On Optimal Arches[1]

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THE RECENT contribution by Wu [1][5] on a perturbation method for solving an optimal-arch buckling problem prompts the present Note on a class of such problems. Attention is restricted to inextensional buckling in their planes of uniformly loaded simply supported arches, Fig. 1; the results are consequently applicable only to arches of opening angle 2α sufficiently large that antisymmetrical buckling is critical, rather than the symmetrical snapping associated with shallow arches.

Normal Pressure

If the load per unit length q is considered to remain normal to the arch, the appropriate Rayleigh quotient for inextensional antisymmetrical buckling is

$$q = \frac{\int_0^L EI\left(\frac{d^2w}{ds^2} + \frac{w}{R^2}\right)^2 ds}{R\int_0^L \left[\left(\frac{dw}{ds}\right)^2 - \left(\frac{w}{R}\right)^2\right] ds} \tag{1}$$

We seek an area distribution $A(s)$ that maximizes q, subject to the volume constraint

$$\int_0^L A(s)ds = V \tag{2}$$

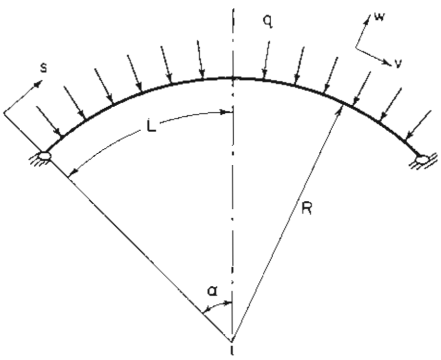

Fig. 1 Arch geometry, displacement, and loading

and a prescribed law of dependence on A of the moment of inertia I. The class of such laws to be considered is

$$\frac{I}{VL} = k\left(\frac{AL}{V}\right)^n \tag{3}$$

(The cases $n = 1, 2, 3$ correspond, respectively, to light-core sandwiches of constant width and depth, solid cross sections of fixed shape, and solid rectangular sections of constant width. Other values of n might describe, at least approximately, more complicated design constraints.) Then, nondimensionally, we have to maximize with respect to τ and minimize with respect to w

$$\lambda = \frac{\int_0^1 \tau^n(w'' + \alpha^2 w)^2 d\xi}{\int_0^1 [(w')^2 - \alpha^2 w^2] d\xi} \tag{4}$$

where

$$\lambda = \frac{qRL}{kEV}, \quad \tau = \frac{AL}{V}, \quad \xi = \frac{s}{L}, \quad \text{and} \quad (\ )' \equiv \frac{d}{d\xi}(\ ).$$

The volume constraint becomes

$$\int_0^1 \tau d\xi = 1 \tag{5}$$

The Euler equations implied by $\delta\lambda = 0$ for admissible δw for simply supported arches are

$$(\tau^n z)'' + \alpha^2\tau^n z + \lambda z = 0$$
$$\tau^n z = 0 \quad \text{at} \quad \xi = 0, 1 \tag{6}$$

where $z = w'' + \alpha^2 w$.

The optimality condition found by varying (4) with respect to τ, with the side condition (5), is

$$\tau^{n-1}z^2 = \text{const} \tag{7}$$

(In obtaining (7), it is recognized that the variation in w produced by a variation in τ does not, to first order, contribute to the variation of λ, since λ is stationary with respect to the eigenfunction w.) Elimination of z from (6) and (7) gives

$$\left(\tau^{\frac{n+1}{2}}\right)'' + \alpha^2\tau^{\frac{n+1}{2}} + \lambda\tau^{-\left(\frac{n-1}{2}\right)} = 0$$
$$\tau(0) = \tau(1) = 0 \tag{8}$$

For the case $n = 1$, the system (8), in conjunction with the constraint (5), has the solution

$$\lambda = \alpha^3\left(2\tan\frac{\alpha}{2} - \alpha\right)^{-1} \tag{9}$$

and

$$\tau = \frac{2\lambda}{\alpha^2}\,\frac{\left(\sin\frac{\alpha\xi}{2}\sin\frac{\alpha(1-\xi)}{2}\right)}{\cos\frac{\alpha}{2}} \tag{10}$$

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The uniform arch has the eigenvalue $\lambda_0 = \pi^2 - \alpha^2$, so that, for $n = 1$,

$$\frac{\lambda}{\lambda_0} = \frac{\alpha^3}{(\pi^2 - \alpha^2)\left(2 \tan \dfrac{\alpha}{2} - \alpha\right)} \tag{11}$$

This relation is plotted in Fig. 2 for $0 \leqq \alpha \leqq \pi$.

For other values of n, closed solutions of (8) do not appear possible. But, by symmetry, $\tau'(1/2) = 0$, and so a first integral of (8) can be found in the form

$$\left(\tau^{\frac{n+1}{2}}\right)' = [\lambda(n+1)\hat{\tau} + \alpha^2\hat{\tau}^{n+1} - \lambda(n+1)\tau - \alpha^2\tau^{n+1}]^{1/2}$$

where $\hat{\tau} \equiv \tau(1/2)$. Then, with the help of the substitutions

$$\hat{\tau}^n = \frac{\lambda(n+1)\epsilon}{\alpha^2} \tag{12}$$

and

$$x = \tau/\hat{\tau} \tag{13}$$

the dependence of λ on α can be expressed parametrically by way of ϵ in the form

$$\lambda = [n+1]\frac{\left[\displaystyle\int_0^1 \frac{x^{\frac{n-1}{2}}\,dx}{(1+\epsilon - x - \epsilon x^{n+1})^{1/2}}\right]^{n+2}}{\left[\displaystyle\int_0^1 \frac{x^{\frac{n+1}{2}}\,dx}{(1+\epsilon - x - \epsilon x^{n+1})^{1/2}}\right]^{n}}$$

$$\alpha = (n+1)\sqrt{\epsilon}\int_0^1 \frac{x^{\frac{n-1}{2}}\,dx}{(1+\epsilon - x - \epsilon x^{n+1})^{1/2}}. \tag{14}$$

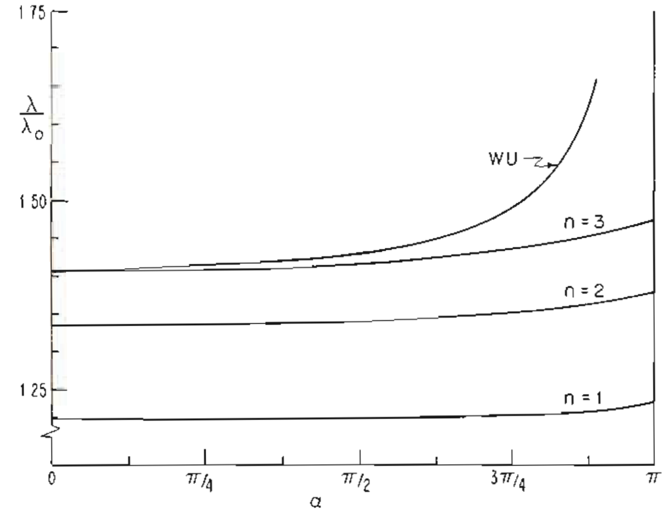

Fig. 2 Optimal buckling loads for arch under normal pressure

Note that $\alpha \to 0$ for $\epsilon \to 0$; and it is readily verified that $\alpha \to \pi$ for $\epsilon \to \infty$. Asymptotic results for λ/λ_0 at these limits are

$$\alpha = 0: \quad \frac{\lambda}{\lambda_0} = \frac{1}{\pi}\frac{(n+2)^n}{(n+1)^{n-1}}\left[\frac{\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n}{2}+1\right)}\right]^2 \tag{15}$$

$$\alpha = \pi: \quad \frac{\lambda}{\lambda_0} = \frac{\pi^{\frac{n+1}{2}}}{2}\left[\frac{\Gamma\left(\dfrac{n+2}{n+1}\right)}{\Gamma\left(\dfrac{n+3}{2n+2}\right)}\right]^{n-1} \tag{16}$$

(At $\alpha = 0$, λ/λ_0 varies from $(12/\pi^2) \approx 1.216$ for $n = 1$ to $(2e/\pi) \approx 1.731$ for $n \to \infty$.)

Numerical integration of (14) for intermediate values of α, together with the end-point values provided by (15) and (16), give the results plotted in Fig. 2 for $n = 2$ and $n = 3$. Also shown are the approximate results of Wu [1] for the case $n = 3$, which, as seen, start losing accuracy rapidly as α approaches π.

The optimum shape $\tau(\xi)$ can be found from numerical integration of

$$\xi = \frac{\displaystyle\int_0^{\tau/\hat{\tau}} \frac{x^{\frac{n-1}{2}}\,dx}{(1+\epsilon - x - \epsilon x^{n+1})^{1/2}}}{2\displaystyle\int_0^1 \frac{x^{\frac{n-1}{2}}\,dx}{(1+\epsilon - x - \epsilon x^{n+1})^{1/2}}} \tag{17}$$

with

$$\hat{\tau} = \frac{\displaystyle\int_0^1 \frac{x^{\frac{n-1}{2}}\,dx}{(1+\epsilon - x - \epsilon x^{n+1})^{1/2}}}{\displaystyle\int_0^1 \frac{x^{\frac{n+1}{2}}\,dx}{(1+\epsilon - x - \epsilon x^{n+1})^{1/2}}} \tag{18}$$

Dead Pressure

It can be shown that if the loads q are constrained to retain their initial directions of applications during buckling, the Rayleigh quotient for antisymmetrical buckling of simply supported circular arches can be written as

$$q = \frac{\displaystyle\int_0^L EI\left(\frac{d\phi}{ds}\right)^2 ds}{R\displaystyle\int_0^L \phi^2 ds} \tag{19}$$

where $\phi = (dw/ds) - (v/R)$. Nondimensionalization gives

$$\lambda = \frac{\int_0^1 \tau^n(\phi')^2 d\xi}{\int_0^1 \phi^2 d\xi} \tag{20}$$

Note that (20) (which does not contain α) has precisely the same form as the Rayleigh quotient for simple-support column buckling. Hence, the results for optimal columns found in [2] for $n = 1$ and in [3] for $n = 2$ carry over directly to the arch under dead loading. In fact, these results are precisely those determined for $\alpha = 0$ in the case of normal pressure. Thus, for *dead* loading, λ/λ_0 for *all* α is given by equation (15). The corresponding optimal shapes are easily found to be given parametrically through a dummy variable ω by

$$\tau/\hat{\tau} = \sin^2 \omega$$

$$\xi = \frac{\Gamma\left(\frac{n+2}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)} \int_0^\omega (\sin \beta)^n \, d\beta \tag{21}$$

where

$$\hat{\tau} = \frac{n+2}{n+1} \tag{22}$$

The integral in (21) can, of course, always be evaluated analytically for integer values of n. For $n = 1$, the very simple relation $\tau/\hat{\tau} = 4\xi(1 - \xi)$ found in [2] is recovered.

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