below one third or at most one half the yield stress. Most certainly, the small-scale yielding assumption loses its validity at loads near the fracture point of semi-brittle or ductile metals. For such materials, dissipation of energy in plastic deformation is comparable to the elastic energy release and the small-scale yielding approximation will break down, as will be clear from the discussion which follows.

In the remainder of this section use is made of a path-independent line integral recently discovered by Rice (1967) (in cartesian coordinates) :

$$J = \int_{\Gamma} (W dy - \sigma_{ij} n_j u_{i,x} ds) \tag{19}$$

where W is the strain energy density for a linear or nonlinear material (with no discontinuous stress–strain behaviour such as unloading) :

$$W = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}. \tag{20}$$

Sanders (1960) has given a related path-independent integral for the linear elastic case. Rice has shown that the line integral J is zero when evaluated about any closed circuit which encloses no singularities on the basis of any solution to the associated equations of plane stress or plane strain. When evaluated about any closed circuit which encloses the crack, it assumes a nonzero value. This value can be identified with the rate of energy release per unit extension of the crack when the prescribed loads or displacements are held fixed. Since this result is for a material which undergoes no unloading, it loses its physical interpretation when applied to a common metal. Nonetheless, because of its path-independence, it proves most useful in the analysis of the stationary crack.

Consider the closed circuit $\Gamma = \Gamma_1 - \Gamma_2 + \Gamma_3 + \Gamma_4$ in Fig. 2. Contributions to the integral along the crack vanish; thus since the total integral vanishes, the integral about Γ_1 is identical to that about Γ_2. If Γ_1 is taken entirely in the elastic region and if small-scale yielding is assumed then this integral can be evaluated. Rice (1967) has evaluated this integral for several loading cases on the basis of the purely elastic solution (i.e. on the assumption of small-scale yielding). He has shown that Irwin's (1960) results [rederived by Brueckner (1958) and Sanders (1960)] for the energy release rate are obtained. For the tensile crack under consideration (in dimensionless quantities)

$$\int_{\Gamma_1} (W dy - \sigma_{ij}\, n_j\, u_{i,x}\, ds) = \pi \sigma^{\infty 2}. \tag{21}$$

Now, the radius of the Γ_2 path is chosen such that it lies within the zone dominated by the fully plastic singularity (11), and this integral is evaluated in terms of ϕ and the unknown amplitude K. This reduction is fairly straightforward. One finds

$$\left.\begin{aligned} W &= \alpha K^{n+1} \frac{n}{n+1}\, r^{(n+1)(s-2)}\, \tilde{\sigma}_e^{\,n+1} \\ \text{and} \quad \sigma_{ij}\, n_j\, u_{i,x} &= K^{n+1}\, r^{(n-1)(s-2)} \{\sin\theta\, [\tilde{\sigma}_r\, (\tilde{u}_\theta - \tilde{u}'_r) - \tilde{\sigma}_{r\theta}\, (\tilde{u}_r + \tilde{u}'_\theta)] \\ &\quad + (n\,(s-2)+1) \cos\theta\, [\tilde{\sigma}_r\, \tilde{u}_r + \tilde{\sigma}_{r\theta}\, \tilde{u}_\theta]\}. \end{aligned}\right\} \tag{22}$$

The radial and tangential displacements, u_r and u_θ, are readily found using any two of the strain-displacement relations :

$$u_r = \alpha K^n\, r^{n(s-2)+1}\, \tilde{u}_r(\theta) = \frac{\alpha K^n\, r^{n(s-2)+1}}{\{n\,(s-2)+1\}}\, \tilde{\sigma}_e^{\,n-1} \{s\,(3-s)\, \phi/2 + \phi''\},$$

$$u'_\theta = \alpha K^n\, r^{n(s-2)+1}\, \tilde{u}'_\theta(\theta) = \alpha K^n\, r^{n(s-2)+1} [\tilde{\sigma}_e^{\,n-1} \{s\,(s-\tfrac{3}{2})\, \phi - \phi''/2\} - \tilde{u}_r].$$

The final form of the path integral on Γ_2 with radius r_2 is

$$\int_{\Gamma_2} (W dy - \sigma_{ij}\, n_j\, u_{i,x}\, ds) = \alpha K^{n+1}\, r_2^{(n+1)(s-2)+1} \cdot I \tag{23}$$

where

$$I = \int_{-\pi}^{\pi} \left\{ \frac{n}{n+1}\, \tilde{\sigma}_e^{\,n+1} \cos\theta - [\sin\theta\, (\tilde{\sigma}_r\, (\tilde{u}_\theta - \tilde{u}'_r) - \tilde{\sigma}_{r\theta}\, (\tilde{u}_r + \tilde{u}'_\theta)) + (n\,(s-2)+1)\, (\tilde{\sigma}_r\, \tilde{u}_r + \tilde{\sigma}_{r\theta}\, \tilde{u}_\theta) \cos\theta] \right\} d\theta. \tag{24}$$

This integral must equal the value obtained from the Γ_1 integration, thus

$$\alpha K^{n+1} r_2^{(n+1)(s-2)+1}. \quad I = \pi \sigma^{\infty 2} \tag{25}$$

and this condition can only be met (as $r_2 \to 0$) if

$$s = \frac{2n+1}{n+1}. \tag{26}$$

This demonstrates the result found approximately by the numerical calculation discussed earlier.

A re-examination of the above derivation shows that (26) will also hold when the plastic zone is not necessarily very small as long as the integral about Γ_1 does not vanish. Its general validity is reinforced by the numerical solution of (18), which was derived under no restriction on the size of the plastic zone except that no unloading occur at the crack tip.

One other important piece of information can also be obtained from Rice's integral—namely, the amplitude

$$K = \left(\frac{1}{\alpha}\right)^{\frac{1}{n+1}} \left(\frac{\pi}{I}\right)^{\frac{1}{n+1}} (\sigma^{\infty})^{\frac{2}{n+1}} \tag{27}$$

where, as seen from (23), I can be evaluated solely from the dominant singularity solution. Values of I and $(\pi/I)^{1/(n+1)}$ are presented in Table 1 for several values of

TABLE 1.

$n =$		3	5	9	13
	I	3·86	3·41	3·03	2·87
Plane stress	$\left(\frac{\pi}{I}\right)^{\frac{1}{n+1}}$	0·949	0·987	1·004	1·006
	E_n	2·16	2·45	2·67	2·76
	I	5·51	5·01	4·60	4·40
Plane strain	$\left(\frac{\pi}{I}\right)^{\frac{1}{n+1}}$	0·869	0·925	0·963	0·976
	E_n	1·28	1·37	1·48	1·52

hardening coefficient n and an extrapolated plot of I is given in Fig. 4. Note that the strain energy density is given by

$$W = \frac{n}{n+1} \frac{\pi}{I} \frac{\sigma^{\infty 2}}{r} \tilde{\sigma}_e^{n+1} \tag{28}$$

and recall that $\tilde{\sigma}_e$ takes on a maximum value of unity.

It will be most appropriate to discuss these results after the plane strain calculations have been reported. Details of the analogous calculations for a crack in a

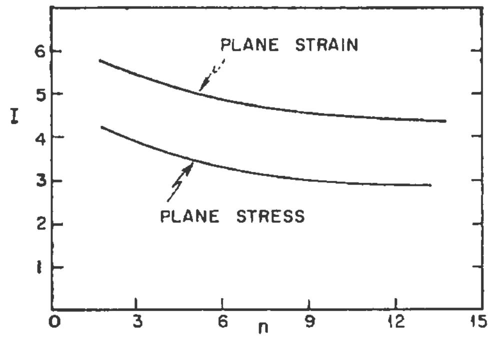

FIG. 4. Values of I as defined by equation (24).

tension field under the condition of plane strain are much the same as those given in the previous section and only the major points are recorded.

5. SINGULAR BEHAVIOUR IN PLANE STRAIN

The equation governing the stress function [defined, as before, by (5)] has a more complicated form than in the previous case and will not be given. On the other hand, the equation governing the dominant singularity is, if anything, slightly less lengthy than its plane stress counterpart and is found to be

$$\left[r^{-2}\frac{\partial^2}{\partial\theta^2} - r^{-1}\frac{\partial}{\partial r} - r^{-1}\frac{\partial\,(r)}{\partial r^2}\right]\cdot\left[\sigma_e^{\,n-1}\left(r^{-1}\phi' + r^{-2}\phi^{\cdot\cdot} - \phi''\right)\right] + 4r^{-2}\left[r\sigma_e^{\,n-1}\phi^{\cdot})'\right]^{\cdot} = 0 \tag{29}$$

where now

$$\sigma_e^{\,2} = \tfrac{3}{4}\left(\sigma_r - \sigma_\theta\right)^2 + 3\sigma_{r\theta}^{\,2}.$$

This formula for the effective stress is only correct in the singular zone where the elastic strains can be neglected.

Equation (29) can be separated if ϕ is taken to be

$$\phi = Kr^s\,\tilde{\phi}\,(\theta) \tag{30}$$

The eigenvalue equation for s and $\tilde{\phi}$ is

$$\left[\frac{\partial^2}{\partial\theta^2} - n\,(s-2)\left\{n\,(s-2)+2\right\}\right]\cdot\left[\tilde{\sigma}_e^{\,n-1}\left\{s\,(2-s)\,\tilde{\phi} + \tilde{\phi}^{\cdot\cdot}\right\}\right] + 4\,(s-1)\left\{n\,(s-2)+1\right\}\left(\tilde{\sigma}_e^{\,n-1}\,\tilde{\phi}^{\cdot}\right)^{\cdot} = 0 \tag{31}$$

where

$$\sigma_e = Kr^{s-2}\,\tilde{\sigma}_e = Kr^{s-2}\left[\tfrac{3}{4}\left(\tilde{\sigma}_r - \tilde{\sigma}_\theta\right)^2 + 3\tilde{\sigma}_{r\theta}^{\,2}\right]^{\frac{1}{2}}$$

and $\tilde{\sigma}_r$, $\tilde{\sigma}_\theta$ and $\tilde{\sigma}_{r\theta}$ are defined in (14).

As discussed previously, this equation was solved numerically subject to the boundary conditions

$$\tilde{\phi}^{\cdot}\,(0) = \tilde{\phi}^{\cdot\cdot\cdot}\,(0) = 0, \qquad \tilde{\phi}\,(\pi) = \tilde{\phi}^{\cdot}\,(\pi) = 0.$$

Similarly,

$$s = \frac{2n+1}{n+1}$$

and θ-variations of the stresses are shown in Figs. 5a and b.

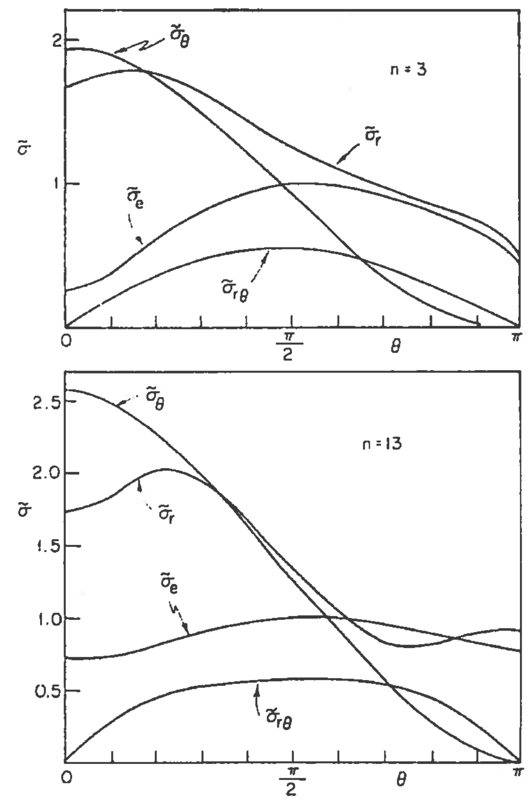

FIG. 5 (a) and (b). θ**-variation of stresses at crack tip for plane strain.**

For small-scale yielding, application of Rice's path-independent integral gives

$$K = \left(\frac{1-\nu^2}{\alpha}\right)^{\frac{1}{n+1}} \left(\frac{\pi}{I}\right)^{\frac{1}{n+1}} (\sigma^\infty)^{\frac{2}{n+1}} \tag{32}$$

(the integral about Γ_1 for plane strain is $\pi(1-\nu^2)\,\sigma^{\infty 2}$ in our nondimensionalization). I is again given by (24) and for plane strain

$$\tilde{u}_r(\theta) = \frac{3\tilde{\sigma}_e^{\,n-1}}{4(2-s)}\left\{s(2-s)\,\tilde{\phi} + \tilde{\phi}''\right\},$$

$$\tilde{u}_\theta'(\theta) = (s-3)\,\tilde{u}_r.$$

Values of I and $(\pi/I)^{1/(n+1)}$ are listed in Table 1 and I is plotted in Fig. 4. As in the previous case, $\tilde{\phi}$ has been normalized so that $\tilde{\sigma}_e$ assumes a maximum value of unity.

Finally, the strain energy within an small radius r_2 in the fully plastic zone is given by

$$\int_0^{r_2} r\,dr \int_{-\pi}^{\pi} W\,d\theta = \sigma^{\infty 2}\, r_2 \cdot \frac{n}{n+1}\,\frac{\pi}{I}\int_{-\pi}^{\pi} \tilde{\sigma}^{n+1}\,d\theta\,(1-\nu^2) \tag{33}$$

and values of

$$E_n = \frac{n}{n+1}\,\frac{\pi}{I}\int_{-\pi}^{\pi} \tilde{\sigma}_e^{\,n+1}\,d\theta \tag{34}$$

are recorded in Table 1 for both plane strain and plane stress.

6. SINGULAR BEHAVIOUR FOR A PIECEWISE-LINEAR STRESS–STRAIN RELATION

A somewhat less realistic representation of the tensile stress–strain behaviour of common metals than the Ramberg–Osgood relation is the piecewise-linear relation shown in Fig. 6. This approximation does model certain features of plastic flow. And since the singularity dominating the behaviour at the crack tip in this

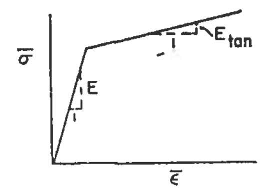

FIG. 6. Piecewise-linear stress–strain relation.

case can be obtained in a simple form with no numerical computation, it is interesting to contrast such results with those based on the Ramberg–Osgood representation.

The stress–strain relation according to deformation theory is easily obtained from the tensile stress–strain curve of Fig. 6. Continuing with the same notation, we find

$$\left.\begin{aligned} &\epsilon_{ij} = (1+\nu)\,\sigma_{ij} - \nu\sigma_{pp}\,\delta_{ij} + \lambda\,(1-\sigma_e^{-1})\,s_{ij} \\ \text{where}\quad &\lambda = \tfrac{3}{2}\left(\frac{E}{E_{\tan}} - 1\right) \qquad \text{if} \quad \sigma_e > 1 \\ \text{and}\quad &\lambda = 0 \qquad \text{if} \quad \sigma_e \leqslant 1. \end{aligned}\right\} \tag{35}$$

The complementary energy functional appropriate to a stress boundary value problem is

$$\int \left\{ \tfrac{1}{3}(1+\nu)\,\sigma_e^2 + \frac{1-2\nu}{6}\,\sigma_{KK} + \frac{\lambda}{3}\,(\sigma_e - 1)^2 \right\} dA. \tag{37}$$

Plane stress calculation

In a yielded region where $\sigma_e > 1$ the equation governing the stress function is

$$\frac{E}{E_{\tan}}\,\nabla^4\phi - \tfrac{1}{3}\,\lambda\left\{\left[r^{-2}\frac{\partial^2}{\partial\theta^2} - r^{-1}\frac{\partial}{\partial r}\right]\cdot\left[\sigma_e^{-1}\,(2r^{-2}\,\phi^{\cdot\cdot} + 2r^{-1}\,\phi' - \phi'')\right]\right.$$
$$\left. + r^{-1}\left[\sigma_e^{-1}\,(2r\phi'' - \phi' - r^{-1}\phi^{\cdot\cdot})\right]'' + 6r^{-2}\left[\sigma_e^{-1}\,r\,(r^{-1}\,\phi^{\cdot})'\right]^{\cdot}\right\} = 0 \tag{38}$$

where $\tilde{\sigma}_e$ is given by (6). Proceeding as before, a solution is sought in the form

$$\phi = Kr^s\,\tilde{\phi}$$

and $\sigma_e = Kr^{s-2}\,\tilde{\sigma}_e$ is introduced. It is anticipated that s will be less than 2 and thus the dominant singularity is governed by the biharmonic term in (38), i.e.,

$$\nabla^4\,\phi = 0. \tag{39}$$

A solution to this equation appropriate to our purposes (with $\phi = \phi^{\cdot} = 0$ on $\pm\,\pi$ together with a symmetric stress distribution) is

$$\phi = Kr^{\frac{3}{2}}\,(\cos\tfrac{1}{2}\,\theta + \tfrac{1}{3}\cos\tfrac{3}{2}\,\theta). \tag{40}$$

The r and θ variations of the stresses derived from this solution are the same (except for the amplitude K) as in the purely elastic solution, and in particular

$$\sigma_e = Kr^{-\frac{1}{2}} (\cos^2 \tfrac{1}{2}\theta + \tfrac{3}{4}\sin^2\theta)^{\frac{1}{2}}. \tag{41}$$

Contrary to the assumption implicit in the derivation of (39), $\bar{\sigma}_e$ is zero for the isolated values $\theta = \pm\pi$. Apparently the representation for the dominant singularity as given by (41) may not be uniformly valid for all θ. However, it should be a good approximation over most of the θ-range and it can be used to calculate the amplitude K for small scale yielding. Here it is necessary to note that Rice's proof of the path independence of (19) does not apply, as given, when the stress–strain relation has a discontinuity in slope such as the one under present consideration. This proof can be easily extended to cover cases in which a demarcation between the yielded and non-yielded zone occurs if it is assumed that the displacement gradients are continuous. Displacement gradients will be continuous across an elastic–plastic interface if $E_{\tan} > 0$.

The path-independent integral (19) is evaluated using the dominant singular solution (40). It is found to be

$$\int_{\Gamma_2} W dy - \sigma_{ij} n_j u_{i,x} ds = 2\pi K^2 \frac{E}{E_{\tan}}$$

which, for small-scale yielding, must equal $\pi\sigma^{\infty 2}$ and thus

$$K = \frac{1}{\sqrt{2}} \left(\frac{E_{\tan}}{E}\right)^{\frac{1}{2}} \sigma^{\infty}. \tag{42}$$

If $E_{\tan} = E$ the purely elastic result is retrieved as it obviously must be. According to this piecewise-linear calculation, then, the stresses at the tip are the same as predicted by linear elasticity but reduced by a factor $\sqrt{(E_{\tan}/E)}$.

It is interesting to note that the strain energy density in the fully plastic regime is independent of $E_{\tan}$ and exactly the same as that predicted by the purely elastic solution; namely,

$$W = \tfrac{1}{4} r^{-1} \sigma^{\infty 2} (\cos^2 \tfrac{1}{2}\theta + \tfrac{3}{4}\sin^2\theta) \tag{43}$$

and the energy within a small radius r_2 is

$$\int_0^{r_2} r dr \int_{-\pi}^{\pi} W d\sigma = \tfrac{7}{16} \pi\sigma^{\infty 2} r_2. \tag{44}$$

Plane strain calculation

Here, again, the dominant singularity is governed by the biharmonic. The appropriate solution is given by (40) and, except for the amplitude, the in-plane stresses are the same as in the linear elastic case. The effective stress for plane strain and the piecewise-linear relation is

$$\sigma_e^2 = (1 - \beta + \beta^2)(\sigma_r - \sigma_\theta)^2 + (1 - 4\beta + 4\beta^2)\sigma_r \sigma_\theta + 3\sigma_{r\theta}^2$$

where

$$\beta = \frac{\nu + \frac{1}{2}\left(\dfrac{E}{E_{\tan}} - 1\right)}{\dfrac{E}{E_{\tan}}}.$$

It is easily seen that σ_e vanishes for $\theta = \pm\pi$ and also for $\theta = 0$ when $\beta = \frac{1}{2}$ (as it very nearly is when $E/E_{\tan} \gg 1$). Thus, this solution may not be valid at these points but, as discussed above, it should be accurate for most θ. Assuming this to be so, the amplitude for small-scale yielding can be obtained as

$$K = \frac{(1-\nu^2)^{\frac{1}{2}}}{\sqrt{2}} \sigma^\infty \left[1 + \tfrac{1}{2}\beta^2 - \tfrac{3}{2}\nu\beta + \tfrac{1}{6}\lambda(4 - 3\beta + 2\beta^2)\right]^{-\frac{1}{2}}.$$

As a check, note that when $\nu = \frac{1}{2}$ and $E/E_{\tan} = 1$, $\beta = \frac{1}{2}$ and K assumes the purely elastic value

$$K = \frac{1}{\sqrt{2}} \sigma^\infty.$$

But more important, when $E/E_{\tan} \gg 1$, $\beta \cong \frac{1}{2}$ and the amplitude is given by

$$K = (1-\nu^2)^{\frac{1}{2}} \sqrt{(\tfrac{2}{3})} \left(\frac{E_{\tan}}{E}\right)^{\frac{1}{2}} \sigma^\infty. \tag{45}$$

Thus, the stresses at the tip are the linear elastic ones times the factor [using (45) for $E_{\tan}/E \ll 1$]

$$\frac{2}{\sqrt{3}}(1-\nu^2)^{\frac{1}{2}} \left(\frac{E_{\tan}}{E}\right)^{\frac{1}{2}}.$$

A comparison with the corresponding factor for plane stress indicates that the stresses will be higher in plane strain than in plane stress—varying from 15 for $\nu = 0$ to 11 percent higher for $\nu = \frac{1}{4}$ to identical values when $\nu = \frac{1}{2}$.

It need hardly be remarked that the $E_{\tan} = 0$ limit of the present solution does not correspond to the perfect-plasticity solution. This case presents some very special problems of its own. RICE (1967) has presented an approximate analysis of the tensile crack using perfect plasticity theory. He employs his integral to connect the slip line field at the crack tip with the far elastic field. At the tip the stresses are bounded and the strains vary as $1/r$.

7. IMPLICATIONS OF THE STRESS ANALYSIS

Certain details of the stress distribution at the crack tip depend rather strongly on the model chosen to represent the material, while others do not. For example, the piecewise-linear calculation indicates an $r^{-\frac{1}{2}}$ variation of the stress; and the Ramberg–Osgood relation predicts an $r^{-1/(n+1)}$ variation as the crack tip is approached. On the other hand, the predictions for the energy density at the crack tip are relatively insensitive to the model. One should not lose sight of the fact that the dominant singularities determined in this paper are only accurate representations of the solutions in a very small portion of the plastic zone. Until the solution for the entire plastic zone and surrounding elastic region is known one can only extrapolate the character of the singularity into the region of comparable elastic and plastic strains.

Two formulas emerge from the present analysis which have been shown by BATEMAN, BRADSHAW and ROOKE (1964) to correlate reasonably well with tests on thin, cracked aluminium sheet. Bateman *et al.* measured through-thickness thinning which occurred in the plastic zone at the crack tip. A formula suggested by DIXON and STRANNIGAN (1963) for the strain component normal to the sheet, namely

$$\epsilon_z \sim \left(\frac{E}{E_{sec}}\right)^{\frac{1}{2}} \left(\frac{\bar{r}}{L}\right)^{-\frac{1}{2}} \frac{\bar{\sigma}^{\infty}}{\bar{\sigma}_y}, \tag{46}$$

was found to give a fairly good fit for the strain ahead of the crack (E_{sec} is the secant modulus). Even better agreement was achieved with a formula suggested by McClintock (1961)

$$\epsilon_z \sim \left(\frac{\bar{\sigma}^{\infty}}{\bar{\sigma}_y}\right)^2 \left(\frac{\bar{r}}{L}\right)^{-1}. \tag{47}$$

It is a simple matter to show that the strain just ahead of the crack as predicted by the piecewise-linear theory of plane stress for small-scale yielding is

$$\frac{\bar{\epsilon}_z}{\epsilon_y} = -\frac{1}{\sqrt{2}} \left(\frac{E}{E_{tan}}\right)^{\frac{1}{2}} \left(\frac{\bar{r}}{L}\right)^{-\frac{1}{2}} \frac{\bar{\sigma}^{\infty}}{\bar{\sigma}_y}. \tag{48}$$

The similarity between (48) and (46) follows from the fact that E_{tan} can be identified with E_{sec} for large strains.

Secondly, the plane stress predictions for the Ramberg–Osgood relation is (apart from a constant which can be obtained from the numerical results),

$$\frac{\bar{\epsilon}_z}{\epsilon_y} \sim \left(\frac{\bar{\sigma}^{\infty}}{\bar{\sigma}_y}\right)^{\frac{2n}{n+1}} \left(\frac{\bar{r}}{L}\right)^{-\frac{n}{n+1}}. \tag{49}$$

For large n this formula takes on the appearance of the one suggested by McClintock.

Without a doubt, the most interesting theoretical result coming out of the present calculations is the prediction of a higher tensile stress singularity at the tip of a 'plane strain crack' than at the tip of a corresponding 'plane stress crack.' It is recalled the corresponding in-plane stresses are identical according to the linear theory. The predictions of the piecewise-linear theory show only a relatively small increase (about 11 percent for $\nu = \frac{1}{2}$). However, as discussed previously, the stresses ahead of the plane strain crack as predicted by the single term (40) may well be in error, and this particular result must be regarded with some suspicion.

The Ramberg–Osgood predictions, on the other hand, show a very clear trend for relatively larger tensile stresses at the tip of a crack under plane strain conditions. Using the small-scale yielding solutions, we find the ratio of the tensile stresses ahead of the cracks to be

$$\frac{\sigma_\theta(\theta = 0)_{p.\,strain}}{\sigma_\theta(\theta = 0)_{p.\,stress}} = \left[\frac{(1 - \nu^2)\, I_{p.\,stress}}{I_{p.\,strain}}\right]^{\frac{1}{n+1}} \cdot \frac{\tilde{\sigma}_\theta(0)_{p.\,strain}}{\tilde{\sigma}_\theta(0)_{p.\,stress}}. \tag{50}$$

This ratio is plotted in Fig. 7 as a function of n.

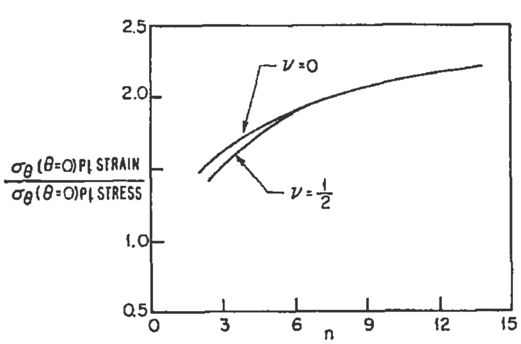

Fig. 7. Ratio of plane strain amplitude of tensile stress singularity ahead of crack to plane stress amplitude.

It is now well known that a ' thin ' cracked plate often fractures at an applied stress significantly higher—a factor of 2, typically—than its ' thick ' counterpart, with all other factors equal. A plate is regarded as ' thin ' if the plastic zone at fracture spans several plate thicknesses while it is ' thick ' if the plastic zone is small compared to the thickness. Experimental results of WINNE and WUNDT (1958) have demonstrated this effect for monotonic loading. BROEK and SCHIJVE (1965) have found that fatigue cracks grow faster in thick aluminium sheets than in thin ones. COTTRELL (1965) and IRWIN (1960) have discussed this effect in some detail. Generally speaking, the thin plate situation is approximated by a plane stress assumption while the behaviour in the interior at the tip of a crack in a thick plate is approximately governed by the conditions of plane strain. Three dimensional effects are prominent under both conditions. Two very different modes of deformation are observed at the tip of a cracked thin sheet of mild steel on the one hand and at the tip of a crack in a thin sheet of a smooth yielding material such as aluminium, say [see, for example, BATEMAN *et al.* (1964), GERBERICH (1964) or HAHN and ROSENFIELD (1965)]. The plane stress analysis of the present paper is not applicable to the mild steel mode of deformation; but it does model, if only approximately, the deformation at the tip of a crack in a material such as aluminium. Thus, for such materials there is a very definite theoretical indication that the plane strain fracture toughness as reflected by the tensile singularity ahead of the crack is significantly lower than the plane stress toughness.

We note that the plastic zone will fully engulf the crack tip in a Ramberg–Osgood-type material. Nothing can be said about the extent of the plastic zone until the full solution is known. The small-scale yielding problem should be amenable to a fairly straightforward numerical analysis with the dominant singularity known.

The stress distribution around the plane stress crack is rather unusual (see Figs. 3a and b). GERBERICH (1964) and SWEDLOW and GERBERICH (1964) have very carefully analysed the strain field at the tips of cracked sheets of aluminium alloys characterized by various hardening properties. Here we present only results which can be compared with their isochromatic pictures of the plastic zone obtained by a photo-elastic coating method. The contours in the isochromatic picture correspond to constant values of the principal strain difference, which in the present theory for plane stress is given by

$$\epsilon_{\mathrm{I}} - \epsilon_{\mathrm{II}} \sim r^{-\frac{n}{n+1}}\, \tilde{\sigma}_e^{\,n-1} \left[(\tilde{\sigma}_r - \tilde{\sigma}_\theta)^2 + 2\tilde{\sigma}_{r\theta}^{\,2}\right]^{\frac{1}{2}}.$$

Sketches of contours of constant principal strain difference, $\epsilon_{\mathrm{I}} - \epsilon_{\mathrm{II}}$, are shown in Fig. 8 for the purely elastic solution (as given by the dominant singularity) and for the plastic solution for $n = 3$ and $n = 13$. The correspondence with the experimental results in the papers of Gerberich and Swedlow is surprisingly good.

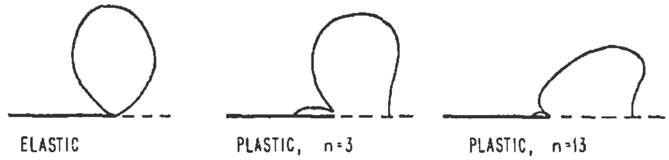

FIG. 8. Sketches of curves of constant principal strain difference ($\epsilon_{\mathrm{I}} - \epsilon_{\mathrm{II}}$) for plane stress.

8. Proportionality of the Deformation Theory Solution

According to our earlier discussion, the loading at any point will be nearly proportional under increasing loading if the stress distribution along the corresponding radial ray of the similarity solution is nearly proportional. The piecewise-linear solution should meet this condition since the proportions of the stress components in the fully plastic zone are almost identical to those in the elastic zone. This may not be so for the Ramberg–Osgood solution. We note now that the proportions in these two zones are quite different on the same ray. Thus a check of the full solution from the point of view of Budiansky's (1959) acceptability criterion seems warranted.

Prof. J. R. Rice has informed me that he and an associate, G. Rosengren, have also just completed a study along the lines contained in this paper. They recognized the validity of (15) and used this as their starting point. These writers have restricted consideration to the plane strain analysis and have presented a fairly complete discussion of this case.

Acknowledgment

This work was carried out while the writer was on leave at Oxford University in the Engineering Science Department under Prof. W. S. Hemp. Use of the facilities of the Oxford University Computing Laboratory is gratefully acknowledged.

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APPENDIX

Numerical Iterative Method for Solving the Nonlinear Eigenvalue Problem

AT THE crack tip the behaviour is governed by (13) for plane stress and (31) for plane strain together with the homogeneous boundary conditions. Each equation is homogeneous in $\tilde{\phi}$ and therefore if $\tilde{\phi}$ is a solution of either equation then so is any multiple of $\tilde{\phi}$. For the purpose of this discussion set $\tilde{\phi}(0) = 1$. Now if values are assigned to s and $\tilde{\phi}^{\cdot\cdot}(0)$ (with $\tilde{\phi}^{\cdot}(0) = \tilde{\phi}^{\cdot\cdot\cdot}(0) = 0$), a solution will be specified. In general, a function so specified will not satisfy the two boundary conditions at $\theta = \pi$ ($\tilde{\phi} = \tilde{\phi}^{\cdot} = 0$). A systematic method for determining the correct values of s and $\tilde{\phi}^{\cdot\cdot}(0)$ similar in spirit to that described by Fox (1962) was employed. Briefly, the end-point values of $\tilde{\phi}$ and $\tilde{\phi}^{\cdot}$ are regarded as functions of s and $\tilde{\phi}^{\cdot\cdot}(0)$.

The four derivatives

$$\frac{\partial\tilde{\phi}(\pi)}{\partial s}, \quad \frac{\partial\tilde{\phi}^{\cdot}(\pi)}{\partial s}, \quad \frac{\partial\tilde{\phi}(\pi)}{\partial\tilde{\phi}^{\cdot\cdot}(0)}, \quad \frac{\partial\tilde{\phi}^{\cdot}(\pi)}{\partial\tilde{\phi}^{\cdot\cdot}(0)}$$

are calculated at the start of each iteration. This step was performed numerically. Then these derivatives were used to step toward the desired values of s and $\tilde{\phi}^{\cdot\cdot}(0)$. Of course, once it is appreciated that $s = (2n + 1)/(n + 1)$ only the iteration on $\tilde{\phi}^{\cdot\cdot}(0)$ need be carried out.

A fourth-order Runge-Kutta method was used and the step size was varied in an appropriate manner.* This procedure converged very rapidly for the plane strain case—usually, with a

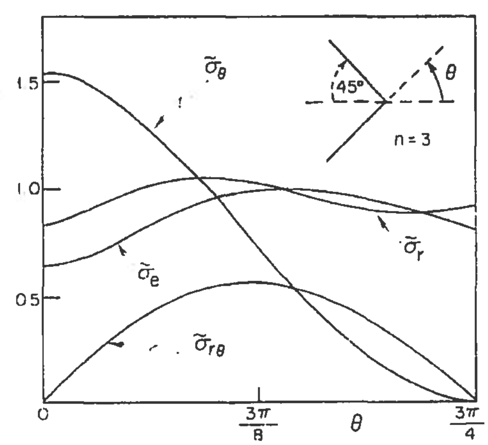

FIG. 9. θ-variations of stresses at tip of a 90° V-notch in plane strain.

*A Runge-Kutta programme which was especially efficient for fourth-order differential equations was kindly supplied by Dr. R. N. Franklin.

reasonably close first guess for s and $\tilde{\phi}''(0)$, in three or four iterations. Convergence was somewhat slower for plane stress. The integral (24) was evaluated with a standard integration procedure.

The numerical method outlined above can also be applied to find the dominant singularity at a V-notch. Of course, the amplitude is undetermined. Plastic deformation at notches has been investigated using slip line theory of perfect plasticity as reported by Hill (1950). Recently, Ewing and Hill (1967) have presented more results, again on the basis of perfect plasticity, on the stress character at notches and on the constraint effect of the notch. We present the stress distribution for only one example—that for the stresses at a 90° notch in a tension field perpendicular to its centre-line. These results, for $n = 3$ and plane strain, are shown in Fig. 9. The order of the singularity is only slightly reduced from the crack case. Recall for the crack with $n = 3$

$$\sigma_e \sim r^{-\frac{1}{4}} \tilde{\sigma}_e$$

while for the notch with a 90° opening angle

$$\sigma_e \sim r^{-0.225} \tilde{\sigma}_e.$$

Note added in proof

Following preparation of this paper we have discovered that J. D. Eshelby, working in a somewhat different context, has also given a path independent integral which is essentially the same as that given by Rice.

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