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<u>ERRATA</u> - "Post-Bifurcation Behavior in the Plastic Range" by J. W. Hutchinson
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<u>Page 171</u> - Equation (4.2) should read as follows:

$$\tilde{\eta}_{ij} = \frac{1}{2}(\tilde{u}_{i,j}+\tilde{u}_{j,i}) + \frac{1}{2}(u^{0k}{}_{,i}\tilde{u}_{k,j}+u^{0k}{}_{,j}\tilde{u}_{k,i})$$

- Equation (4.3) should read as follows:

$$F(\lambda,\tilde{u}) = \int_V \{L_c^{ijk\ell}\tilde{\eta}_{ij}\tilde{\eta}_{k\ell} + \tau^{0ij}\tilde{u}^k{}_{,i}\tilde{u}_{k,j}\}dV$$

<u>Page 180</u> - Equation (6.35) should read as follows:

$$\overset{*}{u}_i = -(1+\beta)^2\lambda_2^2 g_c^{-1}\phi_i\xi^\beta \int_{\overset{*s}{z}_1}^{\overset{*}{z}_1} f(\zeta,\overset{*}{z}_2,\overset{*}{z}_3)d\zeta$$

- Line 4 after Equation (6.35):

".................... i.e. $f(\overset{*s}{z}_1,\overset{*}{z}_2,\overset{*}{z}_3) = 0$. The ...."

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POST-BIFURCATION BEHAVIOR IN THE PLASTIC RANGE

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SUMMARY

IN THE first part of the paper a simple model is used to introduce some of the analytical and physical features of post-bifurcation phenomena in continuous elastic–plastic systems. An analysis is presented for the initial post-bifurcation behavior of a class of elastic–plastic solids subject to loads characterized by a single load parameter. Bifurcations which occur at the lowest possible load are singled out for attention. The theory makes connection with Hill's general theory of bifurcation and uniqueness in elastic–plastic solids and Koiter's general approach to the initial post-buckling behavior of conservative elastic systems. Buckling of an axially compressed column in the plastic range is used to illustrate the theory.

1. INTRODUCTION

THE FIRST possible bifurcation to occur in a solid or structure compressed into the plastic range almost always takes place under *increasing* load according to the notions of SHANLEY (1947) and the general theory of bifurcation and uniqueness of HILL (1958, 1961). This is even true for imperfection-sensitive structures, such as many shells, which have an unstable initial post-bifurcation behavior in the elastic range and bifurcate under *decreasing* load. In the plastic range, nonlinearity in the stress–strain relation in the form of decreasing stiffness with increasing deformation contributes an additional destabilizing effect to the geometrical effects already present in the elastic range. Thus, it seems almost paradoxical that a structure with an unstable bifurcation behavior in the elastic range will undergo a stable bifurcation in the plastic range. An analysis of the compressive buckling of a simple imperfection-sensitive model by HUTCHINSON (1972) shows that while the load does increase above the lowest bifurcation load it does so by only a very small amount. For all practical purposes, the lowest bifurcation load is the maximum support load of the model in the absence of any imperfections. In fact, even the column under axial compression, which has a fully stable post-buckling behavior in the elastic range, becomes unstable under dead loading after just a slight rise in the load above the lowest bifurcation load if account is taken of the decreasing slope of the stress–strain curve (DUBERG and WILDER, 1952).

All this suggests that in many instances the stable portion of the post-bifurcation response is contained within a small neighborhood of the bifurcation point. Consequently, a perturbation expansion developed about the bifurcation point may often provide not only the behavior in the initial stable régime but also the transition to unstable behavior. An approach of this kind is undertaken here. Our analysis is similar in spirit to that employed by KOITER (1945, 1963) in his general theory of elastic post-buckling behavior. Some of the salient features of initial post-bifurcation behavior in the plastic range are first introduced with the aid of a simple model. Our starting point in the general continuum analysis is Hill's bifurcation theory for elastic–plastic solids. We specialize Hill's bifurcation criterion to a class of loadings characterized by a single load parameter, and we direct attention to post-bifurcation behavior associated with the lowest possible bifurcation load. A column under uniaxial compression is used to illustrate the general theory. For the convenience of the reader, an outline of the general analysis together with the introduction of notation and terminology is given in Section 3 following the discussion of the simple model.

2. POST-BIFURCATION BEHAVIOR OF A SIMPLE MODEL

The simple model shown in Fig. 1 is a "continuum" version of a model studied in a previous paper (HUTCHINSON, 1972). It combines the essential features of SHANLEY's (1947) model of a plastic column and VON KÁRMÁN, DUNN and TSIEN's (1940) model of an elastic imperfection-sensitive structure. The rigid-rod model can displace vertically as measured by u and can rotate as measured by θ so that the contraction of a spring attached at a distance x along the horizontal rod is

$$\varepsilon = u+x\theta. \tag{2.1}$$

The springs are taken to be continuously distributed so that the rate of change of the compressive force per unit length is given in terms of the local strain-rate by

$$\dot{s} = E_t\dot{\varepsilon} \tag{2.2}$$

for plastic loading and by

$$\dot{s} = E\dot{\varepsilon} \tag{2.3}$$

within the elastic range or for elastic unloading. The tangent modulus E_t is taken to be a smooth function of s or ε.

Nonlinear geometrical effects are incorporated into the model only through the nonlinear horizontal spring which develops a force $K = k_1L^2\theta^2+k_2L^2\theta^3+\ldots$ under rotation as indicated in Fig. 1. Vertical equilibrium requires that

$$\dot{P} = \int_{-L}^{L} \dot{s}\,dx \tag{2.4}$$

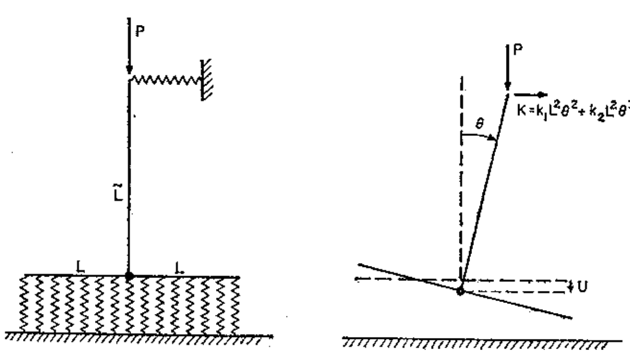

FIG. 1. A simple model for bifurcation of continuous elastic–plastic solids in the plastic range.

while moment equilibrium requires

$$(P\tilde{L}\theta)^{\cdot}+\dot{K}\,\tilde{L}=\int_{-L}^{L}\dot{s}\,x\,dx \tag{2.5}$$

where P is the applied compressive load.

When bifurcation occurs in the elastic range the critical load is $P_c = 2EL^3/(3\,\tilde{L})$ and the bifurcation point is an asymmetrical one (if $k_1 \neq 0$) in which the load and rotation following bifurcation are related by

$$\frac{P}{P_c}=1-\frac{3k_1\tilde{L}}{2EL}\,\theta-\frac{3k_2\tilde{L}}{2EL}\,\theta^2+\ldots. \tag{2.6}$$

For $k_1\theta > 0$ or for $k_1 = 0$ and $k_2 > 0$, the initial post-bifurcation behavior is unstable under dead load.

In the plastic range the first possible bifurcation occurs at the load $P_c = 2E_t^cL^3/(3\,\tilde{L})$ where E_t^c denotes the value of E_t at the bifurcation point. At any point where plastic loading continues,

$$E_t = E_t^c+\left(\frac{dE_t}{ds}\right)_c(s-s_c)+\tfrac{1}{2}\left(\frac{d^2E_t}{ds^2}\right)_c(s-s_c)^2+\ldots. \tag{2.7}$$

Before analyzing the model by properly taking into account the loading-unloading behavior characterized by (2.2) and (2.3), we digress to consider the bifurcation behavior of a model with nonlinear elastic stress-strain behavior characterized by (2.2) (i.e., the unloading branch (2.3) is suppressed). Extending the meaning of HILL's (1961) terminology we will refer to the model with the nonlinear elastic stress-strain relation as the *comparison model*. The lowest bifurcation load of the comparison model is still given by the tangent modulus formula, $P_c = 2E_t^cL^3/(3\,\tilde{L})$, and the first term in the post-bifurcation expansion is found to be

$$\frac{P}{P_c}=1+a_1^e\,\theta+a_2^e\,\theta^2+\ldots, \tag{2.8}$$

where

$$a_1^e=-\frac{3k_1\tilde{L}}{2E_t^cL}\left[1-\frac{L^2}{3\tilde{L}}\left(\frac{dE_t}{ds}\right)_c\right]^{-1} \tag{2.9}$$

and the superscript e is used to distinguish the initial slope of the comparison model from that of the elastic–plastic model which will be introduced later.

If there is no reversal in sign of the strain-rate within some range of positive (or negative) θ then the behavior of the comparison model will be identical to that of the elastic–plastic model in that same range of θ. The condition for this to be true is easily obtained. Strain-rate reversal will first occur at either $x = \pm L$ depending on whether $\dot{\theta} \gtrless 0$, respectively. Using the expansion for the comparison model one finds that

$$\dot{\varepsilon}=\frac{L^2}{3\tilde{L}}\left[a_1^e\pm\frac{3\tilde{L}}{L}\right]\dot{\theta}+O(\theta\dot{\theta}), \tag{2.10}$$

for $x = \pm L$ respectively. Thus, for example, if bifurcation takes place with $\dot{\theta} > 0$ and if $a_1^e > 3\,\tilde{L}/L$, then no strain-rate reversal will occur within some finite (perhaps small) range of positive θ. This can only occur if the bifurcation point of the comparison model is an asymmetrical one. Then due to the geometrical nonlinearity (as determined by k_1—see (2.9)), the load increases proportional to increasing θ at a sufficiently large rate to ensure that $\dot{\varepsilon} > 0$ everywhere on $|x| < L$. If this is the case then strain-rate reversal *will* occur in the comparison model at bifurcation for $\dot{\theta} < 0$. This latter case is by far the more important since on this branch the geometrical nonlinearity has a destabilizing effect; when the comparison model has an asymmetrical bifurcation point we will restrict consideration to this branch. Without loss in generality, we will take $k_1 \geq 0$ so that $a_1^e \leq 0$ and consider *bifurcation under monotonically increasing* θ.

The position of the instantaneous boundary between the plastically loading and elastically unloading regions is denoted by d. The boundary separating the loading region from the unloading region starts out at $d = -L$ and sweeps to the right in Fig. 1 as θ increases. Details of a direct method for obtaining the initial post-bifurcation expansions about P_c are given in the Appendix. The perturbation expansions for P and d are found to be

$$\frac{P}{P_c} = 1 + a_1\theta + a_2\theta^{3/2} + a_3\theta^2 + a_4\theta^{5/2} + \ldots, \tag{2.11}$$

$$\frac{d}{L} = -1 - \tfrac{3}{2}b_2\theta^{1/2} - 2b_3\theta - \tfrac{5}{2}b_4\theta^{3/2} + \ldots, \tag{2.12}$$

where the first few coefficients are given by

$$a_1 = \frac{3\tilde{L}}{L}, \tag{2.13}$$

$$a_2 = \frac{3\tilde{L}}{L} b_2 = -\frac{4\tilde{L}}{L}\left\{\frac{2E_t^c[3\tilde{L} - (dE_t/ds)_c L^2] + 3k_1\tilde{L}}{3(E - E_t^c)L}\right\}^{1/2}, \tag{2.14}$$

$$b_3 = \frac{E_t^c[3\tilde{L} - (dE_t/ds)_c L^2] - k_1\tilde{L}}{3(E - E_t^c)L}, \tag{2.15}$$

$$a_3 = \frac{3b_3\tilde{L}}{L} - \frac{3\tilde{L}^2}{L^2} + 3\tilde{L}\left(\frac{dE_t}{ds}\right)_c - \frac{3k_1\tilde{L}^2}{2E_t^c L^2}. \tag{2.16}$$

The initial post-bifurcation expansion (2.11) is distinguished from counterpart expansions for elastic systems (KOITER, 1945, 1963) and even from those for discrete element elastic–plastic systems (see, for example, SEWELL (1965), AUGUSTI (1968) and HUTCHINSON (1972)) by the presence of terms involving *fractional powers* of the bifurcation amplitude θ. The presence of these terms is associated with the continuous growth of the region of elastic unloading, and it will be shown that they are an essential part of expansions for elastic–plastic continua.

Bifurcation clearly takes place under increasing load since $a_1 > 0$ in (2.11). However, since the third term in the expansion, $a_2\theta^{3/2}$, is only of relative order $\theta^{1/2}$ smaller than $a_1\theta$ (as opposed to a relative order θ between terms for an elastic system), it may already become numerically important at small values of θ. Since a_2 is negative the maximum increase in load above P_c will be slight if $|a_2|$ is large compared to a_1. First, note from (2.6) that in the elastic range the model will have a steeply falling load-deflection behavior if $(k_1\tilde{L}/EL) \gg 1$. If this condition holds, then in the plastic range $|a_2| \gg a_1$ even if $(dE_t/ds)_c = 0$. Note also that a decreasing slope of the stress–strain curve (i.e. $(dE_t/ds)_c < 0$) also contributes to the magnitude of a_2 and further tends to diminish the maximum support load, as expected.

At bifurcation, plastic loading occurs everywhere according to (2.12) except at $x = -L$ where neutral loading occurs (i.e. $\dot{\varepsilon} = 0$). A Shanley-type bifurcation analysis only requires that no unloading take place at the lowest bifurcation load and this limits the possible branching solutions by the condition $a_1 \geq 3\tilde{L}/L$. However when higher order terms in the post-bifurcation expansion are considered, only the one solution (2.11) emanating from P_c is possible. In fact, if $a_1^e < 3\tilde{L}/L$ we can immediately argue that a_1 cannot exceed $3\tilde{L}/L$. Since if it did one can readily show that no elastic unloading would occur in some range of positive θ and thus the comparison model would pertain. But this is a contradiction since we have already shown that the behavior of the comparison model cannot coincide with that of the elastic–plastic model if $a_1^e < 3\tilde{L}/L$. Uniqueness of the lowest branching solution has also been shown for discrete Shanley-type models as discussed by SEWELL (1972).

Initial post-bifurcation responses are depicted in Fig. 2. The sketch on the left illustrates the case where the comparison model has an asymmetrical bifurcation

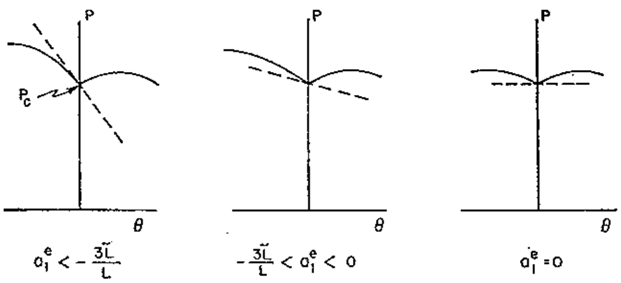

FIG. 2. Load–rotation behavior of simple model in the initial post-bifurcation régime. Dashed line indicates initial slope of comparison model.

point with $a_1^e < -3\tilde{L}/L$ so that for negative θ the initial slope of the elastic–plastic model is the same as that of the comparison model. In the center sketch, $-3\tilde{L}/L < a_1^e < 0$, so the initial response of the elastic–plastic model does not coincide with that of the comparison model on either branch. A case where the comparison model has a symmetrical bifurcation point ($a_1^e = 0$) is shown on the right. In each sketch the initial slope of the comparison model is displayed as a dashed line.

Expansions (2.11) and (2.12) break down in the "elastic limit" as $E_t^c \to E$, as is seen from the fact that a_2 and b_2 approach infinity. This is not unexpected since a_1 in (2.11) is determined by the condition that no unloading occurs at bifurcation and a_1 is positive, independent of E_t^c. The coefficient of θ in the elastic expansion (2.6) is determined from equilibrium considerations alone. In general, reversal of stress does occur at bifurcation in the elastic case and thus the expansion (2.11) for the plastic range does not yield (2.6) in the limit $E_t^c \to E$.

3. FIELD EQUATIONS FOR AN ELASTIC–PLASTIC SOLID AND AN OVERVIEW OF THE POST-BIFURCATION ANALYSIS

3.1 *Field equations*

Let material points in the body be identified by a set of convected coordinates x^i. Let g_{ij} and g^{ij} be the covariant and contravariant components, respectively, of the metric tensor of the undeformed body. We will use the standard convention that superscript indices correspond to the contravariant components of a tensor or vector and subscript indices denote covariant components. Let u_i be the components of the displacement vector referred to the reciprocal base vectors of the undeformed body. The Lagrangian strain tensor is given by

$$\eta_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i})+\tfrac{1}{2}u^k{}_{,i}\,u_{k,j}, \tag{3.1}$$

where the comma denotes covariant differentiation with respect to the metric of the undeformed body and the indices of the components of the displacement vector are raised and lowered using this same metric tensor.

Contravariant components of the symmetric Kirchhoff stress tensor defined with respect to base vectors deforming with the body are written as τ^{ij}. Components of the nominal surface traction vector (force per original area), T^i, are defined with respect to the undeformed base vectors and are related to the Kirchhoff stress tensor by

$$T^i = (\tau^{ij}+\tau^{kj}u^i{}_{,k})n_j, \tag{3.2}$$

where n_i are the components of the unit normal to the surface of the undeformed body referred to the undeformed base vectors. With the choice of variables the principle of virtual work is (see, for example, BUDIANSKY (1969))

$$\int_V \tau^{ij}\,\delta\eta_{ij}\,dV = \int_S T^i\,\delta u_i\,dS, \tag{3.3}$$

where

$$\delta\eta_{ij} = \tfrac{1}{2}(\delta u_{i,j}+\delta u_{j,i})+\tfrac{1}{2}(u^k{}_{,i}\,\delta u_{k,j}+u^k{}_{,j}\,\delta u_{k,i}) \tag{3.4}$$

and where dV and dS are volume and surface elements of the undeformed body.

Convected rates of the contravariant components of the Kirchhoff stress are denoted by $\dot{\tau}^{ij}$; and the strain-rates $\dot{\eta}_{ij}$ can be expressed in terms of the velocities $\dot{u}_i$ using (3.1). The incremental form of the principle of virtual work is (assuming no body forces)

$$\int_V \{\dot{\tau}^{ij}\,\delta\eta_{ij}+\tau^{ij}\,\dot{u}^k{}_{,i}\,\delta u_{k,j}\}\,dV = \int_S \dot{T}^i\,\delta u_i\,dS. \tag{3.5}$$

Surface traction rates are obtained from (3.2) and the incremental equilibrium equations are

$$\dot{\tau}^{ij}{}_{,j}+(\dot{\tau}^{kj}\,u^i{}_{,k})_{,j}+(\tau^{kj}\,\dot{u}^i{}_{,k})_{,j} = 0. \tag{3.6}$$

A class of flow theories discussed by HILL (1967) for solids characterized by a smooth yield surface is considered. Let m^{ij} denote the components of the unit tensor normal to the elastic domain in strain-rate space. Where the yield condition is satisfied, the relation between the stress-rate and the strain-rate is

$$\dot{\tau}^{ij} = L^{ijkl}\dot{\eta}_{kl} \qquad \text{for} \qquad m^{kl}\dot{\eta}_{kl} \geq 0, \tag{3.7}$$

$$\dot{\tau}^{ij} = \mathscr{L}^{ijkl}\dot{\eta}_{kl} \qquad \text{for} \qquad m^{kl}\dot{\eta}_{kl} \leq 0, \tag{3.8}$$

where

$$L^{ijkl} = \mathscr{L}^{ijkl} - \frac{1}{g}\,m^{ij}m^{kl}. \tag{3.9}$$

Here, $\mathscr{L}$ is the current tensor of elastic moduli (for this choice of objective stress-rate) and g is a positive constant which depends on the deformation history and which characterizes the current level of strain-hardening. When the stress lies within the yield surface (3.8) holds.

3.2 *Overview of the general analysis*

With the simple model results serving as a guide, we now give a preliminary discussion of a method for analyzing the initial post-bifurcation behavior in a three-dimensional elastic–plastic solid. Attention is restricted to a body subject to combinations of dead loads and prescribed displacements over its surface which are everywhere proportional to a single load (or deformation) parameter λ. Initially, as λ increases monotonically from zero, there is a unique solution τ^{0ij}, termed the *fundamental solution,* from which bifurcation is first possible at $\lambda = \lambda_c$. For simplicity it will be assumed that there is a unique eigenmode $\overset{(1)}{\tau}{}^{ij}$ associated with λ_c. The eigenmode is normalized in some way and the amplitude of its contribution to the post-bifurcation deflection is denoted by ξ. This amplitude is the expansion variable in the perturbation expansion which is developed about the lowest bifurcation point. There are, in general, at least two distinct branches to the post-bifurcation response which are not analytic continuations of each other (as illustrated for the simple model in Fig. 2). *Therefore, it is convenient to adopt to convention that ξ take on only positive values.* To analyze the opposite-signed deflection we will change the sign of the eigenmode $\overset{(1)}{\tau}{}^{ij}$ and not of its amplitude ξ.

The bifurcation analysis is carried out in Section 4. There it is shown that the initial slope λ_1 (where $\lambda = \lambda_c+\lambda_1\xi+\ldots$) must be sufficiently large to ensure that no elastic unloading occurs at bifurcation. The bifurcation solution is of the form

$$\dot{\tau}^{ij} = \dot{\tau}^{0ij}+\overset{(1)}{\tau}{}^{ij}+\ldots \tag{3.10}$$

or

$$\tau^{ij} = \tau_c^{0ij}+\xi(\lambda_1\overset{\prime}{\tau}{}^{0ij}+\overset{(1)}{\tau}{}^{ij})+\ldots, \tag{3.11}$$

where now

$$\dot{()} = \frac{d()}{d\xi} \tag{3.12}$$

and

$$()' = \left.\frac{d()}{d\lambda}\right|_{\lambda_c}. \tag{3.13}$$

The fundamental solution is a function of λ, and λ is in turn dependent on ξ on the bifurcated path; thus, τ^{0ij} can be regarded as a function of ξ and $\dot{\tau}^{0ij}$ is shorthand for

$$\frac{d\tau^{0ij}}{d\lambda}\frac{d\lambda}{d\xi}.$$

A superscript or subscript c always signifies quantities evaluated at λ_c.

In Section 5 we digress briefly, just as in the analysis of the simple model, to discuss the initial post-bifurcation behavior of the comparison solid. We introduce the non-linear hypo-elastic solid specified by the moduli (3.7) corresponding to the loading branch of the elastic–plastic constitutive relation. Under conditions to be detailed, the critical value of λ and the eigenmode for the comparison solid are identical to the corresponding quantities for the elastic–plastic solid. The initial slope for the hypoelastic comparison solid is denoted by λ_1^{he} so that $\lambda = \lambda_c + \lambda_1^{he}\xi + \dots$. If $\lambda_1^{he} \neq 0$ the bifurcation point of the comparison solid is asymmetric. We then show that if λ_1^{he} is sufficiently large there will be no reversal in the sign of $m^{ij}\dot{\eta}_{ij}$ in some finite range of positive ξ. Within this range the behavior of the elastic–plastic solid coincides with that of the comparison solid and, in particular, $\lambda_1 = \lambda_1^{he}$. However, the more interesting case, and the one on which we will concentrate, is where there is a reversal in sign of $m^{ij}\dot{\eta}_{ij}$ at bifurcation for the comparison solid.

For this latter case we show that the initial slope for the elastic–plastic solid λ_1 must be the *smallest* possible value which ensures that no elastic unloading occurs at bifurcation. That is, there must be at least one point in the body where neutral loading (i.e. $m^{kl}\dot{\eta}_{kl} = 0$) occurs at bifurcation. This point(s) will be denoted by x_c^i. As ξ increases from zero, the surface, termed the *instantaneous neutral loading surface*, which separates the regions of plastic loading and elastic unloading, spreads from x_c^i as depicted in Fig. 3. Note that in the case of the simple model the unloading region grows as the square root of the amplitude of the eigenmode (see (2.12)), and thus the rate of its growth at bifurcation is infinite.

The post-bifurcation expansion is started in Section 6. The lowest order correction to (3.10) involves terms which arise as a result of the elastic unloading and which in most cases make a contribution only within the elastically unloaded region. Local stretched coordinates $\overset{*}{z}_i$ are introduced such that the description of the instantaneous neutral loading surface is independent of ξ in this coordinate system. The lowest order corrections to (3.10) are denoted by $\overset{*}{\tau}^{ij}$ and are expressed as functions of the stretched coordinates. A boundary-layer analysis is used to determine these terms. The initial post-bifurcation expansion is shown to be of the form

$$\lambda = \lambda_c + \lambda_1 \xi + \lambda_2 \xi^{1+\beta} + \dots \tag{3.14}$$

and

$$\dot{\tau}^{ij} = \dot{\tau}^{0ij} + \overset{(1)}{\tau}{}^{ij} + \xi^{\beta}\overset{*}{\tau}{}^{ij} + \dots \tag{3.15}$$

where $0 < \beta < 1$.

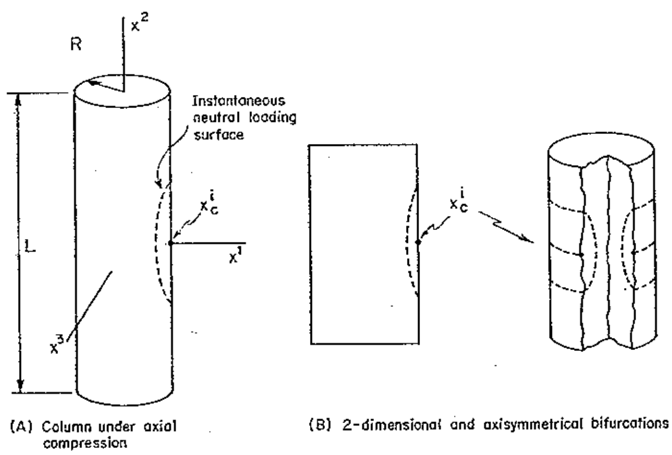

FIG. 3. Illustrations of the occurrence of the initial unloading points and the spread of the elastically unloaded region.

The coefficient λ_2 and the exponent β are left undetermined by the boundary-layer analysis. These are determined in Section 7 with the aid of the principle of virtual work. By examining the lowest order terms in the expansion of the principle of virtual work one can identify β and also obtain a general formula for λ_2 in terms of the eigenmode. The analysis is applied to the plastic buckling of a solid cylindrical column under uniaxial compression in Section 8. An approximate calculation of the maximum support load of the column is made using the initial post-bifurcation expansion.

4. Hill's Bifurcation Criterion Specialized to a Class of Problems Involving Monotonically Applied Loads

To lay the groundwork for the post-bifurcation analysis, some of the details of a specialized application of Hill's (1958, 1961) general bifurcation analysis are briefly discussed in this section. Over a portion of the surface of the body, S_T, dead load surface tractions $T^i = \lambda T_S^i$ are prescribed and over the remainder of the surface, S_u, displacements $u_i = \lambda u_i^S$ are prescribed, where T_S^i and u_i^S may have spatial variation but are independent of the load parameter λ. Our investigation is restricted to bifurcations which occur prior to the occurrence of any limit point in the fundamental solution. A superscript 0 is used to label all quantities of the fundamental solution and it is to be understood that these quantities are associated with monotonically increasing λ.

Denote the increment in the fundamental solution for $\lambda \to \lambda+\dot{\lambda}$ by $\dot{u}_i^0$, $\dot{T}^{0i}$, $\dot{\tau}^{0ij}$ and $\dot{\eta}_{ij}^0$. Suppose that at λ there is also a second increment solution, the bifurcation solution $\dot{u}^i$, $\dot{T}^i$, $\dot{\tau}^{ij}$ and $\dot{\eta}_{ij}$. Introduce the differences between the two solutions according to $\tilde{u}_i = \dot{u}_i - \dot{u}^0$, $\tilde{T}^i = \dot{T}^i - \dot{T}^{0i}$, $\tilde{\tau}^{ij} = \dot{\tau}^{ij} - \dot{\tau}^{0ij}$ and $\tilde{\eta}_{ij} = \dot{\eta}_{ij} - \dot{\eta}_{ij}^0$. Then, following the usual construction in uniqueness proofs,

$$0 = \int_S \tilde{T}^i \tilde{u}_i \, dS = \int_V \{\tilde{\tau}^{ij}\tilde{\eta}_{ij} + \tau^{0ij}\tilde{u}^k{}_{,i}\,\tilde{u}_{k,j}\}\, dV \equiv H, \tag{4.1}$$

where $\tilde{\eta}_{ij}$ and $\tilde{u}_i$ are connected by

$$\tilde{\eta}_{ij} = \tfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i}) + \tfrac{1}{2}(u^{0k}{}_{,i}\,u_{k,j} + u^{0k}{}_{,j}\,u_{k,i}). \tag{4.2}$$

At the instant of bifurcation define $\mathbf{L}_c$ at every point in the body to be equal to $\mathbf{L}$ where the yield condition is currently satisfied, independent of the sign of $m^{ij}\dot{\eta}_{ij}$, and to be $\mathcal{L}$ where the stress lies within the yield surface.† If both the increment in the fundamental solution $\dot{u}_i^0$ and the increment in the bifurcation solution $\dot{u}_i$ have the property that plastic loading (or possibly neutral loading) takes place at every point where the yield condition is satisfied, then $\mathbf{L}_c$ *are* the actual operative moduli for both solutions. Then H, which is defined by the last equality in (4.1), is equal to F where

$$F(\lambda, \tilde{u} = \int_V \{L_c^{ijkl}\tilde{\eta}_{ij}\tilde{\eta}_{kl} + \tau^{0ij}\,{}^k{}_{,i}\,\tilde{u}_{k,j}\}\, dV. \tag{4.3}$$

For the class of solids characterized by (3.7) and (3.8), Hill (1958) has shown that when the two solution increments do not share the common plastic loading region described above then $F < H$. Thus, if the quadratic functional F satisfies

$$F(\lambda, \tilde{u}) > 0$$

† By definition, $\mathbf{L}_c$ does not depend on the loading–unloading condition. Hill (1961) refers to $\mathbf{L}_c$ as the moduli of an elastic comparison solid. Hill and Sewell (1960), Como (1971), and Sewell (1972) have given brief discussions somewhat along the lines of this section.

for *all* nonvanishing admissible $\tilde{u}_i$ with $\tilde{u}_i = 0$ on S_u, then the uniqueness of $\dot{u}_i^0$ is obviously ensured.

In some applications it is convenient to employ the deformed configuration at bifurcation as the reference configuration rather than that of the undeformed body. The form of F remains unchanged. But then, dV is the volume element of the deformed body at bifurcation; the comma denotes covariant differentiation with respect to the deformed configuration; and u_i^0 must be set to zero in (4.2). HILL (1961, Eq. (3.3)) gives this functional with the deformed configuration as reference.

Let λ_c be the *lowest* value of λ for which there exists an admissible field $\overset{(1)}{u}_i$ with with associated fields $\overset{(1)}{\eta}_{ij}$ and $\overset{(1)}{\tau}{}^{ij}$ such that

$$F(\lambda_c, \overset{(1)}{u}) = 0. \tag{4.4}$$

This eigenmode satisfies the variational equation $\delta F = 0$ and the associated field equations together with the homogeneous boundary conditions, $\overset{(1)}{u}_i = 0$ on S_u and $\overset{(1)}{T}{}^i = 0$ on S_T. The eigenmodal quantities are connected by

$$\overset{(1)}{\eta}_{ij} = \tfrac{1}{2}(\overset{(1)}{u}_{i,j} + \overset{(1)}{u}_{j,i}) + \tfrac{1}{2}(u^{0c}_{k,i}\,\overset{(1)}{u}{}^k_{,j} + u^{0c}_{k,j}\,\overset{(1)}{u}{}^k_{,i}) \quad \text{and} \quad \overset{(1)}{\tau}{}^{ij} = L_c^{ijkl}\,\overset{(1)}{\eta}_{kl}, \tag{4.5}$$

where the c labels quantities evaluated at λ_c. For simplicity, only the case where $\overset{(1)}{u}_i$ is unique, apart from an arbitrary amplitude, will be discussed. The eigenmode is taken to be normalized in some definite way and the deflection in the mode is written as $\xi \overset{(1)}{u}_i$.† As discussed in Section 3, the amplitude of the eigenmode, ξ, is taken to be non-negative and is selected as the independent variable in the initial post-bifurcation expansion; the sign of $\overset{(1)}{u}_i$ will be changed to analyze the opposite-signed bifurcation.

A bifurcation solution is considered in the form

$$\lambda = \lambda_c + \lambda_1 \xi, \qquad \dot{u}_i = \dot{u}_i^0 + \overset{(1)}{u}_i = \lambda_1 \acute{u}_i^0 + \overset{(1)}{u}_i, \tag{4.6}$$

where $(\cdot)$ is specified by (3.12) and $(')$ by (3.13). With this choice, $F = 0$; however, H will not vanish (and thus $\dot{u}_i$ is not a solution) unless *both* solutions $\dot{u}_i^0$ and $\dot{u}_i$ have the property that no elastic unloading occurs at every point in the body where the yield condition is currently satisfied. In our analysis we restrict consideration to problems where there is *no unloading associated with the fundamental solution.* Moreover, it will be assumed that a number Δ greater than zero can always be found such that at every point where the yield condition is currently satisfied

$$m_c^{ij} \acute{\eta}_{ij}^0 \geq \Delta. \tag{4.7}$$

When the fundamental solution meets these restrictions it is always possible to choose λ_1 such that the proposed bifurcation solution (4.6) also loads everywhere. Because, then,

$$m_c^{ij} \dot{\eta}_{ij} = m_c^{ij}(\lambda_1 \acute{\eta}_{ij}^0 + \overset{(1)}{\eta}_{ij}) \geq 0, \tag{4.8}$$

where the last inequality can obviously be met if λ_1 is large enough.

† An operational definition of ξ is given later in conjunction with (6.12).

Unless $m_c^{ij} \overset{(1)}{\eta}_{ij}$ is everywhere positive, (4.8) implies that $\lambda_1 > 0$.† The fact that the bifurcation solution is a linear combination of the fundamental solution increment and the eigenmode in the form (4.6) with $\lambda_1 > 0$ implies that the lowest bifurcation takes place under increasing applied load as first discussed by SHANLEY (1947) and as generalized by Hill. To summarize, the problem for the lowest bifurcation load has been reduced to an eigenvalue problem (4.4) precisely of the form for an elastic body with instantaneous moduli $\mathbf{L}_c$. The initial slope λ_1 must be sufficiently large to ensure that no elastic unloading occurs at any point in the body where the yield condition is currently satisfied.

5. BEHAVIOR OF THE COMPARISON SOLID AND DETERMINATION OF λ_1

The fundamental solution of the elastic–plastic solid has been assumed to display no elastic unloading in the range of λ of interest. Consider a comparison solid characterized by the loading branch (3.7) of the elastic–plastic constitutive relation. Clearly, the behavior of the comparison solid and the elastic–plastic solid coincide at least until bifurcation. Furthermore, the lowest bifurcation load λ_c and the eigenmode are also obviously the same for both solids. The moduli $\mathbf{L}$ must be expanded about λ_c to obtain the initial post-bifurcation expansion of the comparison solid. We assume that $\mathbf{L}$ and m^{ij} can be regarded as functions of the stress alone, at least in the neighborhood of λ_c if not globally, so that

$$L^{ijkl} = L_c^{ijkl} + (\tau^{mn} - \tau_c^{0mn}) \left.\frac{\partial L^{ijkl}}{\partial \tau^{mn}}\right|_c + \ldots \tag{5.1}$$

and

$$m^{ij} = m_c^{ij} + (\tau^{mn} - \tau_c^{0mn}) \left.\frac{\partial m^{ij}}{\partial \tau_{mn}}\right|_c + \ldots . \tag{5.2}$$

These variable moduli are not derivable from a potential (assuming the plasticity theory is not a deformation theory) and thus the constitutive relation for the comparison solid is nonlinear hypo-elastic.

Denote the initial slope for the comparison solid by λ_1^{he} so that

$$\lambda = \lambda_c + \lambda_1^{he}\xi + O(\xi^2). \tag{5.3}$$

The initial post-bifurcation expansion is obtained by an approach similar to that initiated by KOITER (1945, 1963). This approach is discussed in some detail in Section 7 where an explicit formula (7.11) is given for λ_1^{he} in terms of the eigenmode.

If the calculated value of λ_1^{he} is sufficiently large such that $m^{ij}\dot{\eta}_{ij} > 0$ in some range of positive ξ, then on this bifurcation branch the behavior of the elastic–plastic solid will coincide with that of the comparison solid in the same range. To see the condition for this, consider the expansion of $m^{ij}\dot{\eta}_{ij}$ about the bifurcation point. First, using (5.2),

$$m^{ij} = m_c^{ij} + \xi(\lambda_1^{he}\tau'^{0kl} + \overset{(1)}{\tau}{}^{kl}) \left.\frac{\partial m^{ij}}{\partial \tau^{kl}}\right|_c + \ldots \tag{5.4}$$

and then $\dot{\eta}_{ij} = (\lambda_1^{he}\eta'^{0}_{ij} + \overset{(1)}{\eta}_{ij}) + O(\xi)$, one finds

$$m^{ij}\dot{\eta}_{ij} = m_c^{ij}(\lambda_1^{he}\eta'^{0}_{ij} + \overset{(1)}{\eta}_{ij}) + O(\xi). \tag{5.5}$$

† One can conceive of problems in which $m_c^{ij}\overset{(1)}{\eta}_{ij}$ is positive throughout the current yielded region. However, usually in problems of interest this quantity is positive in part of the yielded region and negative in the rest so that λ_1 must be positive.

Thus, if λ_1^{he} is greater than the smallest value needed to ensure that the lowest order term in (5.5) is everywhere positive in the current yielded region, then the behaviors of the two solids coincide in some range of positive ξ, and $\lambda_1 = \lambda_1^{he}$.

As mentioned in conjunction with the simple model and again in Section 3, the above possibility is generally of much less interest than when the value of λ_1^{he} is such that

$$m_c^{ij}(\lambda_1^{he}\dot{\eta}_{ij}^0 + \overset{(1)}{\eta}_{ij}) < 0 \tag{5.6}$$

over part of the current yielded region (i.e. $\lambda_1^{he} < \lambda_1$). Henceforth, attention is restricted to the analysis of situations in which (5.6) holds.

If (5.6) holds we can now immediately argue that the initial slope of the elastic–plastic solid, λ_1, must be the *smallest* value consistent with (4.8). For if λ_1 were larger, then by continuity there would be some range of positive ξ where $m^{ij}\dot{\eta}_{ij}$ would be greater than zero; and consequently, the behavior of the elastic–plastic solid would initially coincide with that of the comparison solid so that $\lambda_1 = \lambda_1^{he}$. But this possibility is contradicted by (5.6). Thus, λ_1 must satisfy

$$\text{minimum over current yielded region } \{m_c^{ij}(\lambda_1\dot{\eta}_{ij}^0 + \overset{(1)}{\eta}_{ij})\} = 0. \tag{5.7}$$

Neutral loading must occur in at least one point in the body at bifurcation where the minimum in (5.7) is attained. The region of elastic unloading grows from this point(s) with increasing ξ as described in Section 6.

6. LOWEST ORDER BOUNDARY-LAYER TERMS

In this section we determine the lowest order perturbation on the bifurcation solution

$$\dot{u}_i - \dot{u}_i^0 = \overset{(1)}{u}_i \quad \text{or} \quad u_i - u_i^{0c} = \xi(\lambda_1\dot{u}_i^0 + \overset{(1)}{u}_i). \tag{6.1}$$

In many respects our approach will be similar to that adopted by KOITER (1945, 1963) in his general theory of elastic stability. Like Koiter, we employ the amplitude of the eigenmode, ξ, as the independent variable on the post-bifurcation path. We investigate the sequence of statical equilibrium states by taking ξ to increase monotonically from zero. Koiter made use of the potential energy functional in his study of conservative elastic systems. Our analysis will employ the principle of virtual work in much the same way that BUDIANSKY and HUTCHINSON (1964), BUDIANSKY (1966), FITCH (1968) and COHEN (1968) employed this principle in their specialized versions of Koiter's theory.

As discussed in Section 3, a situation is envisioned in which an elastic unloading region spreads out from a point(s) where the minimum in (5.7) is attained. We will work out all the details for the case where the neutral loading point of the bifurcation solution occurs at a single point on a traction-free portion of a smooth surface. This situation is depicted in Fig. 4 where the initial neutral loading point is denoted by x_c^i. Many problems of interest fall in this category, including the solid cylindrical column to which the general analysis will be applied in Section 8.

For reasons which will become increasingly clear as the analysis proceeds, introduce a local rectangular Cartesian coordinate system z_i centered at x_c^i as shown in

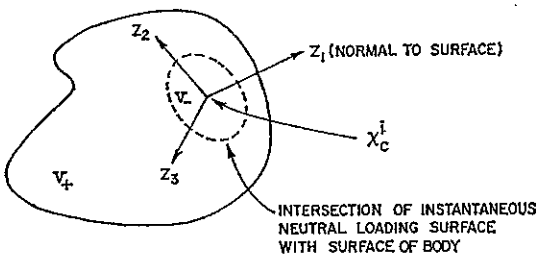

FIG. 4. Local Cartesian triad at initial neutral loading point and instantaneous region of elastic unloading.

Fig. 4. Choose the z_1 axis such that it lies along the outward normal to the surface *at bifurcation*. The z_2 and z_3 axes are mutually perpendicular and tangential to the surface at bifurcation. Later in this section a set of stretched coordinates $\overset{*}{z}_i$ will be chosen such that the description of the growing neutral loading surface in terms of these coordinates will be independent of ξ, to the lowest order. We have seen that for the simple model such a coordinate would be of the form $\overset{*}{z} \approx \xi^{-1/2}z$. These same stretched coordinates together with a boundary-layer analysis will enable us to obtain a perturbation representation for the stress and strain fields in the vicinity of the expanding region of elastic unloading.

Guided by the results for the simple model we entertain the possibility that the lowest order perturbation on the bifurcation solution will involve the expansion parameter ξ to a fractional power. Assuming that the lowest order contributions are of order ξ^β, we can write without any approximation

$$\lambda = \lambda_c + \lambda_1 \xi + \lambda_2 \xi^{1+\beta} + \tilde{\lambda}(\xi) \tag{6.2}$$

or

$$\dot{\lambda} = \lambda_1 + (1+\beta)\lambda_2 \xi^\beta + \frac{d\tilde{\lambda}}{d\xi} \tag{6.3}$$

where $\lim_{\xi \to 0} (\xi^{-1-\beta}\tilde{\lambda}) = 0$. Here, it is anticipated that $0 < \beta < 1$; this will be verified *a posteriori* in Section 7.

We can also express the difference between the bifurcated solution-rate and the fundamental solution-rate at the same value of λ quite generally as

$$\begin{Bmatrix} \dot{w} \\ \dot{\eta} \\ \dot{\tau} \end{Bmatrix} - \begin{Bmatrix} \dot{w}^0 \\ \dot{\eta}^0 \\ \dot{\tau}^0 \end{Bmatrix} = \begin{Bmatrix} \overset{(1)}{w} \\ \overset{(1)}{\eta} \\ \overset{(1)}{\tau} \end{Bmatrix} + \xi^\beta \begin{Bmatrix} \overset{a}{w}(\xi, x) \\ \overset{a}{\eta}(\xi, x) \\ \overset{a}{\tau}(\xi, x) \end{Bmatrix}, \tag{6.4†}$$

where w stands for the displacement gradients, i.e.

$$w_{ij} = u_{i,j}. \tag{6.5}$$

Outside the boundary layer the displacement-rate quantities are of the same order as the displacement gradient-rates. However, inside the boundary layer they are of

† From here on, indices will sometimes be deleted to give a less encumbered notation when this creates no ambiguity.

different order, as will be shown. In the boundary layer it is convenient to work with w, rather than u, since the perturbations on w are of the same order as those on η and τ.

We now consider two different limiting processes. First, let $\xi \to 0$ with x^i fixed (but $x^i \neq x^i_c$) and introduce the definitions

$$\lim_{\xi \to 0,\ x \text{ fixed}} [\overset{a}{w}, \overset{a}{\eta}, \overset{a}{\tau}] = [\bar{w}(x), \bar{\eta}(x), \bar{\tau}(x)]. \tag{6.6}$$

Next, let $\xi \to 0$ with $\overset{*}{z}_i$ fixed and introduce boundary-layer terms according to

$$\lim_{\xi \to 0,\ \overset{*}{z} \text{ fixed}} [\overset{a}{w}, \overset{a}{\eta}, \overset{a}{\tau}] = [\overset{*}{w}(\overset{*}{z}), \overset{*}{\eta}(\overset{*}{z}), \overset{*}{\tau}(\overset{*}{z})]. \tag{6.7}$$

In the limit (6.6) the point in question is fixed and the boundary-layer region shrinks down to zero size at x^i_c. In the second limit (6.7) the point in question moves towards x^i_c in fixed position relative to the shrinking boundary-layer region in the limiting process.

The remainder of this section is divided into two subsections. In the first it will be shown that the lowest order additions to the bifurcation solution are restricted to the boundary layer so that $(\bar{w}, \bar{\eta}, \bar{\tau}) = 0$. In the second the boundary-layer analysis will be carried out.

6.1 *Demonstration that* $(\bar{w}, \bar{\eta}, \bar{\tau}) = 0$

To show that only the boundary-layer terms appear to order ξ^β where β is anticipated to be less than unity, we substitute the representation (6.4) into the principle of virtual work and examine the lowest order, nonvanishing terms. The bifurcation solution satisfies (3.5); at the same value of λ the fundamental solution satisfies

$$\int_V \{\dot{\tau}^{0ij}(\delta u_{i,j} + u^0_{k,i}\, \delta^k_{u,j}) + \tau^{0ij}\dot{u}^0_{k,i}\delta u^k{}_{,j}\}\, dV = \int_S \dot{T}^i \delta u_i\, dS. \tag{6.8}$$

Eliminate the right-hand side of (3.5) using (6.8) and rearrange the resulting expression to the form most suited to our purposes:

$$\int_V \{(\dot{\tau}^{ij} - \dot{\tau}^{0ij})\, \delta\eta_{ij} + \tau^{ij}(\dot{u}_{k,i} - \dot{u}^0_{k,i})\, \delta u^k{}_{,j} + \dot{\tau}^{0ij}(u_{k,i} - u^0_{k,i})\, \delta u^k{}_{,j} + \\ + (\tau^{ij} - \tau^{0ij})\dot{u}^0_{k,i}\delta u^k{}_{,j}\}\, dV = 0, \tag{6.9}$$

where $\delta\eta_{ij}$ is given by (3.4). Using (6.1) and (6.4) and relations such as

$$\tau = \tau^0 + \xi \overset{(1)}{\tau} + \ldots \qquad \text{and} \qquad \tau = \tau^0_c + \xi(\lambda_1 \acute{\tau}^0 + \overset{(1)}{\tau})\ldots,$$

expand out (6.9) and collect terms of order zero and order ξ^β with the result

$$\int_V \{\overset{(1)}{\tau}{}^{ij}\, \delta^c\eta_{ij} + \tau^{0ij}_c \overset{(1)}{u}_{k,i}\, \delta u^k{}_{,j}\}\, dV + \xi^\beta \int_V \{\overset{a}{\tau}{}^{ij}\, \delta^c\eta_{ij} + \tau^{0ij}_c \overset{a}{u}_{k,i}\, \delta u^k{}_{,j}\}\, dV + \ldots = 0, \tag{6.10}$$

where it is convenient to make the definition

$$\delta^c\eta_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) + \tfrac{1}{2}(u^{0c}_{k,i}\delta u^k{}_{,j} + u^{0c}_{k,j}\delta u^k{}_{,i}) \tag{6.11}$$

and as before the c denotes evaluation at λ_c.

The lowest order expression in (6.10) is just the variational statement of the eigenvalue problem associated with (4.4) and thus this term vanishes for all admissible δu.

Now divide (6.10) by ξ^β and take the limit as $\xi \to 0$. Note that, although the boundary-layer terms (6.7) enter into the integrand to order ξ^β, they will not make a contribution to the integral in the limit since the boundary-layer region shrinks to zero. Thus,

$$\int_V \{\bar{\tau}^{ij}\, \delta^c \eta_{ij} + \tau_c^{0ij}\, \bar{u}_{k,i}\, \delta u^k{}_{,j}\}\, dV = 0.$$

By substituting (6.4) into the relation between the strain-rates and displacement-rates and then taking the limit as $\xi \to 0$ for fixed x^i, one obtains the same relation between $\bar{\eta}$ and $\bar{u}$ (or $\bar{w}$) as is satisfied by the eigenmodal quantities $\overset{(1)}{\eta}$ and $\overset{(1)}{u}$ in (4.5). In much the same way, (6.4) is substituted into the constitutive relation (3.7) appropriate for plastic loading. Then, taking the limit as $\xi \to 0$ for fixed x^i, using (5.1), one again finds that the eigenmodal relation (4.5) is satisfied by $\bar{\tau}$ and $\bar{\eta}$. The variational equation for the barred quantities, together with the auxiliary equations, are identical to the equations for the eigenmode. It has been assumed that the eigenvalue problem has a unique mode and this implies that the barred solution must be some multiple of the eigenmode; that is, $\bar{u} = \zeta \overset{(1)}{u}$.

In Section 4, ξ was introduced as the amplitude of the eigenmodel contribution to the bifurcated solution. Since we have written $u - u^0 = \xi \overset{(1)}{u} + \ldots$ or, equivalently, $\dot{u} - \dot{u}^0 = \overset{(1)}{u} + \ldots$, the higher order contributions must be orthogonal to $\overset{(1)}{u}$ in some sense. The most convenient orthogonality requirement for our purposes is

$$\int_V \tau_c^{0ij}[(\dot{u}_{k,i} - \dot{u}^0_{k,i}) - \overset{(1)}{u}_{k,i}]\, \overset{(1)}{u}{}^k{}_{,j}\, dV = 0. \tag{6.12}$$

This presupposes that $\int \tau_c^0 \overset{(1)}{u} \overset{(1)}{u}\, dV$ is not zero so that $\overset{(1)}{u}$ cannot be orthogonal to itself. However, it can be shown that, to the extent the post-bifurcation analysis is carried out in this paper, the $\lambda - \xi$ relation is the same for *any bona fide* orthogonality condition. By substituting (6.4) into (6.12), dividing by ξ^β and taking the limit as $\xi \to 0$ one finds that ζ must be zero.

6.2 *Boundary-layer analysis*

The equation for the instantaneous neutral loading surface emanating from x_c^i is $m^{ij}\dot{\eta}_{ij} = 0$. Using (5.2), (6.3) and (6.4) and $\dot{\eta}^0 = \lambda_1 \acute{\eta}^0 + (1+\beta)\lambda_2 \xi^\beta \acute{\eta}^0 + \ldots$, this equation can be written as

$$0 = m^{ij}\dot{\eta}_{ij} = m_c^{ij}(\lambda_1 \acute{\eta}^0_{ij} + \overset{(1)}{\eta}_{ij}) + \xi^\beta[(1+\beta)\lambda_2\, m_c^{ij}\acute{\eta}^0_{ij} + m_c^{ij}\overset{a}{\eta}_{ij}] + \ldots, \tag{6.13}$$

where higher order terms than ξ^β have been dropped. Recall that, by (5.7), the zeroth order term in the above equation vanishes at x_c^i and furthermore is positive elsewhere in the body. Expand this term in a Taylor series about x_c^i using the z_i coordinates to get

$$m_c^{ij}(\lambda_1 \acute{\eta}^0_{ij} + \overset{(1)}{\eta}_{ij}) = C_1 z_1 + \sum_{m=1}^{3} \sum_{n=1}^{3} C_{mn} z_m z_n + \ldots, \tag{6.14}$$

where

$$C_1 = \frac{\partial}{\partial z_1}\, [m_c^{ij}(\lambda_1 \acute{\eta}^0_{ij} + \overset{(1)}{\eta}_{ij})]\Big|_{z_i=0}$$

and

$$C_{mn} = \tfrac{1}{2} \frac{\partial}{\partial z_m} \frac{\partial}{\partial z_n}\, [m_c^{ij}(\lambda_1 \acute{\eta}^0_{ij} + \overset{(1)}{\eta}_{ij})]\Big|_{z_i=0}. \tag{6.15}$$

Because this function attains its smallest value in the body at $z_i = 0$, $C_2 = C_3 = 0$ and $C_1 \leq 0$. In what follows we will carry out the analysis for the case of $C_1 < 0$ and later comment on the case where $C_1 = 0$.

Divide (6.13) by $(1+\beta)\lambda_2 \xi^\beta$ and define stretched coordinates by

$$\overset{*}{z}_1 = -\frac{\xi^{-\beta} z_1}{(1+\beta)\lambda_2} \quad \text{and} \quad (\overset{*}{z}_2, \overset{*}{z}_3) = \frac{\xi^{-\beta/2}}{\{-(1+\beta)\lambda_2\}^{1/2}}(z_2, z_3), \tag{6.16}$$

where $(1+\beta)\lambda_2$ has been introduced into the definition of the stretched coordinates for later convenience and it has been anticipated that λ_2 is negative. Now take the limit as $\xi \to 0$ with $\overset{*}{z}_i$ fixed. The result is the lowest order equation for the neutral loading surface in terms of the stretched coordinates, i.e.

$$(1+\beta)\lambda_2 f(\overset{*}{z}) + m_c^{ij}(x_c)\overset{*}{\eta}_{ij} = 0, \tag{6.17}$$

where

$$f(\overset{*}{z}) = m_c^{ij}\acute{\eta}_{ij}^0\big|_{x^i_c} - C_1 \overset{*}{z}_1 - C_{22}\overset{*}{z}{}_2^2 - C_{33}\overset{*}{z}{}_3^2 - 2C_{23}\overset{*}{z}_2\overset{*}{z}_3. \tag{6.18}$$

Next, the relations between $\overset{*}{\tau}$ and $\overset{*}{\eta}$ are obtained. In the portion of the boundary layer which has elastically unloaded (i.e. inside the neutral loading surface (6.17) and denoted by V_-) $\dot{\tau}^{ij} = \mathscr{L}^{ijkl}\dot{\eta}_{kl}$. Expand this equation, using (6.4) and

$$\dot{\tau}^0 = \lambda_1 \acute{\tau}^0 + (1+\beta)\lambda_2 \xi^\beta \acute{\tau}^0 + \ \ldots, \text{etc.},$$

to find

$$\lambda_1 \acute{\tau}^{0ij} + \overset{(1)}{\tau}{}^{ij} + (1+\beta)\lambda_2 \xi^\beta \acute{\tau}^{0ij} + \xi^\beta \overset{a}{\tau}{}^{ij} + \ldots =$$
$$\mathscr{L}^{ijkl}[\lambda_1 \acute{\eta}_{kl}^0 + \overset{(1)}{\eta}_{kl} + (1+\beta)\lambda_2 \xi^\beta \acute{\eta}_{kl}^0 + \xi^\beta \overset{a}{\eta}_{kl} + \ldots]. \tag{6.19}$$

Since

$$\acute{\tau}^{0ij} = L_c^{ijkl}\acute{\eta}_{kl}^0, \qquad \overset{(1)}{\tau}{}^{ij} = L_c^{ijkl}\overset{(1)}{\eta}_{kl} \qquad \text{and} \qquad \mathscr{L}^{ijkl} - L_c^{ijkl} = g_c^{-1} m_c^{ij} m_c^{kl} \tag{6.20}$$

(6.19) can be rewritten as

$$\xi^\beta \overset{a}{\tau}{}^{ij} = g_c^{-1} m_c^{ij}[m_c^{kl}(\lambda_1 \acute{\eta}_{kl}^0 + \overset{(1)}{\eta}_{kl})] + \xi^\beta \mathscr{L}^{ijkl}\overset{a}{\eta}_{kl} + \xi^\beta (1+\beta)\lambda_2 g_c^{-1} m_c^{ij} m_c^{kl}\acute{\eta}_{kl}^0 + \ldots. \tag{6.21}$$

Now divide (6.21) by ξ^β and let $\xi \to 0$ with $\overset{*}{z}_i$ fixed. Using (6.7), (6.14), (6.16) and (6.18) we obtain

$$\overset{*}{\tau}{}^{ij} = \mathscr{L}^{ijkl}\overset{*}{\eta}_{kl} + (1+\beta)\lambda_2 g_c^{-1} m_c^{ij} f(\overset{*}{z}). \tag{6.22}$$

Using the same procedure, it is a straightforward matter to show that in the portion of the boundary layer where loading occurs (outside the neutral loading surface and denoted by V_+),

$$\overset{*}{\tau}{}^{ij} = L_c^{ijkl}(x_c)\overset{*}{\eta}_{kl}. \tag{6.23}$$

Now we derive the boundary conditions and equilibrium equations for the boundary-layer quantities. To do this it is most convenient to work in the local rectangular Cartesian coordinates z_i whose z_1 axis is perpendicular to the surface of the body at x_c at bifurcation, as indicated in Fig. 4. *For the remainder of this section the fixed Cartesian system will be used as the reference system instead of the undeformed system, and all indices will be subscripted.*

We analyze the case where the boundary conditions are prescribed zero surface tractions in the vicinity of x_c. On S_T (from 3.2))

$$0 = [\dot{t}_{ij} + \dot{t}_{kj}(u_{i,k} - u^{0c}_{i,k}) + \tau_{kj}\dot{u}_{i,k}]n_j^c, \tag{6.24}$$

where n_j^c is the unit normal at bifurcation. Substitute (6.4) into (6.24) and use the fact that the fundamental solution itself satisfies (6.24), along with the conditions $\tau^{0c}_{ij} n_j^c = 0$ and $\overset{(1)}{\tau}_{ij} n_j^c = 0$ on S_T, to obtain $\xi^\beta \overset{a}{\tau}_{ij} n_j^c + \ldots = 0$. Dividing by ξ^β and letting $\xi \to 0$ with $\overset{*}{z}_i$ fixed gives

$$\overset{*}{\tau}_{ij} n_j^c(x_c) = 0 \tag{6.25}$$

or, since $n_1^c(x_c) = 1$ and $n_2^c(x_c) = n_3^c(x_c) = 0$,

$$\overset{*}{\tau}_{11} = \overset{*}{\tau}_{12} = \overset{*}{\tau}_{13} = 0 \quad \text{on} \quad S_T. \tag{6.26}$$

The equilibrium equations (3.6) become

$$\dot{t}_{ij,j} + [\dot{t}_{kj}(u_{i,k} - u^{0c}_{i,k}) + \tau_{kj}\dot{u}_{i,k}]_{,j} = 0. \tag{6.27}$$

Here again, (6.4) is substituted into (6.27), and use is made of the fact that the fundamental solution satisfies (6.27), of the equilibrium equation satisfied by the eigenmode, and of $\tau^{0c}_{ij,j} = 0$, to arrive at

$$\xi^\beta[\overset{a}{\tau}_{ij,j} + \tau^{0c}_{kj}\overset{a}{w}_{ik,j}] + \ldots = 0. \tag{6.28}$$

The comma denotes differentiation with respect to the z_i coordinates. Now change variables to the stretched coordinates (6.16) using

$$\frac{\partial}{\partial z_1} = -\frac{\xi^{-\beta}}{(1+\beta)\lambda_2}\frac{\partial}{\partial \overset{*}{z}_1}, \quad \left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\right) = \frac{\xi^{-\beta/2}}{\{-(1+\beta)\lambda_2\}^{1/2}}\left(\frac{\partial}{\partial \overset{*}{z}_2}, \frac{\partial}{\partial \overset{*}{z}_3}\right). \tag{6.29}$$

Note that each quantity in the resulting equation which is differentiated by $\overset{*}{z}_1$ is of zeroth order while the terms differentiated by $\overset{*}{z}_2$ and $\overset{*}{z}_3$ are of order $\xi^{\beta/2}$. Therefore, in the limit as $\xi \to 0$ with $\overset{*}{z}_i$ fixed, we find that the boundary-layer equilibrium equations are

$$\frac{\partial \overset{*}{\tau}_{i1}}{\partial \overset{*}{z}_1} = 0. \tag{6.30}$$

The second term in (6.28) makes no contribution since $\tau^{0c}_{11} = \tau^{0c}_{12} = \tau^{0c}_{13} = 0$ *at* x_c.

The character of the boundary-layer solution is extremely simple. To motivate this solution we note that the region of unloading described by (6.17) is a thin sliver whose thickness in the z_1 direction is of order $\xi^{\beta/2}$ compared to its extent in the z_2 and z_3 directions. In the remainder of this section we will show that the strain-rates tangential to the plane of z_2 and z_3 in V_- are the same as they would be in the absence of unloading so that $\overset{*}{\eta}_{22} = \overset{*}{\eta}_{33} = \overset{*}{\eta}_{23} = 0$. Furthermore, $\overset{*}{\eta}_{11}$, $\overset{*}{\eta}_{12}$ and $\overset{*}{\eta}_{13}$ are chosen such that $\overset{*}{\tau}_{11} = \overset{*}{\tau}_{12} = \overset{*}{\tau}_{13} = 0$ throughout V_-, so that the traction conditions (6.26) and equilibrium equations (6.30) are satisfied. The boundary-layer displacement-rates will also be given. Outside V_- the boundary-layer quantities vanish for the case under examination.

If we anticipate that $\overset{*}{\eta}_{22} = \overset{*}{\eta}_{33} = \overset{*}{\eta}_{23} = 0$, then (6.22) for the vanishing of $\overset{*}{\tau}_{11}$, $\overset{*}{\tau}_{12}$ and $\overset{*}{\tau}_{13}$ in V_- provides three equations for $\overset{*}{\eta}_{11}$, $\overset{*}{\eta}_{12}$ and $\overset{*}{\eta}_{13}$. Introduce the quantities

ϕ_i which are the solution to the matrix equation

$$\begin{bmatrix} \mathscr{L}_{1111} & \mathscr{L}_{1112} & \mathscr{L}_{1113} \\ \mathscr{L}_{1112} & \mathscr{L}_{1212} & \mathscr{L}_{1213} \\ \mathscr{L}_{1113} & \mathscr{L}_{1213} & \mathscr{L}_{1313} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = - \begin{bmatrix} m^c_{11} \\ m^c_{12} \\ m^c_{13} \end{bmatrix} \tag{6.31}$$

and define α_{ij} to be

$$\alpha_{ij} = \begin{bmatrix} \phi_1 & \frac{1}{2}\phi_2 & \frac{1}{2}\phi_3 \\ \frac{1}{2}\phi_2 & 0 & 0 \\ \frac{1}{2}\phi_3 & 0 & 0 \end{bmatrix}. \tag{6.32}$$

Then, $\overset{*}{\eta}$ is

$$\overset{*}{\eta}_{ij} = (1+\beta)\lambda_2 g_c^{-1} \alpha_{ij} f(\overset{*}{z}). \tag{6.33}$$

Substituting back into (6.22) gives

$$\overset{*}{\tau}_{ij} = (1+\beta)\lambda_2 g_c^{-1}[\mathscr{L}_{ijkl}\alpha_{kl} + m^c_{ij}] f(\overset{*}{z}). \tag{6.34}$$

By using (6.33) in (6.17), one notes immediately that the equation for the neutral loading surface associated with the assumed solution reduces to $f(\overset{*}{z}) = 0$. Consequently, $\overset{*}{\eta}$ and $\overset{*}{\tau}$ vanish on the neutral loading surface. Also, since $m^{ij}_c \dot{\eta}^0_{ij} > 0$ and $C_1 < 0$ by assumption, $f = 0$ together with (6.16) and (6.18) imply that λ_2 must be negative if the neutral loading surface is to spread into the body, as anticipated.

The final step necessary to establish the validity of the assumed solution is to demonstrate that the strain-rate quantities $\overset{*}{\eta}_{ij}$ can be derived from a boundary-layer displacement-rate field which vanishes on $f(\overset{*}{z}) = 0$. That field is

$$\overset{*}{u}_i = -(1+\beta)^2_2 \lambda_2^2 g_c^{-1} \phi_i \xi^\beta \int_{\overset{*}{z}_l}^{\overset{*}{z}_1} s f(\zeta, \overset{*}{z}_2, \overset{*}{z}_3)\, d\zeta \tag{6.35}$$

which can be verified directly using the definition of $\overset{*}{u}_i$ as an appropriately defined limit of $\overset{a}{u}_i$ as $\xi \to 0$ with fixed $\overset{*}{z}_i$, together with the relation between the strain-rates and the displacement-rates and (6.29). The lower limit in (6.35) is the value of $\overset{*}{z}_1$ on the neutral loading surface for given values of $\overset{*}{z}_2$ and $\overset{*}{z}_3$, i.e. $f(\overset{*}{z}_1, \overset{*}{z}_2, \overset{*}{z}_3) = 0$. The boundary-layer displacement gradients are found to be

$$\overset{*}{w}_{ij} = (1+\beta)\lambda_2 g_c^{-1} \phi_i \delta_{1j} f(\overset{*}{z}), \tag{6.36}$$

where δ_{ij} is the Kronecker delta.

7. DETERMINATION OF β AND λ_2

As mentioned in the preliminary discussion in Section 3, β and λ_2 are determined by an examination of the lowest order nonvanishing terms in the principle of virtual work. We have shown that only the boundary layer terms enter to order ξ^β. Now write quite generally

$$\begin{Bmatrix} \dot{w} \\ \dot{\eta} \\ \dot{\tau} \end{Bmatrix} - \begin{Bmatrix} \dot{w}^0 \\ \dot{\eta}^0 \\ \dot{\tau}^0 \end{Bmatrix} = \begin{Bmatrix} \overset{(1)}{w} \\ \overset{(1)}{\eta} \\ \overset{(1)}{\tau} \end{Bmatrix} + \xi^\beta \begin{Bmatrix} \overset{*}{w} \\ \overset{*}{\eta} \\ \overset{*}{\tau} \end{Bmatrix} + \begin{Bmatrix} \overset{b}{w}(\xi, x) \\ \overset{b}{\eta}(\xi, x) \\ \overset{b}{\tau}(\xi, x) \end{Bmatrix}, \tag{7.1}$$

where it can be asserted that, for either fixed x^i or fixed $\overset{*}{z}_i$,

$$\lim_{\xi \to 0} \xi^{-\beta}[\overset{b}{w}, \overset{b}{\eta}, \overset{b}{\tau}] = 0. \tag{7.2}$$

Substitute (7.1) in the principle of virtual work (6.9) to get the variational equation for $\overset{b}{u}$, $\overset{b}{\eta}$ and $\overset{b}{\tau}$:

$$0 = \int_V \{\overset{b}{\tau}{}^{ij}\, \delta^c\eta_{ij} + \tau_c^{0ij}\overset{b}{u}_{k,i}\, \delta u^k{}_{,j}\}\, dV + \xi^\beta \int_V \{\overset{*}{\tau}{}^{ij}\, \delta^c\eta_{ij} + \tau_c^{0ij}\overset{*}{w}_{ki}\, \delta u^k{}_{,j}\}\, dV +$$

$$+2\xi \int_V \{(\lambda_1 \dot{\tau}^{0ij} + \overset{(1)}{\tau}{}^{ij})\overset{(1)}{u}_{k,i}\, \delta u^k{}_{,j} + \lambda_1 \overset{(1)}{\tau}{}^{ij}\dot{u}^0_{k,i}\, \delta u^k{}_{,j}\}\, dV + O(\xi\overset{b}{\tau}, \xi^\beta \overset{*}{w}\overset{b}{\tau}, \ldots). \tag{7.3}$$

Using the boundary-layer solutions (6.34) and (6.36) and the definitions of the stretched coordinates (6.16), the second term in (7.3) can be expressed as

$$\xi^\beta \int \{\overset{*}{\tau}\, \delta^c\eta + \tau_c^0 \overset{*}{w}\, \delta u\}\, dV = \xi^{3\beta}\lambda_2^3(1+\beta)^3 g_c^{-1}[\mathscr{L}_{ijkl}\,\alpha_{kl}\, \delta^c\eta_{ij} + m_{ij}^c\, \delta^c\eta_{ij}]_{x_c} \int_{\overset{*}{V}_-} f\, d\overset{*}{z}_1\, d\overset{*}{z}_2\, d\overset{*}{z}_3$$

$$\equiv \xi^{3\beta}\lambda_2^3\, Q(\delta u), \tag{7.4}$$

where all nonboundary-layer terms are evaluated at x_c (this entails relative errors which go to zero in the subsequent limiting process). The last equality defines Q, which is a function of $\delta u_{i,j}(x_c)$. All tensors on the right-hand side of (7.4) are referred to the base bectors of the z_i system. In the vicinity of x_c, the surface of the body at bifurcation can be represented by $2z_1 + z_2^2/R_{22} + z_3^2/R_{33} + 2z_2 z_3/R_{23} + \ldots = 0$, where R_{22} and R_{33} are instantaneous radii of curvature at x_c. Thus, $\overset{*}{V}_-$ represents the volume in $\overset{*}{z}$-space enclosed by the two surfaces

$$f(\overset{*}{z}) = 0 \qquad \text{and} \qquad 2\overset{*}{z}_1 + \frac{1}{R_{22}}\overset{*}{z}{}_2^2 + \frac{1}{R_{33}}\overset{*}{z}{}_3^2 + \frac{2}{R_{23}}\overset{*}{z}_2\overset{*}{z}_3 = 0. \tag{7.5}$$

By (7.4), the second term in (7.3) is of the order $\xi^{3\beta}$. The third term is of order ξ. Since (7.3) is the variational equilibrium equation for $\overset{b}{u}$, $\overset{b}{\eta}$ and $\overset{b}{\tau}$, the first term in (7.3) must be of order $\xi^{3\beta}$ or ξ, whichever is the lower. Now by considering the three possibilities (i.e. $\beta > \frac{1}{3}$, $\beta = \frac{1}{3}$ and $0 < \beta < \frac{1}{3}$), we show that $\beta = \frac{1}{3}$ leading to a balance of the first three terms in (7.3).

First, suppose $\beta > \frac{1}{3}$ so the first term in (7.3) is of order ξ. Define the following limit:

$$\lim_{\xi \to 0,\, x\,\text{fixed}} \xi^{-1}[\overset{b}{u}, \overset{b}{\eta}, \overset{b}{\tau}] = [2\overset{(2)}{u}(x), 2\overset{(2)}{\eta}(x), 2\overset{(2)}{\tau}(x)]. \tag{7.6}†$$

While the first term in (7.3) must be of order ξ if $\beta > \frac{1}{3}$, this does not imply there is no boundary layer component in $\overset{b}{\tau}$ of lower order. However, by (7.2), such a component must be of higher order than ξ^β and thus will make a contribution to the integral in the first term of (7.3) of order higher than $\xi^{3\beta}$. For our purposes we need only the limit for fixed x defined in (7.6). Divide (7.3) by ξ and let $\xi \to 0$ to obtain

† The factor of 2 is introduced in this definition so that in the specialization to an elastic solid these quantities are identical to terms introduced by Budiansky and Hutchinson (1964) and Budiansky (1966).

the variational equation for $\overset{(2)}{u}$, $\overset{(2)}{\eta}$ and $\overset{(2)}{\tau}$,

$$2\int_V \{\overset{(2)}{\tau}{}^{ij}\,\delta^c\eta_{ij}+\tau_c^{0ij}\,\overset{(2)}{u}_{k,i}\,\delta u^k{}_{,j}\}\,dV+$$
$$+2\int_V \{(\lambda_1\,\dot{\tau}^{0ij}+\overset{(1)}{\tau}{}^{ij})\overset{(1)}{u}_{k,i}\,\delta u^k{}_{,j}+\lambda_1\,\overset{(1)}{\tau}{}^{ij}\dot{u}^0_{k,i}\,\delta u^k{}_{,j}\}\,dV = 0. \quad (7.7)$$

To obtain the connection between $\overset{(2)}{\eta}$ and $\overset{(2)}{u}$ write the expression for $\dot{\eta}-\dot{\eta}^0$ using (3.1) and arrange this expression so that it involves the differences $\dot{u}-\dot{u}^0$ and $u-u^c$ similar to what was done in (6.9). Then substitute in the representation (7.1) and make use of (4.5_1) to eliminate the lowest order term. Divide the resulting equation by ξ and let $\xi\to 0$ *with fixed* x with the result

$$\overset{(2)}{\eta}_{ij} = \tfrac{1}{2}(\overset{(2)}{u}_{i,j}+\overset{(2)}{u}_{j,i})+\tfrac{1}{2}(u^{0c}_{k,i}\overset{(2)}{u}{}^k{}_{,j}+u^{0c}_{k,j}\overset{(2)}{u}{}^k{}_{,i})+$$
$$+\tfrac{1}{2}\overset{(1)}{u}_{k,i}\overset{(1)}{u}{}^k{}_{,j}+\tfrac{1}{2}\lambda_1(\dot{u}^0_{k,i}\overset{(1)}{u}{}^k{}_{,j}+\dot{u}^0_{k,j}\overset{(1)}{u}{}^k{}_{,i}). \quad (7.8)$$

A similar calculation for $\dot{\tau}-\dot{\tau}^0$ using the moduli for plastic loading (5.1), the representation (7.1), (4.5), and the limit *with fixed* x leads to

$$\overset{(2)}{\tau}{}^{ij} = L_c^{ijkl}\,\overset{(2)}{\eta}_{kl}+\tfrac{1}{2}(\lambda_1\,\dot{\tau}^{0mn}+\overset{(1)}{\tau}{}^{mn})\left.\frac{\partial L^{ijkl}}{\partial\tau^{mn}}\right|_c\overset{(1)}{\eta}_{kl}+\tfrac{1}{2}\lambda_1\,\overset{(1)}{\tau}{}^{mn}\left.\frac{\partial L^{ijkl}}{\partial\tau^{mn}}\right|_c\dot{\eta}^0_{kl}. \quad (7.9)$$

Now let $\delta u_{i,j} = \overset{(1)}{u}_{i,j}$ in (7.7) and note that $\delta^c\eta_{ij} = \overset{(1)}{\eta}_{ij}$ by (6.11) and (4.5). The identity

$$\int_V \{\overset{(2)}{\tau}{}^{ij}\,\overset{(1)}{\eta}_{ij}+\tau_c^{0ij}\,\overset{(2)}{u}_{k,i}\overset{(1)}{u}{}^k{}_{,j}\}\,dV = \int_V \{\tfrac{1}{2}\overset{(1)}{\tau}{}^{ij}\,\overset{(1)}{u}_{k,i}\,\overset{(1)}{u}{}^k{}_{,j}+\lambda_1\,\overset{(1)}{\tau}{}^{ij}\dot{u}^0_{k,i}\overset{(1)}{u}{}^k{}_{,j}+$$
$$+\tfrac{1}{2}(\lambda_1\,\dot{\tau}^{0mn}+\overset{(1)}{\tau}{}^{mn})\left.\frac{\partial L^{ijkl}}{\partial\tau^{mn}}\right|_c\overset{(1)}{\eta}_{ij}\overset{(1)}{\eta}_{kl}+\tfrac{1}{2}\lambda_1\,\overset{(1)}{\tau}{}^{mn}\left.\frac{\partial L^{ijkl}}{\partial\tau^{mn}}\right|_c\dot{\eta}^0_{ij}\overset{(1)}{\eta}_{kl}\}\,dV \quad (7.10)$$

can be obtained using the variational equation for the eigenmodal quantities,

$$\int \{\overset{(1)}{\tau}\,\delta^c\eta+\tau^{0c}\,\overset{(1)}{u}\,\delta u\}\,dV = 0, \qquad \text{with} \qquad \delta u = \overset{(2)}{u},$$

together with (7.8), (7.9) and the property $L^{ijkl} = L^{klij}$. Thus, under the supposition that $\beta > \frac{1}{3}$, one obtains the following expression involving λ_1 and the eigenmode:

$$A+\lambda_1 B = 0, \quad (7.11)$$

where

$$A = \int_V \{3\,\overset{(1)}{\tau}{}^{ij}\,\overset{(1)}{u}_{k,i}\overset{(1)}{u}{}^k{}_{,j}+\overset{(1)}{\tau}{}^{mn}\left.\frac{\partial L^{ijkl}}{\partial\tau^{mn}}\right|_c\overset{(1)}{\eta}_{ij}\overset{(1)}{\eta}_{kl}\}\,dV \quad (7.12)$$

$$B = \int_V \{2\dot{\tau}^{0ij}\,\overset{(1)}{u}_{k,i}\overset{(1)}{u}{}^k{}_{,j}+4\,\overset{(1)}{\tau}{}^{ij}\dot{u}^0_{k,i}\overset{(1)}{u}{}^k{}_{,j}+\dot{\tau}^{0mn}\left.\frac{\partial L^{ijkl}}{\partial\tau^{mn}}\right|_c\overset{(1)}{\eta}_{ij}\overset{(1)}{\eta}_{kl}+\overset{(1)}{\tau}{}^{mn}\left.\frac{\partial L^{ijkl}}{\partial\tau^{mn}}\right|_c\dot{\eta}^0_{ij}\overset{(1)}{\eta}_{kl}\}\,dV. \quad (7.13)$$

The assumption that $\beta > \frac{1}{3}$ is tantamount to disregarding elastic loading to this order since the boundary-layer terms have dropped out completely. Thus, with λ_1 replaced by λ_1^{he} in (7.11), this equation gives the initial slope of the hypo-elastic comparison solid discussed in Section 5. This expression for λ_1^{he} reduces to an expression given by FITCH (1968) and COHEN (1968) for elastic systems with constant moduli and by BUDIANSKY and HUTCHINSON (1964) and BUDIANSKY (1966) when pre-bifurcation deflections can be neglected. Equation (7.11) can only hold for λ_1 if $\lambda_1 = \lambda_1^{he}$; but we have restricted attention to problems for which $\lambda_1^{he} < \lambda_1$. Thus, β cannot exceed $\frac{1}{3}$.

A similar calculation can be carried out for β anticipated to satisfy $0 < \beta < \frac{1}{3}$ so that the first and second terms in (7.3) are of the same order. The result of this calculation is $\lambda_2^3 Q(\overset{(1)}{u}) = 0$ from which one must conclude that $\lambda_2 = 0$ since, in general, $Q(\overset{(1)}{u}) \neq 0$ (as will be seen by example in the Section 8). In other words, $0 < \beta < \frac{1}{3}$ implies that the boundary-layer terms vanish and the representation (7.1) is not possible for this range of β.

Now try $\beta = \frac{1}{3}$. Define the limit (7.6) as before and carry out the sequence of operations used to arrive at (7.11). The term $\lambda_2^3 Q(\delta u)$ is now added to the left-hand side of (7.7). Equations (7.8) and (7.9) still hold as does (7.10). Equation (7.11) is replaced by

$$\lambda_2^3 Q(\overset{(1)}{u}) = -A - \lambda_1 B = -(\lambda_1 - \lambda_1^{he})B, \tag{7.14}$$

where A and B are again given by (7.12) and (7.13) and $Q(\overset{(1)}{u})$ is obtained by replacing $\delta^c\eta$ by $\overset{(1)}{\eta}$ in (7.4). In summary, (7.14) is the expression for λ_2 for the case of an initial neutral loading point x_c occurring at an isolated point on the traction-free surface of a body when the coefficient C_1 in (6.15) is not zero.

Analogous expressions can be obtained for other cases. When the neutral loading points at bifurcation occur on a line on the surface as in the two examples cited in Fig. 3(b), only two coordinates are stretched and one finds $\beta = \frac{2}{5}$. If the bifurcation behavior involves only one spacial coordinate (as in the case of the simple model of Section 2) then $\beta = \frac{1}{2}$. If the initial neutral loading point is isolated as in Fig. 4 but with $C_1 = 0$, all three coordinates must be stretched proportionally to $\xi^{-\beta/2}$. Then one finds $\beta = \frac{2}{5}$ rather than $\beta = \frac{1}{3}$. In this case the boundary-layer analysis is more difficult to carry out since now the boundary-layer terms do not vanish outside the region of elastic unloading.

8. POST-BIFURCATION BEHAVIOR OF A COMPRESSED SOLID CYLINDRICAL COLUMN

Buckling of a solid cylindrical column under axial compression will be used to illustrate the application of the general post-bifurcation analysis given in the preceding sections.

The so-called tangent modulus load corresponding to the first possible bifurcation of an axially compressed, simply supported, solid cylindrical column is

$$P_c = -\pi R^2 \sigma_c = \frac{\pi^3}{4} E_t^c \frac{R^4}{L^2}. \tag{8.1}$$

Here, R and L are the radius and length of the column, and σ_c and E_t^c are the axial stress and tangent modulus at bifurcation. This formula is based on the well-known Euler–Bernoulli approximations for columns. HILL and SEWELL (1960) have obtained an improved extimate of P_c which takes into account the shear stiffness of the column.

For a solid characterized by isotropic elastic moduli, P_c as given by (8.1) is accurate as long as the parameter $(\mu/E_t^c)(R/L)^2$ is not too large, where μ is the elastic shear modulus. For example, if this parameter is not greater than $\frac{1}{10}$, then for a given value of E_t^c, (8.1) underestimates the lowest bifurcation load by no more than about 5 per cent.

In this section the approximate eigenmode associated with (8.1) will be used in (5.7) to evaluate λ_1 and again in (7.14) to evaluate λ_2. The approximate eigenmodal displacement fields are

$$\overset{(1)}{u}_1 = R\cos\left(\frac{\pi x_2}{L}\right), \qquad \overset{(1)}{u}_2 = \frac{\pi R x_1}{L}\sin\left(\frac{\pi x_2}{L}\right), \qquad \overset{(1)}{u}_3 = 0. \tag{8.2}$$

The Cartesian coordinate system is shown in Fig. 3. For the present calculation it is convenient to use the configuration of the body at bifurcation as the reference state so that R and L are the radius and length at bifurcation. The formulae given in the previous sections are based on the undeformed state as the reference. These formulae can be converted to the bifurcation state reference simply by setting $u^{0c} = 0$ and by taking the volume element to be that in the new reference state. In addition, we will not bother to distinguish between different stress measures. This is permissible when the instantaneous moduli are large compared to the stress level—a condition that will clearly hold for all but stubby columns since, by (8.1), σ/E_t is of order $(R/L)^2$. The symbol σ will be used as an abbreviation for τ_{22}.

The eigenmodal contribution to the deflection is $\xi\overset{(1)}{u}$ so that, with the normalization of (8.2), $\xi = 1$ corresponds to a one radius contribution to the lateral deflection at the center of the column. Nonzero components of the eigenmodel strain are

$$\overset{(1)}{\eta}_{22} = \frac{\pi^2 R x_1}{L^2}\cos\left(\frac{\pi x_2}{L}\right), \qquad \overset{(1)}{\eta}_{11} = \overset{(1)}{\eta}_{33} = -\bar{\upsilon}_c\overset{(1)}{\eta}_{22}, \tag{8.3}$$

where $\bar{\upsilon}_c$ is the instantaneous lateral contraction ratio associated with a uniaxial increment in stress $\dot{\sigma}$, to be specified later. The expression for $\overset{(1)}{\eta}_{22}$ is obtained from (8.2). The other components $\overset{(1)}{\eta}_{11}$ and $\overset{(1)}{\eta}_{33}$ are consistent with the eigenmodal stresses (i.e. $\overset{(1)}{\tau}_{22} = E_t^c\overset{(1)}{\eta}_{22}$ with other components zero) which are assumed as an approximation for the eigenmode. This approach is slightly different from, but equivalent to, deriving all the strains from the Euler–Bernoulli displacement fields (8.2) and *then* approximating the stress–strain relation.

The load parameter λ is taken to be the compressive load normalized by its value at bifurcation (8.1), $\lambda = P/P_c$. Components of the unit tensor normal to the elastic domain in strain-rate space at bifurcation are $m_{22}^c = -2/\sqrt{6}$ and $m_{11}^c = m_{33}^c = 1/\sqrt{6}$; one also finds $\acute{\eta}_{22}^0 = \sigma_c/E_t^c$ and $\acute{\eta}_{11}^0 = \acute{\eta}_{33}^0 = -\bar{\upsilon}_c\acute{\eta}_{22}^0$ where $'()$ is still defined by (3.13) Using (5.7), one finds the initial neutral loading point is given by $x_1^c = R, x_2^c = x_3^c = 0$, as shown in Fig. 3, and that the initial slope is given by $\lambda_1 = 4$. Furthermore,

$$m_{ij}^c(\lambda_1\acute{\eta}_{ij}^0 + \overset{(1)}{\eta}_{ij}) = \frac{2(1+\bar{\upsilon}_c)}{\sqrt{6}}\left(\frac{\pi R}{L}\right)^2\left[1 - \left(\frac{x_1}{R}\right)\cos\left(\frac{{}_1x_2}{L}\right)\right] \tag{8.4}$$

and, from (6.15) and (6.18),

$$f(\overset{*}{z}) = \frac{2(1+\bar{\upsilon}_c)}{\sqrt{6}}\left(\frac{\pi R}{L}\right)^2\left[\frac{1}{4} + \frac{\overset{*}{z}_1}{R} - \frac{1}{2}\left(\frac{\pi}{L}\right)^2\overset{*}{z}{}_2^2\right]. \tag{8.5}$$

We take the elastic moduli to be isotropic at bifurcation so that (3.9) becomes

$$L^c_{ijkl} = \frac{E}{1+v}\left\{\tfrac{1}{2}(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk}) + \frac{v}{1-2v}\,\delta_{ij}\,\delta_{kl}\right\} - \frac{1}{g_c}\,m^c_{ij}\,m^c_{kl}, \tag{8.6}$$

where E is Young's modulus and v is Poisson's ratio. The solution of (6.31) gives $\phi_1 = -(1+v)(1-2v)/[\sqrt{6}(1-v)E]$ and $\phi_2 = \phi_3 = 0$. Using (6.32) and (7.4) with $\beta = \frac{1}{3}$ gives

$$\lambda_2^3\, Q(\overset{(1)}{u}) = -(4\lambda_2/3)^3(\pi R/L)^4 \Gamma E \int_{\overset{*}{V}_-} \left[\tfrac{1}{4} + \frac{\overset{*}{z}_1}{R} - \tfrac{1}{2}\left(\frac{\pi}{L}\right)^2 \overset{*}{z}{}_2^2\right] d\overset{*}{z}_1\, d\overset{*}{z}_2\, d\overset{*}{z}_3, \tag{8.7}$$

where

$$\Gamma = (1+\bar{v}_c)(2+\bar{v}_c - v - 2v\bar{v}_c)/[3(1-v)g_c E] \tag{8.8}$$

and, by (7.5), $\overset{*}{V}_-$ is the volume enclosed by the two surfaces $f(\overset{*}{z}) = 0$ and $2\overset{*}{z}_1 + \overset{*}{z}{}_3^2/R = 0$. The integral in (8.7) can be evaluated in closed form so that (8.7) becomes

$$\lambda_2^3\, Q(\overset{(1)}{u}) = -\lambda_2^3 \Gamma E({}_1R/L)^4 R^2 L/81. \tag{8.9}$$

From (8.6) one finds that the contraction ratio, $\bar{v}_c$, and g_c can be expressed in terms of the tangent modulus according to

$$\bar{v}_c = [1-(1-2v)(E^c_t/E)]/2 \quad \text{and} \quad g_c E = (1+v)[E-(1-2v)E^c_t/3]/[E-E^c_t]. \tag{8.10}$$

It is a straightforward matter to evaluate the remaining terms in (7.14). First of all, $\lambda_1^{he} = 0$ by symmetry. Note also that, since the approximate eigenmode involves only the one non zero component $\overset{(1)}{\tau}_{22}$, it is not necessary to evaluate the full tensor $(\partial \mathbf{L}/\partial \tau)_c$; only the single quantity $(dE_t/d\sigma)_c$ enters. The result of this calculation is

$$-(\lambda_1 - \lambda_1^{he})B = (1+q)E^c_t\left(\frac{\pi R}{L}\right)^4 \pi R^2 L \quad \text{with} \quad q = \left(\frac{\pi R}{2L}\right)^2\left(\frac{dE_t}{d\sigma}\right)_c, \tag{8.11}$$

where, here, terms of order $(R/L)^2$ relative to those retained have been dropped. Finally, combining (8.9) and (8.11) according to (7.14) gives

$$\lambda_2 = -3\left\{\frac{3\pi E^c_t(1+q)}{\Gamma E}\right\}^{1/3}. \tag{8.12}$$

Written out in full, the load–deflection relation is

$$\frac{P}{P_c} = 1 + 4\xi - 3\left\{\frac{3\pi E^c_t(1+q)}{\Gamma E}\right\}^{1/3}\xi^{4/3} + \ldots, \tag{8.13}$$

where it is recalled that $\xi = 1$ corresponds to an eigenmodal contribution to the lateral deflection at the center of the column of one radius. The distance d the neutral loading surface has penetrated into the column on its mid-plane ($x_2 = 0$) is found from (8.5) and (6.16) to be

$$d = \left\{\frac{3\pi E^c_t(1+q)}{\Gamma E}\right\}^{1/3} R\xi^{1/3} + \ldots. \tag{8.14}$$

For most metals the rate of change of the tangent modulus, $dE_t/d\sigma$, is on the order of (1 ÷ strain) in the plastic range. Typical values of q, defined by (8.11), may be as large as from 10 to 100, depending on the slenderness of the column. Thus, through the $\xi^{4/3}$-order term in (8.13), the rate of change of the tangent modulus has a significant effect on the initial post-bifurcation behavior.

The parameter Γ, defined by (8.8), can be expressed as a function of Poisson's ratio v and E_t^c/E†. Values of the factor which multiplies $(1+q)^{1/3}$ in the coefficient of the $\xi^{4/3}$-order term in (8.13) are listed in Table 1 for $v = \frac{1}{3}$ and a range of values of E_t^c/E. Except for very low values of E_t^c/E, the coefficient of the $\xi^{4/3}$-order term is sufficiently large that this term becomes numerically significant compared to the second, 4ξ, at small values of ξ. In other words, the initial slope is a good approximation to the actual slope only for loads slightly above P_c. In the "elastic limit", as $E_t^c/E \to 1$, the expansion breaks down for precisely the same reasons cited for the simple model.

If just the first three terms of (8.13) are used to estimate the maximum support load of the column one finds

$$\frac{dP}{d\xi} = 0 \Rightarrow \xi = \frac{\Gamma E}{3\pi E_t^c(1+q)} \tag{8.15}$$

and

$$\frac{P_0^{\max}}{P_c} = 1 + \frac{\Gamma E}{3\pi E_t^c(1+q)}. \tag{8.16}$$

Values of $\Gamma E/(3\pi E_t^c)$ are included in Table 1 and plots of $P_0^{\max}/P_c$ as a function of E_t^c/E for several values of q are shown in Fig. 5. Also shown in Fig. 5 is a plot of the reduced modulus load P^{rm} normalized by P_c. This is the load at which a straight column with instantaneous moduli E_t^c would deflect under no first order change in axial load.‡

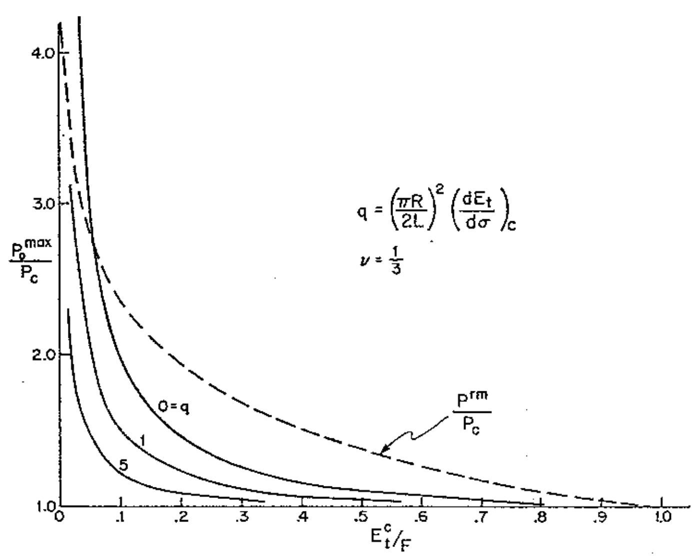

FIG. 5. Approximate maximum support load for an axially compressed column and comparison with the reduced modulus load for a column with constant tangent modulus E^c.

† When $v = \frac{1}{2}$, Γ assures its simplest form: $\Gamma = 1 - (E_t^c/E)$.

‡ The reduced modulus load (VON KÁRMÁN, 1910) for the solid cylindrical column shown in Fig. 5 is calculated using the usual engineering approximation for columns. Details are omitted here.

TABLE 1. *Variation of parameters with* E_t^c/E

$\frac{E_t^c}{E}$	$3\left(\frac{3\pi E_t^c}{\Gamma E}\right)^{1/3}$	$\frac{\Gamma E}{3\pi E_t^c}$
0·05	2·35	2·07
0·1	3·02	0·981
0·2	3·96	0·435
0·3	4·74	0·253
0·5	6·30	0·108
0·7	8·39	0·046
0·9	13·17	0·012
0·95	16·90	0·006
1·0	∞	0

The effect of the rate of change of the tangent modulus on the maximum support load is brought out in (8.16) and in Fig. 5. For most applications with typical structural materials, (8.16) implies that the maximum support load is only slightly larger than the lowest bifurcation load. Experiments (SHANLEY, 1947) and numerical calculations (DUBERG and WILDER (1952), MALVICK and LEE (1965), and HOFF (1967)) have previously suggested that same conclusion.

Equation (8.13) is an asymptotic expression valid for small ξ. It is approximate only in that λ_1 and λ_2 are evaluated using an approximate eigenmode rather than the exact mode. On the other hand, (8.16) is an approximate formula which is not asymptotic in any sense because the maximum support load occurs at a *finite*, although perhaps very small, value of ξ. Obviously, (8.16) can be an accurate estimate of the maximum support load only if the first three terms in (8.13) provide a good approximation of the load–deflection relation up until the maximum load point. Cited below are two reasons why the numerical accuracy of (8.16) may be suspect even though it seems to give a correct qualitative picture.

From Fig. 5, note that the prediction of (8.16) with $q = 0$ falls below the reduced modulus curve, except for very small E_t/E. Calculations based on two-flange column models indicate that the reduced modulus load is approached asymptotically for large lateral deflections if E_t is constant (see, for example, DUBERG and WILDER (1952)). While apparently this has not been verified for a solid column, it is a plausible result which suggests that the maximum load prediction (8.16) with $q = 0$ is incorrect. That is, with constant E_t, no maximum load point may exist at any (small) finite value of ξ and the maximum load prediction of (8.16) with $q = 0$ is a fictitious result stemming from the termination of the series expansion.

Further, note that, according to (8.14) and (8.15), the neutral loading surface has penetrated a distance $d = R$ into the column at the point where the maximum load is attained for any value of q. It may well be that the perturbation expansion is not accurate for such a large encroachment of the elastic unloading region. In addition, numerical results of MALVICK and LEE (1965) and discussion by SEWELL (1972) suggest that the maximum load should occur before the neutral loading surface has penetrated as far as half way through the column. Here again, this suggests that (8.13) may not have a sufficient range of validity to yield a numerically accurate prediction for the maximum load.

9. CONCLUDING REMARKS

The initial post-bifurcation expansions corroborate previous experience gleaned from experiments and model problems: namely, that the initial slope of the load–deflection curve (required for bifurcation to occur at the lowest possible load) governs in only a very small neighborhood of the bifurcation point; and rates of change of the instantaneous moduli at bifurcation have a major influence on the post-bifurcation behavior. It is an open question as to how large is the range of validity of these expansions and whether they can be used to give reasonably accurate predictions of the maximum support load. Certainly, the range of validity will vary from problem to problem. Experience with similar expansions in the elastic range indicates that in most applications only a few terms are necessary to give an accurate picture of a significant portion of the initial post-bifurcation behavior (e.g. HUTCHINSON and KOITER (1970)).

While our discussion has emphasized bifurcation under compressive load, the analysis applies equally well to bifurcation under tensile-type loadings where bifurcation usually occurs under increasing elongation rather than increasing load.

Some of the restrictions made in the course of the analysis are easily removed. In particular, the case of multiple eigenmodes associated with the lowest bifurcation load can be treated with little additional complexity. To carry the expansions to higher order terms beyond what is done in Sections 6 and 7 would, in general, entail greatly increased effort. In this connection, it should be noted that, due to symmetry of the structure and the eigenmode, the cubic terms, (7.12), and thus also λ_1^{he} will vanish identically in many applications. As a consequence in such cases, the expansion, to the order it has been carried out here, does not bring in the geometrical nonlinearities which determine whether bifurcation in the elastic range is stable or unstable. These nonlinearities can be expected to be significant to the plastic post-bifurcation behavior as well.

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APPENDIX

ANALYSIS OF THE SIMPLE MODEL

Let θ be the independent variable on the post-bifurcation branch so that $(\dot{\ }) = d(\ )/d\theta$. The position ($x = d$) of the instantaneous boundary between the loading and unloading regions is where $\dot{\varepsilon} = 0$, and from (2.1)

$$d = -\dot{u}. \tag{A.1}$$

For the moment, E_t is regarded as a function of ε. Equation (2.4) can be rewritten as

$$\dot{P} = \int_{-L}^{d} E(\dot{u}+x)\,dx + \int_{d}^{L} \times \left[E_t^c + \left(\frac{dE_t}{d\varepsilon}\right)_c (\Delta u + x\theta) + \tfrac{1}{2}\left(\frac{d^2E_t}{d\varepsilon^2}\right)_c (\Delta u + x\theta)^2 + \ldots\right](\dot{u}+x)\,dx, \tag{A.2}$$

where $\Delta u = u - u_c$. The integrations in (A.2) are now carried out to give

$$\dot{P} = E\dot{u}(d+L) + E_t^c\dot{u}(L-d) + \frac{(E-E_t^c)}{2}(d^2-L^2) + \left(\frac{dE_t}{d\varepsilon}\right)_c [\Delta u\dot{u}(L-d) + \tfrac{1}{2}\Delta u(L^2-d^2) + \tfrac{1}{2}\dot{u}\theta(L^2-d^2) + \tfrac{1}{3}\theta(L^3-d^3)] + \tfrac{1}{2}\left(\frac{d^2E_t}{d\varepsilon^2}\right)_c [\Delta u^2\dot{u}(L-d) + (\ )] + \ldots. \tag{A.3}$$

In the same way (2.5) can be written as

$$(P\tilde{L}\theta)^{\cdot}+\dot{K}\tilde{L} = \tfrac{1}{2}(E-E_t^c)\dot{u}(d^2-L^2)+\tfrac{1}{3}E(d^3+L^3)+\tfrac{1}{3}E_t^c(L^3-d^3)+$$
$$+\left(\frac{dE_t}{d\varepsilon}\right)_c[\tfrac{1}{2}\Delta u\dot{u}(L^2-d^2)+\tfrac{1}{3}\Delta u(L^3-d^3)+\tfrac{1}{3}\dot{u}\theta(L^3-d^3)+\tfrac{1}{4}\theta(L^4-d^4)]+$$
$$+\tfrac{1}{2}\left(\frac{d^2E_t}{d\varepsilon^2}\right)_c[\tfrac{1}{2}\Delta u^2\dot{u}(L^2-d^2)+(\,)]+\dots \qquad \text{(A.4)}$$

Consider bifurcation with monotonically increasing θ with $a_1^e \leq 0$. For bifurcation at any value of the load *greater* than the lowest possible bifurcation load (i.e. $P_c > 2E_t^cL^3/(3\tilde{L})$) a perturbation expansion can be developed involving only integral powers of θ. However, this expansion breaks down as $P_c \to 2E_t^cL^3/(3\tilde{L})$. It is found that the system of differential equations (A.1), (A.3) and (A.4) has a singular point at $\theta = 0$ when bifurcation takes place at the lowest bifurcation load, $P_c = 2E_t^cL^3/(3\tilde{L})$. By admitting the possibility of terms in the expansion with fractional powers of θ, it is found that the expansions must include half-integer powers of θ.

The expansion for P is of the form of (2.11); Δu is expressed as

$$\frac{\Delta u}{L} = b_1\theta+b_2\theta^{3/2}+b_3\theta^2+b_4\theta^{5/2}+\dots \qquad \text{(A.5)}$$

and d, from (A.1), is given by (2.12). The neutral loading point must occur at $d = -L$ at bifurcation and thus from (2.12) $b_1 = 1$. Expansions (2.11), (2.12) and (A.5) are substituted into (A.3) and like-power terms in θ are collected with the result that

$$a_1 = 3\tilde{L}/L, \qquad a_2 = 3b_2\tilde{L}/L, \qquad a_3 = \frac{3b_3\tilde{L}}{L}-\frac{27(E-E_t^c)\tilde{L}b_2^2}{32E_t^cL}+\frac{2\tilde{L}}{E_t^c}\left(\frac{dE_t}{d\varepsilon}\right)_c, \qquad \text{(A.6)}$$

etc. Similarly, (A.4) implies

$$\left.\begin{aligned} a_1 &= \frac{27(E-E_t^c)b_2^2}{32E_t^c}+\frac{L}{E_t^c}\left(\frac{dE_t}{d\varepsilon}\right)_c-\frac{3k_1\tilde{L}}{2E_t^cL}, \\ a_2 &= \frac{9(E-E_t^c)}{5E_t^c}[b_2b_3+\tfrac{3}{16}b_2^3]+\frac{b_2L}{E_t^c}\left(\frac{dE_t}{d\varepsilon}\right)_c, \text{ etc.} \end{aligned}\right\} \qquad \text{(A.7)}$$

First, b_2 is obtained from (A.7$_1$); from (2.12) it is seen that the negative root of b_2^2 is chosen since the neutral loading point must move away from $d = -L$ with increasing values of d. From here on the equations are solved sequentially with the results given by (2.13) to (2.16).