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Elastic Postbuckling Behavior of Stiffened and Barreled Cylindrical Shells[1]

The initial postbuckling behavior of axially stiffened cylindrical shells is studied with a view to ascertaining the extent to which various effects such as stiffener eccentricity, load eccentricity, and barreling influence the imperfection-sensitivity of these structures to buckling. In most cases, when these effects result in an increase in the buckling load of the perfect structure, they increase its imperfection-sensitivity as well. In some instances, however, barreling can significantly raise the buckling load of the shell while reducing its imperfection-sensitivity. The analysis, which is based on Koiter's general theory of postbuckling behavior and is made within the context of Kármán-Donnell-type theory, takes into account nonlinear prebuckling deformations and different boundary conditions.

Introduction

Recent buckling studies of stiffened cylindrical shells have pointed to a number of potentially useful ways to substantially increase the resistance of these structures to buckling at little or no weight expense. Some of the most interesting of these have been discussed by Almroth and Bushnell [1][2] and surveyed by Stein [2]. Included are such effects as stiffener eccentricity, load eccentricity and meridional curvature or barreling. It is not uncommon that computed buckling loads are raised by a factor of two and even more if these effects are properly taken advantage of. Tests have already verified the superior strength of outside-stiffened cylinders over their inside-stiffened counterparts. While most studies to date have not been based on optimal configurations, there seems to be little doubt that exploitation of these effects will lead to more efficient structures.

In this paper we examine the extent to which three of these effects, which have such a large influence on the classical buckling load, influence the structure's postbuckling behavior and its sensitivity to imperfections. Previous shell studies seem to indicate that it is not uncommon to find increased sensitivity going along with increased buckling loads. This was established in some instances for outside vs. inside axially stiffened cylinders under axial compression in a study which neglected the prebuckling deformations [3]. Here, a consistent approach is followed which takes into account the nonlinear prebuckling deformations. The analysis is based on Koiter's general theory of postbuckling behavior [4–6] and is made within the context of Kármán-Donnell-type shell theory. All calculations are exact although numerical results follow from numerical integration of ordinary differential equations.

A general outline of the analysis is presented in the next section. No attempt has been made to fill in all details since many of them are similar to those covered in Budiansky's analysis for the case in which the prebuckling deformations can be ignored [7].

Buckling and Postbuckling Analysis

Generalized Kármán-Donnell equations governing the elastic deformation of stiffened cylindrical shells have been given in References [3] and [8]. These equations are formulated on the basis of nonlinear strain measures of the Donnell-Mushtari-Vlasov type with the aid of the principle of virtual work. Stiffener properties are "smeared out" to arrive at effective bending and stretching stiffnesses for the skin-stiffener combination. The governing equations can be reduced to an equation for moment equilibrium and one compatibility equation written in terms of the normal outward deflection of the shell W and a stress function F, namely,

$$L_D[W] + L_Q[F] = F_{,xx}W_{,yy} + F_{,yy}W_{,xx} - 2F_{,xy}W_{,xy} \quad (1)$$

$$L_H[F] - L_Q[W] = W_{,xy}{}^2 - W_{,xx}W_{,yy} \quad (2)$$

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[2] Numbers in brackets designate References at end of paper.

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Nomenclature

a, b = postbuckling coefficients: see equation (9)
A_{xx}, A_{xy} = see equation (4)
A_s = cross-sectional area of stringer
$\bar{b}$ = imperfection-sensitivity measure
B_{xx} = see equation (4)
D = $Et^3/12(1 - \nu^2)$
D_{xx}, D_{xy}, D_{yy} = effective bending stiffnesses
d_s = stringer spacing
E = Young's modulus
e_s = distance from stringer centroid to skin middle surface—positive for an external stiffener
F = stress function
H = see Fig. 4
H_{xx}, H_{xy}, H_{yy} = effective stretching compliances
I_s = stringer moment of inertia
L = cylinder length
L_D, L_H, L_Q = differential operator defined in equation (3)
M_s = stringer bending moment
M_x, M_y, M_{xy} = skin bending moments
N_s = stringer contribution to resultant membrane stress
N_x, N_y, N_{xy} = resultant membrane stresses in skin
n = number of circumferential wavelengths
P = compressive axial load

(Continued on next page)

Table 1

$$(D_{xx},D_{xy},D_{yy}) = \frac{Et^3}{12(1-\nu^2)}(\bar{D}_{xx},\bar{D}_{xy},\bar{D}_{yy}) \; ; \; (Q_{xx},Q_{xy},Q_{yy}) = t(\bar{Q}_{xx},\bar{Q}_{xy},\bar{Q}_{yy})$$

$$(H_{xx},H_{xy},H_{yy}) = \frac{1}{Et}(\bar{H}_{xx},\bar{H}_{xy},\bar{H}_{yy}) \; ; \; (B_{xx},B_{xy},B_{yx},B_{yy}) = Et^2(\bar{B}_{xx},\bar{B}_{xy},\bar{B}_{yx},\bar{B}_{yy})$$

$A_{xx}=\nu\alpha_s/\alpha_o$	$A_{yy}=\nu\alpha_r/\alpha_o$	$A_{xy}=(1+\alpha_r)/\alpha_o$	$A_{yx}=(1+\alpha_s)/\alpha_o$
$\bar{B}_{xx}=\alpha_s\gamma_s(1+\alpha_r)/\alpha_o$	$\bar{B}_{yy}=\alpha_r\gamma_r(1+\alpha_s)/\alpha_o$	$\bar{B}_{xy}=\nu\alpha_s\alpha_r\gamma_r/\alpha_o$	$\bar{B}_{yx}=\nu\alpha_r\alpha_s\gamma_s/\alpha_o$
$\bar{D}_{xx}=1+\beta_s+\{12(1-\nu^2)\alpha_s(1+\alpha_r)\gamma_s^2\}/\alpha_o$	$\bar{D}_{yy}=1+\beta_r+\{12(1-\nu^2)\alpha_r(1+\alpha_s)\gamma_r^2\}/\alpha_o$	$\bar{D}_{xy}=1+\{12(1-\nu^2)\nu\alpha_s\alpha_r\gamma_s\gamma_r\}/\alpha_o$	$\bar{D}_{yx}=\bar{D}_{xy}$
$\bar{H}_{xx}=\{1+\alpha_s(1-\nu^2)\}/\alpha_o$	$\bar{H}_{yy}=\{1+\alpha_r(1-\nu^2)\}/\alpha_o$	$\bar{H}_{xy}=(1+\nu)-\nu/\alpha_o$	$\bar{H}_{yx}=\bar{H}_{xy}$
$\bar{Q}_{xx}=\nu\alpha_s\gamma_s/\alpha_o$	$\bar{Q}_{yy}=\nu\alpha_r\gamma_r/\alpha_o$	$\bar{Q}_{xy}=-[\alpha_s\gamma_s(1+(1-\nu^2)\alpha_r)+\alpha_r\gamma_r(1+(1-\nu^2)\alpha_s)]/2\alpha_o$	$\bar{Q}_{yx}=\bar{Q}_{xy}$

$\alpha_s = A_s/d_st \; , \; \beta_s = EI_s/Dd_s \; , \; \gamma_s = e_s/t$

$\alpha_r = A_r/d_rt \; , \; \beta_r = EI_r/Dd_r \; , \; \gamma_r = e_r/t$

$\alpha_o = (1+\alpha_s)(1+\alpha_r) - \nu^2\alpha_s\alpha_r$

s - stringers

r - rings

where x and y are the axial and circumferential coordinates in the shell middle surface. The linear differential operators are defined by

$$L_D[\;] = D_{xx}[\;]_{,xxxx} + 2D_{xy}[\;]_{,xxyy} + D_{yy}[\;]_{,yyyy}$$

$$L_Q[\;] = Q_{xx}[\;]_{,xxxx} + 2Q_{xy}[\;]_{,xxyy} + Q_{yy}[\;]_{,yyyy} + \frac{1}{R}[\;]_{,xx} + \frac{1}{R_x}[\;]_{,yy} \tag{3}$$

$$L_H[\;] = H_{xx}[\;]_{,xxxx} + 2H_{xy}[\;]_{,xxyy} + H_{yy}[\;]_{,yyyy}$$

where R and R_x are the circumferential and meridional radii of curvature and formulas for the effective bending and stretching stiffnesses were given in Reference [3] and are listed here in Table 1.[3]

In this paper consideration has been restricted to cylinders (and slightly barreled cylinders) with an isotropic skin and axial stiffening. Only three parameters are needed to characterize the stiffening properties if the torsional rigidity of the stringers is ignored. These are the area ratio A_s/d_st, the bending stiffness ratio EI_s/Dd_s and the eccentricity ratio e_s/t where A_s and I_s are the area and moment of inertia of the stringer, d_s is the distance between stringers and e_s, the distance from the skin middle surface to the stringer centroid, is taken to be positive when the stiffener is on the outer surface of the shell. The resultant membrane stresses in the skin N_x, N_y and N_{xy} and the averaged resultant membrane stress in the axial stringers N_s are related to the stress function and the normal displacement by (see Table 1 for the coefficients)

$$N_x + N_s = F_{,yy}; \quad N_y = F_{,xx}; \quad N_{xy} = -F_{,xy}$$
$$N_x = A_{xx}F_{,xx} + A_{xy}F_{,yy} + B_{xx}W_{,xx} \tag{4}$$

Fitch [9] and Cohen [10] have given general developments of Koiter's initial postbuckling theory which are directly applicable to the nonlinear shell theory used in this study. Their treatments are extensions of the analysis given in References [11] and [12] to include the effect of nonlinear prebuckling behavior. In the brief outline of the initial postbuckling analysis which follows, the general theory will not be repeated but results from it will be translated directly into the W-F notation of Kármán-Donnell theory. A precise specification of the different boundary conditions is given in the Appendix.

The *axisymmetric prebuckling deformation* of the perfect shell can be written as

$$W^0 = w^0(x, P) \tag{5}$$

[3] Equations (1–3) are slightly modified from the equations given in Reference [3] to include the effect of meridional curvature. They are restricted to shells which are shallow in the axial direction.

Nomenclature

P_C = classical buckling load (buckling load of perfect structure)

P_S = buckling load (maximum support load) of imperfect structure

$(P_C)_{\text{unstiff.}}$ = $2\pi[3(1-\nu^2)]^{-1/2}Et^2$ (classical buckling load of an infinitely long unstiffened cylinder)

q = distance from point of load application to skin middle surface—positive when outside middle surface

Q_{xx}, Q_{xy}, Q_{yy} = see equation (3)

R = circumferential radius of curvature (cylinder radius)

R_x = meridional radius of curvature

t = skin thickness

U = axial displacement of shell middle surface

V = circumferential displacement

W = normal displacement

x = axial coordinate

y = circumferential coordinate

Z = $(1-\nu^2)^{1/2}L^2/Rt$

α = see equation (20)

δ = buckling displacement amplitude

$\bar{\delta}$ = initial imperfection amplitude

Δ = average endshortening

ν = Poisson's ratio (taken to be 0.3 in the calculations)

θ = initial slope of postbuckling load vs. endshortening curve

$\theta_C{}^0$ = slope of prebuckling load vs. endshortening curve at critical valne

$$F^0 = -\frac{1}{2} y^2 \frac{P}{2\pi R} + f^0(x, P) \tag{5 (Cont.)}$$

where P is the total compressive load applied to the cylinder. Even though the prebuckling behavior is nonlinear, the two fourth order differential equations governing w^0 and f^0 are linear in these quantities. These two equations can, in turn, be reduced to a single fourth order equation for w^0:

$$C_A w^{0''''} + \left(C_B + \frac{P}{2\pi R}\right) w^{0''} + C_C w^0 = \frac{\nu A_{xy} P}{2\pi E t H_{xx} R^2} + \frac{P}{2\pi R R_x} \tag{6}$$

together with an auxiliary equation for f^0:

$$H_{xx} f^{0''} = Q_{xx} w^{0''} + \frac{1}{R} w^0 - \frac{\nu A_{xy} P}{2\pi E t R} \tag{7}$$

where $(\ )' \equiv \frac{d(\ )}{dx}$ and

$$C_A = D_{xx} + \frac{Q_{xx}^2}{H_{xx}}; \quad C_B = \frac{2Q_{xx}}{R H_{xx}}; \quad C_C = \frac{1}{R^2 H_{xx}} \tag{8}$$

The *classical buckling load* of the perfect structure is denoted by P_C and in all cases examined here is the load at which a nonaxisymmetric bifurcation from the prebuckling state occurs. An asymptotic perturbation expansion of the solution, valid in the neighborhood of this bifurcation point, can be obtained in the form

$$\frac{P}{P_C} = 1 + a(\delta/t) + b(\delta/t)^2 + \ldots \tag{9}$$

$$W = W^0 + \delta W^{(1)} + \delta^2 W^{(2)} + \ldots;$$
$$F = F^0 + \delta F^{(1)} + \delta^2 F^{(2)} + \ldots \tag{10}$$

where δ is the amplitude of the buckling mode $W^{(1)}$. The prebuckling solution is also expanded about the critical load so that

$$W^0 = W_C{}^0 + (P - P_C)\dot{W}_C{}^0 + \tfrac{1}{2}(P - P_C)^2 \ddot{W}_C{}^0 + \ldots$$
$$F^0 = F_C{}^0 + (P - P_C)\dot{F}_C{}^0 + \tfrac{1}{2}(P - P_C)^2 \ddot{F}_C{}^0 + \ldots \tag{11}$$

where $W_C{}^0 \equiv W^0(x, P_C)$, $\dot{W}_C{}^0 \equiv \left.\frac{\partial W^0}{\partial P}\right|_{P=P_C}$, $\ddot{W}_C{}^0 \equiv \left.\frac{\partial^2 W}{\partial P^2}\right|_{P=P_C}$, etc.
This expansion generates a sequence of linear boundary value problems. The problem for $\dot{W}_C{}^0$ and $\dot{F}_C{}^0$ follows directly from equations (5)-(7):

$$C_A \dot{w}_C{}^{0''''} + \left(C_B + \frac{P_C}{2\pi R}\right) \dot{w}_C{}^{0''} + C_C \dot{w}_C{}^0 = \frac{\nu A_{xy}}{2\pi E t H_{xx} R^2} + \frac{1}{2\pi R R_x} - \frac{1}{2\pi R} w_C{}^{0''} \tag{12}$$

$$H_{xx} \dot{f}_C{}^{0''} = Q_{xx} \dot{w}_C{}^{0''} + \frac{1}{R} \dot{w}_C{}^0 - \frac{\nu A_{xy}}{2\pi E t R} \tag{13}$$

The *classical buckling problem* is a linear eigenvalue problem governed by the following equations:

$$L_D[W^{(1)}] + L_Q[F^{(1)}] + \frac{P_C}{2\pi R} W_{,xx}{}^{(1)} - f_C{}^{0''} W_{,yy}{}^{(1)} - w_C{}^{0''} F_{,yy}{}^{(1)} = 0 \tag{14}$$
$$L_H[F^{(1)}] - L_Q[W^{(1)}] + w_C{}^{0''} W_{,yy}{}^{(1)} = 0$$

Separated solutions for the buckling mode can be obtained in the form

$$W^{(1)}(x, y) = w^{(1)}(x) \cos(ny/R)$$
$$F^{(1)}(x, y) = f^{(1)}(x) \cos(ny/R) \tag{15}$$

where n is the integer number of circumferential wavelengths associated with the buckling mode. The resulting ordinary differential equations which follow directly from equations (14) and (15) are not given here. Boundary conditions are listed in the Appendix.

The *second order boundary value problem* is governed by (here we anticipate $a = 0$)

$$L_D[W^{(2)}] + L_Q[F^{(2)}] + \frac{P_C}{2\pi R} W_{,xx}{}^{(2)} - f_C{}^{0''} W_{,yy}{}^{(2)} - w_C{}^{0''} F_{,yy}{}^{(2)} = -\tfrac{1}{2}(n/R)^2\{(f^{(1)}w^{(1)})'' + \cos(2ny/R)(f^{(1)''}w^{(1)} + f^{(1)}w^{(1)''} - 2f^{(1)'}w^{(1)'})\} \tag{16}$$
$$L_H[F^{(2)}] - L_Q[W^{(2)}] + w_C{}^{0''} W_{,yy}{}^{(2)} = \tfrac{1}{2}(n/R)^2\{\tfrac{1}{2}(w^{(1)2})'' + \cos(2ny/R)(w^{(1)''}w^{(1)} - w^{(1)'2})\}$$

These equations can be reduced to two systems of ordinary differential equations with the separation

$$W^{(2)} = w_\alpha(x) + w_\beta(x) \cos(2ny/R)$$
$$F^{(2)} = f_\alpha(x) + f_\beta(x) \cos(2ny/R) \tag{17}$$

Solutions of the above boundary value problems are sufficient to yield the most important information about the initial postbuckling behavior and the associated imperfection-sensitivity. In every case considered in this paper the first postbuckling coefficient, a, in equation (9) is identically zero since this relation is independent of the sign of the buckling displacement. Therefore, the initial relationship between the load and the buckling displacement amplitude hinges on the sign and magnitude of b. If b is negative the load carrying capacity diminishes following buckling and the shell is imperfection-sensitive, while if b is positive the structure retains some ability to support increased loads once bifurcation has taken place. Throughout this paper we have consistently identified δ with the amplitude of the buckling mode displacement by normalizing the maximum value of $W^{(1)}$ to be unity. Thus in equation (9), δ/t is the ratio of the buckling amplitude to the skin thickness and not the effective thickness of the skin-stiffener combination. The formula for b is (Fitch [9])

$$b = \frac{F^{(2)*}(W^{(1)}, W^{(1)}) + 2F^{(1)*}(W^{(1)}, W^{(2)})}{P_C[\dot{F}_C{}^{0*}(W^{(1)}, W^{(1)}) + 2F^{(1)*}(\dot{W}_C{}^0, W^{(1)})]} \tag{18}$$

where the following shorthand notation has been used

$$A^*(B, C) = \int_S [A_{,xx}B_{,y}C_{,y} + A_{,yy}B_{,x}C_{,x} - A_{,xy}(B_{,x}C_{,y} + B_{,y}C_{,x})]dS \tag{19}$$

and the integration is over the entire middle surface of the shell.

The *initial slope of the generalized load-deflection curve* just following bifurcation indicates further information concerning the extent to which buckling can be expected to be gradual or sudden under the two limiting conditions of loading, i.e., prescribed load and prescribed generalized displacement. In every example discussed in this paper the generalized displacement is the average end displacement, and for the cylindrical shell examples this can be wrtten as

$$\Delta = \int_S (U_{,x} - qW_{,xx})dS = \int_S (\epsilon_x - \tfrac{1}{2}W^2_{,x} - qW_{,xx})\,dS$$

where U is the axial displacement and q is the loading eccentricity. A Taylor expansion of Δ, using (9)-(11), yields

$$\frac{\Delta}{\Delta_C} = 1 + \alpha\left(\frac{P}{P_C} - 1\right) + \ldots \tag{20}$$

with a fairly lengthy expression for α in terms of $W^{(1)}$, $F^{(1)}$, etc., which will not be given here. A similar calculation yields an expression for α for the barreled cylinders.

Finally, a *measure of the imperfection-sensitivity* of the structure is most easily obtained by considering the effect of an initial deviation of the shell middle surface from the perfect configuration in the shape of the buckling mode. Thus with an initial imperfection $\bar{W} = \bar{\delta} W^{(1)}$, the maximum support load P_S is related to the classical buckling load P_C and the imperfection amplitude $\bar{\delta}$ by the asymptotic formula

$$
\begin{aligned}
P_S/P_C &= 1 - 3(2)^{-2/3}(-\bar{b})^{1/3}|\bar{\delta}/t|^{2/3} + \ldots \\
&\cong 1 - 1.89(-\bar{b})^{1/3}|\bar{\delta}/t|^{2/3} + \ldots
\end{aligned}
\tag{21}
$$

Here again, the imperfection amplitude has been normalized with respect to the skin thickness t. When the prebuckling state is a purely membrane one, $\bar{b} = b$, but in the present case in which prebuckling deformations cannot be ignored[4]

$$
\bar{b} = \frac{b\{F_C^{0*}(W^{(1)}, W^{(1)}) + F^{(1)*}(W_C^{0}, W^{(1)})\}^2}{\{P_C{}^0[\dot{F}_C{}^{0*}(W^{(1)}, W^{(1)}) + 2F^{(1)*}(\dot{W}_C{}^0, W^{(1)})]\}^2}
\tag{22}
$$

To complete the outline of the analysis sketched above we close this section with a brief discussion of the numerical method employed to evaluate the results given in the remaining sections. In nondimensional form the several boundary value problems and associated boundary conditions are specified by the following nondimensional quantities: EI_s/Dd_s, A_s/d_st, e_s/t, q/t, R/R_x, Z, nL/R and ν. The ordinary differential equations for $w_C{}^0$, $\dot{w}_C{}^0$, $w^{(1)}$, w_α, w_β, etc., are reduced to finite difference form and solved by use of a well-known Gaussian elimination scheme due to Potters [13]. The eigenvalue problem for the classical buckling load and mode can be solved in the usual way in which the lowest eigenvalue associated with integer values of n is found, and then the lowest of them all is identified with the classical load. Results were obtained by treating nL/R as a continuous variable and thus it was not necessary to specify the length-radius ratio of the shell. This procedure is consistent with the fact that the critical value of nL/R turns out to be a fairly large number and for an L/R of order unity or less n itself will be fairly large. Therefore, the results can be regarded as exact for any value of L/R such that the associated value of n is integer and approximate for other values of L/R with an error in the buckling load of order $1/n^2$ and an error in b, $\bar{b}$ and α of order $1/n$. In any case, application of the Donnell-Mushtari-Vlasov strain measures is restricted to shells for which $n > 5$, say. In every example presented in this paper, a single buckling mode is associated with the critical eigenvalue and the expansion (9)–(11) was made in anticipation of this fact.

The expressions for b, $\bar{b}$ and α can easily be reduced to ordinary integrations over the x coordinate and these integrations were performed with a standard numerical integration scheme. As a check on accuracy, some examples from reference [3], in which the prebuckling deformation was identically zero and for which closed form series solutions were available, were run as special cases of the present procedure. With sixty integration stations over the half length of the shell, accuracy to within about one tenth of one percent could be obtained for the quantities b, $\bar{b}$ and α. Slightly less accuracy should be expected for the cases in which prebuckling deformations are important. In this paper all results are for a Poisson's ratio of 0.3.

Effect of Stringer Eccentricity on Postbuckling Behavior

Two examples have been chosen to illustrate the effect of stringer eccentricity on the buckling load and on the initial postbuckling behavior of an axially stiffened cylinder under axial compression. The buckling load, the imperfection-sensitivity parameter $\bar{b}$ and the initial slope θ of the load-end shortening curve are plotted in Fig. 1 as a function of the length parameter

[4] There is a discrepancy between Fitch's result [9] for this expression and that of Cohen [10]. We have rederived the general result using each of the two formal procedures used by these authors and find Fitch's expression to be correct.

Table 2

Z	e_s/t	e_r/t	Z_r/R_r	$(P_C)_{stiff.}/(P_C)_{unstiff.}$	nL/R	b	$\bar{b}$	$\theta^\dagger$	$\theta_C{}^\dagger$
500	6	*	0	[illegible]	10.45	-.256	-.064	-135	43
1000	"	*		[illegible]	17.0	-.046	-.040	-135	"
2000	"	*		[illegible]	22.0	-.026	-.032	-135	"
3000		*	"	3.56	24.3	-.024	-.024	-135	"
5000	"	*	"	[illegible]	27.2	[illegible]	-.016	-135	"
500	-6	*	"	4.33	11.3	-.0065	-.0077	-135	"
1000	"	*		2.64	14.7	-.0094	-.0075	-133	"
2000	"	*	"	1.96	18.8	-.013	-.013	-133	"
3000	"	*	"	1.79	21.8	-.013	-.013	-133	"
5000	"	*	"	1.71	26.3	-.012	-.012	-134	"
1000	6	*	10	7.44	15.7	-.046	-.033	-135	"
"	"	*	20	8.64	15.0	-.044	-.026	-135	"
"	"	*	30	9.68	14.5	-.040	-.020	-135	"
"	"	*	50	10.59	13.9	-.033	-.013	-135	"
"	"	*	70	[illegible]	13.4	-.026	-.0055	-135	44
"	"	*	[illegible]	12.40	12.8	-.022	-.0036	-135	44
"	-6	*	10	3.28	14.2	-.016	-.017	-135	45
"	"	*	20	4.15	13.7	-.021	-.022	-134	"
"	"	*	30	5.20	13.3	-.023	-.023	-135	"
"	"	*	40	6.18	13.4	-.020	-.020	-134	"
"		*	50	6.85	14.6	-.012	-.012	-133	"
"	"	*	60	7.31	15.2	-.0086	-.0053	-131	"
300	6	0	0	3.79	6.5	-.012	-.0042	-142	36
500	"	"		5.04	10.6	-.029	-.013	-136	43
750	"	"	"	3.75	13.1	-.034	-.021	-133	44
1000	"	"	"	3.20	14.3	-.030	-.023	-133	45
300	-6	"	"	1.16	9.9	.011	.011	20	"
500	"	"	"	1.04	11.1	-.0024	-.0035	-17	"
750	"	"	"	.996	12.5	-.0039	[illegible]	-90	"
1000	"	"	"	[illegible]	13.5	-.0094	-.099	[illegible]	"
300	"	-6	"	[illegible]	7.0	-.020	-.017	[illegible]	[illegible]
500	"	"	"	[illegible]	9.4	-.013	-.015	[illegible]	45
750	"	"	"	1.72	11.3	-.014	-.014	-131	[illegible]
1000	"	"	"	[illegible]	12.4	-.014	-.014	[illegible]	45

* clamped boundary conditions

† given to closest degree

Z for a cylinder which is both simply supported and loaded at the skin middle surface. The same parameters are displayed in Fig. 2 for a completely clamped cylinder. Numerical data from which these figures were plotted are given in Table 2 along with numerical data for other quantities such as b, nL/R and $\theta_C{}^0$, the slope of the prebuckling load-end shortening curve at the critical load. The parameters $EI_s/Dd_s = 100$, $A_s/d_st = 1$ and $e_s/t = \pm 6$ characterize a cylinder with a moderate amount of stiffening. To emphasize the role of stiffening, we have normalized the buckling load of the stiffened shell $(P_C)_{stiff}$ by the "classical" buckling load of a long unstiffened cylinder with the same radius R and skin thickness t, i.e.,

$$
(P_C)_{unstiff} = 2\pi[3(1 - \nu^2)]^{-1/2}Et^2
$$

Location of the stringers on the outer surface of the shell clearly enhances the resistance to buckling. Such eccentricity effects were first noted by van der Neut [14] and have been further studied by a number of investigators, most recently by Block [15], [16] and Almroth and Bushnell [1] who, as in the present study, account for the prebuckling deformations.

Judging from the lower two plots in Figs. 1 and 2 the outside-stiffened cylindrical shell should be the more imperfection-sensitive of the two configurations at least over some of the range of Z. A similar conclusion was reached in Reference [3] on the basis of an analysis which neglected prebuckling deformations. The shell considered in Fig. 1 is identical to one of the examples analyzed in Reference [3] and a comparison of the two sets of results indicates that prebuckling deformations must be correctly accounted for if accurate quantitative results are to be obtained. However, the trends indicated by the simpler analysis of Reference [3] are correct.

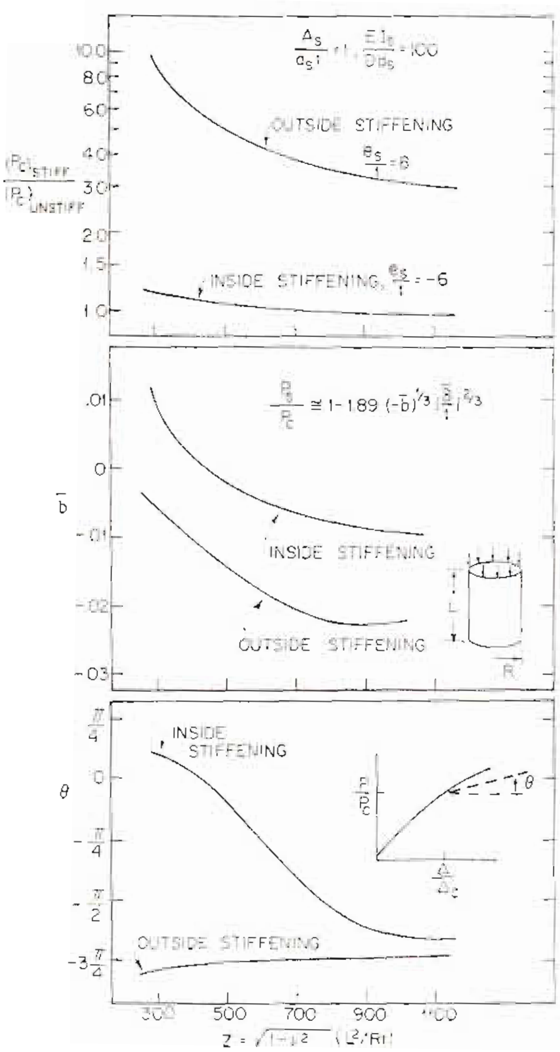

Fig. 1 The effect of stringer eccentricity on the buckling and postbuckling behavior of axially stiffened cylindrical shells which are simply supported at the skin middle surface and loaded in axial compression

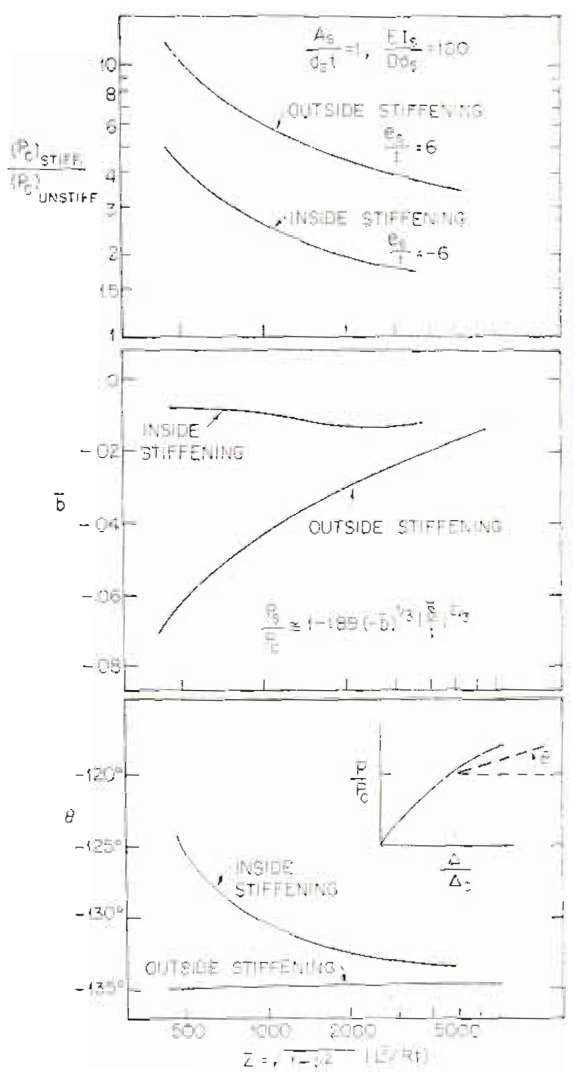

Fig. 2 The effect of stringer eccentricity on the buckling and postbuckling behavior of clamped, axially stiffened cylindrical shells which are loaded in axial compression

The results shown in Fig. 2 indicate that the imperfection-sensitivity of this shell as measured by $\bar{b}$ diminishes steadily as Z becomes large. Nevertheless, the initial slope θ of the load-end shortening curve is very negative which suggests that buckling (of a nearly perfect shell, at least) will not be gradual even under prescribed end displacement even though the sensitivity to imperfections in shape is very low. This example emphasizes that $\bar{b}$ and θ must, in general, be regarded as measures of two different characteristics of a structure. Except for the torsional stiffness of the stringers which has been neglected in this study, the parameters characterizing the shell in Fig. 2 correspond quite closely to some of the test specimens discussed by Card and Jones [17]. The values of Z associated with these specimens were well above 1000, and the buckling behavior of the outside-stiffened cylinders did seem to be consistent with the theoretical prediction of catastrophic buckling under prescribed end displacement coupled with relatively low imperfection-sensitivity.

Effect of Load Eccentricity on Postbuckling Behavior

The simply supported cylinders discussed in conjunction with Fig. 1 were taken to be supported at the skin middle surface with the axial load acting through the skin middle surface as well. Thus the axial load also induces an end moment about the effective centroid of the skin-stiffener combination. This moment, in turn, induces a compressive hoop stress in the inside-stiffened cylinder which tends to lower the shell's buckling load. If instead, the inside-stiffened cylinder were supported at, for example, the centroid of the stringers, then the prebuckling moment would have the opposite effect. Block [15], [16] and Almroth and Bushnell [1] have demonstrated the importance of this effect by way of a number of examples.

In Fig. 3 we present one example which illustrates the extent to which the load eccentricity influences the initial postbuckling behavior. The inside-stiffened shell marked "loaded at skin centroid" is the same as that considered in Fig. 1. The curves labeled "loaded at stringer centroid" are for the same shell but, as discussed above, simply supported at the centroid of the stringers (see the Appendix for a complete specification of the boundary conditions). Loading eccentricity is clearly a very important factor in determining the buckling load particularly in the lower range of Z. At the same time, the imperfection-sensitivity and the tendency for catastrophic buckling go up sharply along with the buckling load, at least in this one example.

Effect of Barreling on Postbuckling Behavior

The example we have chosen to illustrate the effect of a small meridional curvature on the buckling and postbuckling behavior of a clamped, axially stiffened cylindrical shell under axial compression is shown in Fig. 4. The normalized buckling load together with the parameters $\bar{b}$ and θ are plotted for one value of the length parameter Z as a function of the parameter which characterizes the amount of barreling.

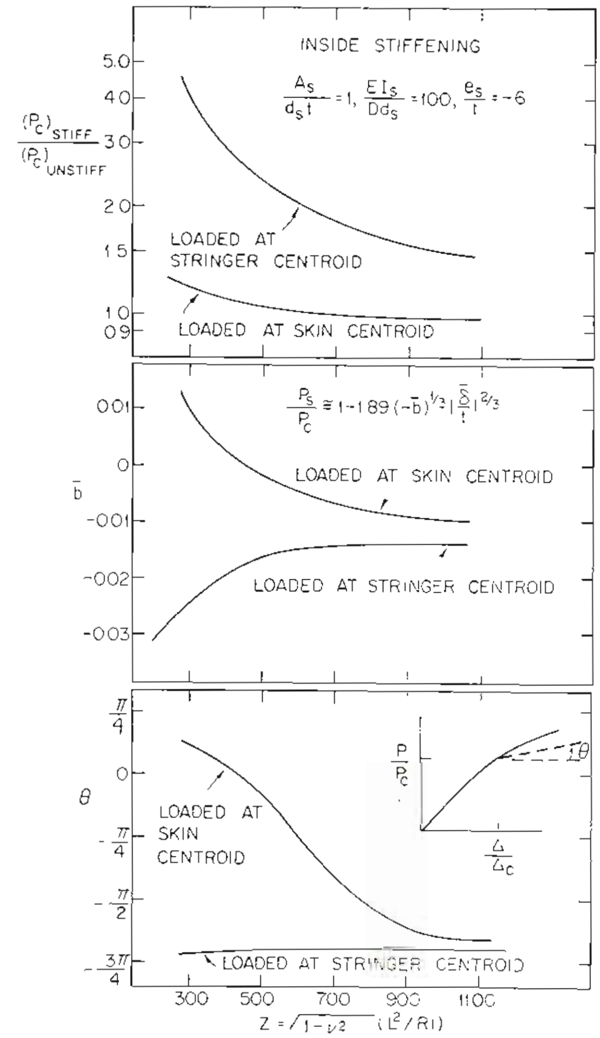

Fig. 3 The effect of load eccentricity on the buckling and postbuckling behavior of axially stiffened cylindrical shells which are simply supported at either the centroid of the skin or the centroid of the stringers and subject to axial compression

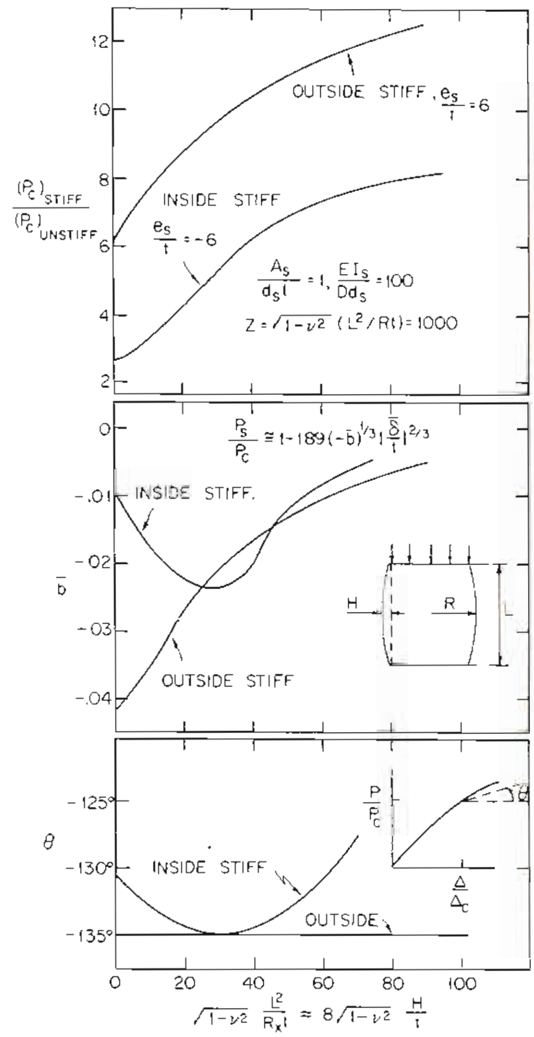

Fig. 4 The effect of barreling on the buckling and postbuckling behavior of clamped, axially stiffened cylindrical shells subject to axial compression

$$\sqrt{1-\nu^2}\,\frac{L^2}{R_x t} \cong 8\,\sqrt{1-\nu^2}\,\frac{H}{t}$$

The second expression above holds when the rise H at the equator is small compared to the length L. Even slight barreling, corresponding to a rise at the equator of only several skin thicknesses, has a pronounced effect on the buckling load.

The outside-stiffened shell is a good deal more imperfection-sensitive than the inside-stiffened one in this example when neither is barreled. Any amount of barreling seems to diminish the sensitivity of the outside-stiffened cylinder, while it increases the sensitivity of the inside-stiffened one over the range in which its buckling load rises rapidly with increasing barreling. However, once the barreling is sufficiently large to cause little further increase in the classical load, then even greater barreling seems to have as its main effect a decrease in the imperfection-sensitivity.

Almroth and Bushnell [1] and Stein [2] discussed examples in which barreling has a much larger effect on the classical buckling load of an inside-stiffened cylindrical shell than its outside-stiffened counterpart so that with a slight amount of barreling the inside-stiffened shell actually has a higher buckling load. We have studied a simply supported shell of this type. The postbuckling behavior predicted is similar to that shown in Fig. 4 for the clamped shells, except that the increase in imperfection-sensitivity of the inside-stiffened cylinder is even much more pronounced in the range in which the buckling load rises rapidly with only small increases in barreling.

Concluding Remarks

As a rough rule, it would appear that most mechanisms of the type discussed here for increasing the buckling load of a stiffened cylindrical shell increase to some extent its imperfection-sensitivity as well. A similar tendency has been noted for other shell configurations—see, for example, the survey by Budiansky and Hutchinson [18] and the paper by Koiter [6]. An important exception to this rule seems to occur in some instanees when barreling is employed to increase buckling resistance as has been discussed in the previous section. More extensive studies of this aspect of barreling would probably be worthwhile.

In general, stiffening lowers imperfection-sensitivity. Some tests on carefully prepared cylinders with moderately heavy stiffening suggest that loads close to those predicted for the perfect structures can be obtained, although it is not entirely clear what boundary conditions actually have been enforced in a number of these tests [1], [2]. More lightly stiffened cylinders show much greater discrepancies between predictious for the perfect shells and tests [1]. Recently the assertion has been made to the effect that the classical buckling analysis should adequately predict the actual buckling loads of cylindrical shell structures with practical levels of stiffening [19], [20]. The present results suggest that perhaps such a blanket assertion may be incautious until further studies and tests to determine optimal configurations of stiffening, barreling, etc., have been carried out. This note of caution seems to be further justified since catastrophic buckling has been observed in a number of the tests reported in the recent literature.

APPENDIX

Boundary Conditions

Clamped Boundary Conditions

$$W = W_{,x} = 0 \qquad x = 0, L \tag{23}$$

$$V = 0 \qquad x = 0, L \tag{24}$$

$$\left.\begin{aligned} U &= 0 \qquad x = 0 \\ U_{,y} &= 0 \qquad x = L \end{aligned}\right\} \tag{25}$$

Equations (23) and (24) imply $\epsilon_y = 0$ or in terms of W and F:

$$(1 - \nu A_{xx})F_{,xx} - \nu A_{xy}F_{,yy} - \nu B_{xx}W_{,xx} = 0 \tag{26}$$

Equations (23), (24), (25) imply $2\epsilon_{xy,y} - \epsilon_{y,x} = 0$ or in terms of W and F:

$$(-2(1 + \nu) + \nu A_{xy})F_{,xyy} - (1 - \nu A_{xx})F_{,xxx} + \nu B_{xx}W_{,xxx} = 0 \tag{27}$$

Thus, the clamped boundary conditions, written in terms of W and F, are equations (23), (26), and (27). The boundary conditions for each of the sequence of boundary value problems is obtained by substitution of the perturbation expansion into these conditions. It should also be mentioned that the requirement that the tangential displacements U and V be single-valued in any circuit about the circumference of the shell can be shown to be satisfied for each of the boundary value problems considered.

Simple Support Boundary Conditions

$$W = 0 \qquad x = 0, L \tag{28}$$

$$V = 0 \qquad x = 0, L \tag{29}$$

$$N_x + N_s = F_{,yy} = -P/2\pi R \qquad x = 0, L \tag{30}$$

$$M_{xx} + M_s/d_s + (N_x + N_s)(e_s - q) - N_s e_s = 0 \qquad x = 0, L \tag{31}$$

The load eccentricity q and the stringer eccentricity e_s are measured from the skin middle surface and are taken to be positive in the outer direction. Equation (29) implies (26). Equation (31) can be written as

$$(D + EI_s/d_s)W_{,xx} + F_{,yy}(q - e_s) + e_s(A_{xx}F_{,xx} + A_{xy}F_{,yy} + B_{xx}W_{,xx}) = 0 \tag{32}$$

Thus, equations (26), (28), (30), and (32) are the boundary conditions considered here. Substitution of the perturbation expansion into these equations yields the conditions for each problem; for example,

$$\left.\begin{aligned} & w^0 = 0 \\ & (D + EI_s/d_s)w^{0''} - (q - e_s)P/2\pi R \\ & \quad + e_s(A_{xx}f^{0''} - A_{xy}P/2\pi R + B_{xx}w^{0''}) = 0 \end{aligned}\right\} x = 0, L$$

and

$$W^{(i)} = W_{,xx}^{(i)} = F_{,yy}^{(i)} = F_{,xx}^{(i)} = 0 \qquad x = 0, L$$

for $i = 1, 2$.

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