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FINITE STRAIN ANALYSIS OF ELASTIC-PLASTIC SOLIDS AND STRUCTURES

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ABSTRACT

A review is given of one approach to the formulation of equations for elastic-plastic solids at finite strains which lends itself to numerical analysis. A generalization of J_2 flow theory to large strains is given which is in a form convenient for applications. Several aspects of the analysis of necking in tension are discussed from this point of view. Applications of the formulation to nonlinear plate and shell theory are also discussed.

INTRODUCTION

Most of the nonlinear theories of plates and shells are Lagrangian in character in that they employ as a reference configuration the undeformed state of the structure. In the construction of these theories it is common practice to start with a set of strain measures and strain-displacement relations (which are usually approximate in some sense), to introduce conjugate stress quantities, and to then postulate a variational principle of virtual work in terms of the variables of the theory. Equilibrium equations are obtained as the Euler equations of the variational principle. In this way the variables of the ad hoc theory are connected by exact variational equations and one or another of these principles is usually at the heart of any scheme for discretizing the equations. Budiansky (1) has emphasized the common mathematical structure shared by such ad hoc theories and a particular form of the nonlinear field equations for three-dimensional solid bodies which employs the Lagrangian strain tensor and the undeformed configuration of the body as reference. This formulation as it pertains to elastic-plastic solids will be briefly reviewed here. A finite strain version of J_2 flow theory will be discussed which fits nicely into the Lagrangian formulation. Some recent results for the problem of necking of a bar in tension will serve to illustrate the possibilities which are opened up by the application of numerical analysis methods to problems involving finite strain complications. A relatively straightforward way to incorporate certain finite strain aspects into the elastic-plastic analysis of thin plates and shells is also discussed.

A LAGRANGIAN FORM OF THE FIELD EQUATIONS FOR ELASTIC-PLASTIC SOLIDS

Material points are identified by a set of convected coordinates x^i. Following the standard convention, superscripted indices denote contravariant components of a tensor and subscripted components the covariant components. Let g_{ij} and G_{ij} be the metric tensors of the undeformed and deformed configurations and let g^{ij} and G^{ij} be their respective inverses. Denote base vectors in the undeformed body by $\underset{\sim}{e}_i$ and their reciprocals by $\underset{\sim}{e}^i = g^{ij}\underset{\sim}{e}_j$. Similarly, the base vectors in the deformed body are denoted by $\bar{\underset{\sim}{e}}_i$ and $\bar{\underset{\sim}{e}}^i = G^{ij}\bar{\underset{\sim}{e}}_j$. Denote the displacement vector from the undeformed configuration by $\underset{\sim}{u} = u_i\underset{\sim}{e}^i = u^i\underset{\sim}{e}_i$ where $u^i = g^{ij}u_j$. The Lagrangian strain tensor is

$$\eta_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}\, u^k{}_{,i}u_{k,j} \,, \tag{1}$$

where the comma denotes covariant differentiation with respect to the undeformed metric.

The exact statement of the principle of virtual work based on the undeformed configuration is (1, 2, 3)

$$\int_V \tau^{ij}\delta\eta_{ij}dV = \int_S T^i\delta u_i dS \,, \tag{2}$$

where

$$\delta\eta_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) + \frac{1}{2}(u^k{}_{,i}\delta u_{k,j} + u^k{}_{,j}\delta u_{k,i}) \,. \tag{3}$$

Here, dV and dS are the volume and surface elements of the undeformed body, τ^{ij} are the contravariant components of the symmetric Kirchhoff stress defined with respect to the deformed base vectors, and $\underset{\sim}{T} = T^i\underset{\sim}{e}_i$ is the surface traction vector per unit undeformed area. With $\underset{\sim}{n} = n_i\underset{\sim}{e}^i$ denoting the unit normal to a surface element in the undeformed body, the surface traction $\underset{\sim}{T}$ acting on this surface element in the deformed body is

$$\underset{\sim}{T} = (\tau^{ij} + \tau^{mj}u^i{}_{,m})n_j\underset{\sim}{e}_i \,. \tag{4}$$

Let $g = |g_{ij}|$ and $G = |G_{ij}|$. The contravariant components of the Cauchy stress are given by

$$\sigma^{ij} = (g/G)^{1/2}\tau^{ij} \,. \tag{5}$$

The surface traction vector per unit current area $\bar{\underset{\sim}{T}}$ acting on a surface whose current unit normal is $\bar{\underset{\sim}{n}} = \bar{n}_i\bar{\underset{\sim}{e}}^i$ is given by

$$\bar{\underset{\sim}{T}} = \sigma^{ij}\bar{n}_i\bar{\underset{\sim}{e}}_j \,. \tag{6}$$

The incremental form of the principle of virtual work is

$$\int_V \{\dot{\tau}^{ij}\delta\eta_{ij} + \tau^{ij}\dot{u}^k{}_{,j}\delta u_{k,i}\} dV = \int_S \dot{T}^i \delta u_i dS \ , \tag{7}$$

and the associated equilibrium equations are

$$\dot{\tau}^{ij}{}_{,j} + (\dot{\tau}^{kj}u^i{}_{,k})_{,j} + (\tau^{kj}\dot{u}^i{}_{,k})_{,j} = 0 \ . \tag{8}$$

Hill (4) has discussed the general framework for the classical rate-constitutive relations for elastic-plastic solids with smooth yield surfaces at finite strain. Using the convected rate of the contravariant components of the Kirchhoff stress, the rate-constitutive relation can be expressed in the general form

$$\dot{\tau}^{ij} = L^{ijk\ell}\dot{\eta}_{k\ell} \tag{9a}$$

where

$$L^{ijk\ell} = \mathscr{L}^{ijk\ell} - \frac{\alpha}{g} m^{ij} m^{k\ell} \ . \tag{9b}$$

For stresses within the yield surface $\alpha = 0$ and for stresses on the yield surface

$$\alpha = 1 \quad \text{if} \quad m^{ij}\dot{\eta}_{ij} \geq 0 \quad \text{and} \quad \alpha = 0 \quad \text{if} \quad m^{ij}\dot{\eta}_{ij} < 0 \ . \tag{10}$$

Here, $\mathscr{L}$ is the current tensor of elastic moduli for this choice of stress-rate and it is assumed that $\mathscr{L}^{ijk\ell} = \mathscr{L}^{k\ell ij}$. The tensor of instantaneous moduli for loading is $\underset{\sim}{L}$ and $\underset{\sim}{m}$ is the current unit tensor normal to the yield surface in strain-rate space. The current level of strain hardening is determined by g and the strain-rate is given by

$$\dot{\eta}_{ij} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) + \frac{1}{2}(u^k{}_{,j}\dot{u}_{k,i} + u^k{}_{,i}\dot{u}_{k,j}) \ . \tag{11}$$

Introduce the functional of $\dot{\underset{\sim}{u}}$,

$$I = \frac{1}{2}\int_V \{\dot{\tau}^{ij}\dot{\eta}_{ij} + \tau^{ij}\dot{u}^k{}_{,i}\dot{u}_{k,j}\} dV - \int_{S_T} \dot{T}^i \dot{u}_i dS \ , \tag{12}$$

where $\dot{\underset{\sim}{T}}$ is prescribed on S_T and $\dot{\underset{\sim}{u}}$ on S_u and where the stress-rates $\dot{\tau}^{ij}$ are regarded to be a function of the strain-rates through (9) and (10). The variational principle governing the incremental boundary value problem is (5)

$$\delta I = 0 \tag{13}$$

for all admissible δu_i which vanish on S_u . Equations (12) and (13) reduce to the well-known principle for the classical small strain and small rotation theory.

The above variational equation provides the theoretical foundation for a variety of possible numerical solution methods. Chen (6) used a

Kantorovich approximation method in conjunction with this variational equation to analyze necking in a bar. Needleman (7) used the principle as the basis for a finite element method solution to a large strain problem related to void growth and coalescence in metals. The same method was applied to the tensile necking problem (8) and some results from this calculation will be discussed in a later section.

Oden (9) has given an extensive review of the work on the development of finite element methods for the large strain analysis of elastic solids. Hibbitt, Marcal and Rice (10) have discussed the formation of finite element equations based on a Lagrangian formulation for elastic-plastic solids which is essentially identical to that reviewed above. The choice of a Lagrangian based numerical scheme as opposed to a Eulerian scheme, for example, is dictated by a number of considerations. Since the variational functional (12) is based on the undeformed configuration, the finite element (or finite difference) grid remains fixed. For this reason, the Lagrangian approach can be attractive if the undeformed configuration is a simple one. In the simplest finite element scheme, used by Needleman, the displacement fields within triangular elements are taken to be linear functions of the reference coordinates and thus the strains, stresses and moduli are constant within each element. At each stage of the calculation procedure the moduli must be updated in a straightforward way which can be illustrated by one possible prescription for the moduli in the next section. As in any elastic-plastic calculation, the loading-unloading behavior associated with an incremental step must in general be handled in an iterative fashion.

A FINITE STRAIN GENERALIZATION OF J_2 FLOW THEORY

Small strain formulations of strain-hardening plasticity involve the stress deviator s_{ij} and the J_2 invariant where in Cartesian coordinates

$$s_{ij} = \tau_{ij} - \frac{1}{3}\tau_{pp}\delta_{ij} \quad \text{and} \quad J_2 = \frac{1}{2} s_{ij}s_{ij} \ , \tag{14}$$

where δ_{ij} is the Kronecker delta. It is usually unnecessary to give a precise definition to the stress measure in small strain formulations and for the moment the precise meaning of τ_{ij} will be left ambiguous. In one of the most widely used plasticity theories, J_2 flow theory, the strain-rate is given in terms of the stress-rate by

$$\dot{\eta}_{ij} = \frac{1}{E}[(1+\nu)\delta_{ik}\delta_{j\ell} + \nu\delta_{ij}\delta_{k\ell}]\dot{\tau}_{k\ell} + \frac{\alpha f}{E} s_{ij}\dot{J}_2 \ , \tag{15}$$

where

$$\left.\begin{aligned} &\alpha = 1 \quad \text{if} \quad \dot{J}_2 = s_{ij}\dot{\tau}_{ij} \geq 0 \quad \text{and} \quad J_2 = (J_2)_{max} \\ &\alpha = 0 \quad \text{if} \quad \dot{J}_2 < 0 \quad \text{or} \quad J_2 < (J_2)_{max} \end{aligned}\right\} \tag{16}$$

In (15) E is Young's modulus, ν is Poisson's ratio and f is a function of J_2 which can be chosen to make (15) coincide with any monotonic proportional loading history.

The inversion of (15) is

$$\dot{\tau}_{ij} = \frac{E}{1+\nu}[\delta_{ik}\delta_{j\ell} + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{k\ell}]\dot{\eta}_{k\ell} - \frac{E}{1+\nu}\frac{\alpha}{g} s_{ij}s_{k\ell}\dot{\eta}_{k\ell} \ , \tag{17}$$

where if $J_2 = (J_2)_{max}$ $\alpha = 1$ if $s_{ij}\dot{\eta}_{ij} \geq 0$ and $\alpha = 0$ if $s_{ij}\dot{\eta}_{ij} < 0$. Also, f and g are connected by

$$\bar{g}' = f/[(1+\nu) + 2fJ_2] \tag{18}$$

The expression for the moduli in the small strain formulation is thus

$$L_{k\ell ij} = L_{ijk\ell} = \frac{E}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{k\ell} - \frac{\alpha}{g}\,s_{ij}s_{k\ell}\right] \tag{19}$$

If a uniaxial tension curve is used to determine f and g , one finds that they are given by

$$[1 + \frac{4}{3}\,J_2 f]^{-1} = (g-2J_2)/[g - \frac{2}{3}(1-2\nu)J_2] = \frac{E_t}{E}\,, \tag{20}$$

where E_t is the tangent modulus which is regarded as a function of J_2 through the connection with the tensile stress, $J_2 = \sigma^2/3$

There are many possible ways to generalize the above relation to a finite strain formulation (11, 12, 13). The one selected for discussion is a special case of Hill's (4) general class (9) and (10) and has a form particularly suitable to a Lagrangian approach. It is a slightly modified version of a relation proposed by Budiansky (14). As in the small strain version the theory employs a J_2 invariant of the stress to describe the yield surface and thus does not account for any Bauschinger effect. It is also assumed that the strains are not so large that appreciable elastic anisotropy develops.

The contravariant components of the Kirchhoff stress τ^{ij} will be used in the formulation and a deviator stress tensor is defined according to

$$s^{ij} = \tau^{ij} - \frac{1}{3}\,G^{ij}G_{k\ell}\tau^{k\ell} \tag{21}$$

where $\underset{\sim}{G}$ is the metric tensor in the deformed system as previously introduced so that with this definition $G_{ij}s^{ij} = 0$. We take J_2 to be defined in terms of the stress deviator by

$$J_2 = \frac{1}{2}\,G_{ik}G_{j\ell}s^{ij}s^{k\ell}. \tag{22}$$

If the coordinate system in the deformed body happens to be Cartesian then (21) and (22) have the same form as (14). Since the undeformed configuration is used as reference the deformed configuration will not, in general, be Cartesian and the general tensor formulation of (21) and (22) is necessary.

If the Cauchy stress (5) is used in forming J_2 in place of the Kirchhoff stress, the invariant will differ from (22) by a multiplicative factor $G/g = (d\bar{V}/dV)^2$, where $d\bar{V}/dV$ is the deformed volume per unit undeformed volume. The volume change in the relation given below arises entirely from the elastic part of the strain-rate. As long as the hydrostatic pressure is very small compared to the elastic bulk modulus, there is little experimental evidence to point to one choice over the other in the formulation of a yield criterion as discussed by Lee (12).

With J_2 defined by (22) it can be shown that the rate of change of

J_2 is

$$\dot{J}_2 = G_{ik}G_{j\ell}s^{k\ell}\overset{*}{\tau}^{ij} \tag{23}$$

Here the $\overset{*}{\tau}^{ij}$ are the contravariant components of the symmetric Jaumann rate of change of the Kirchhoff stress which are related to the convected rate $\dot{\tau}^{ij}$ by

$$\overset{*}{\tau}^{ij} = \dot{\tau}^{ij} + G^{ik}\tau^{j\ell}\dot{\eta}_{k\ell} + G^{jk}\tau^{i\ell}\dot{\eta}_{k\ell} \, . \tag{24}$$

The generalization of (15) we will use is

$$\dot{\eta}_{ij} = \frac{1}{E}[(1+\nu)G_{ik}G_{j\ell} + \nu G_{ij}G_{k\ell}]\overset{*}{\tau}^{k\ell} + \frac{\alpha f}{E} G_{ik}G_{j\ell}s^{k\ell}\dot{J}_2 \tag{25}$$

with

$$\left.\begin{aligned} \alpha &= 1 \quad \text{if} \quad \dot{J}_2 \geq 0 \quad \text{and} \quad J_2 = (J_2)_{max} \\ \alpha &= 0 \quad \text{if} \quad \dot{J}_2 \leq 0 \quad \text{or} \quad J_2 < (J_2)_{max} \end{aligned}\right\} \tag{26}$$

In (25) f is regarded as a function of J_2, and E and ν are taken to be fixed constants corresponding to their values in the undeformed state. The second part of (25) is regarded as the plastic strain-rate; and since $G_{ij}s^{ij} = 0$, the plastic volume change is zero.

In the absence of plastic deformation (25) is a hypo-elastic relation in that the relation cannot be integrated to give the strains in terms of the stresses. Curiously, though, it is possible to write the work done by the stresses per unit original volume in terms of the stresses (and the deformed metric tensor) as

$$\int_0 \tau^{ij} d\eta_{ij} = \frac{1}{2E}[(1+\nu)J_2 + \frac{1}{3}(1-2\nu)(G_{ij}\tau^{ij})^2] \, . \tag{27}$$

The inversion of (25) is

$$\overset{*}{\tau}^{ij} = \frac{E}{1+\nu}[G^{ik}G^{j\ell} + \frac{\nu}{1-2\nu} G^{ij}G^{k\ell}]\dot{\eta}_{k\ell} - \frac{E}{1+\nu}\frac{\alpha}{g} s^{ij}s^{k\ell}\dot{\eta}_{k\ell} \tag{28}$$

with

$$\left.\begin{aligned} \alpha &= 1 \quad \text{if} \quad s^{k\ell}\dot{\eta}_{k\ell} \geq 0 \quad \text{and} \quad J_2 = (J_2)_{max} \\ \alpha &= 0 \quad \text{if} \quad s^{k\ell}\dot{\eta}_{k\ell} < 0 \quad \text{or} \quad J_2 < (J_2)_{max} \end{aligned}\right\} \tag{29}$$

The same relation (18) holds between f and g as in the small strain formulation. Using (24) the rate-constitutive relation can be cast into the form (9) involving $\dot{\tau}^{ij}$ and appropriate to the present formulation, i.e.,

$$\dot{\tau}^{ij} = L^{ijk\ell}\dot{\eta}_{k\ell} \, . \tag{30}$$

The instantaneous moduli are

$$L^{ijk\ell} = L^{k\ell ij} = \frac{E}{1+\nu}\left[\frac{1}{2}(G^{ik}G^{j\ell} + G^{i\ell}G^{jk}) + \frac{\nu}{1-2\nu}\, G^{ij}G^{k\ell} - \frac{\alpha}{g}\, s^{ij}s^{k\ell}\right] - \frac{1}{2}\left[G^{ik}\tau^{j\ell} + G^{jk}\tau^{i\ell} + G^{i\ell}\tau^{jk} + G^{j\ell}\tau^{ik}\right] \tag{31}$$

with α obeying (29).

If data from a uniaxial stress-strain curve is used to determine f and g one finds by specializing (25) to pure tension that, instead of (20),

$$[1+\frac{4}{3}\, J_2 f]^{-1} = (g-2J_2)/[g-\frac{2}{3}(1-2\nu)J_2] = \left(\frac{G}{g}\right)^{1/2}\left[\frac{E_t}{E}+\frac{\sigma}{E}(1-2\bar{\nu})\right] \quad . \tag{32}$$

The tensile data in this equation is considered to be known as a function of the true stress σ . In simple tension $J_2 = (G/g)\sigma^2/3$. The tangent modulus E_t is now defined as $E_t = d\sigma/d\varepsilon$, where ε is the logarithmic, or natural, tensile strain. The instantaneous contraction ratio is defined to be $\bar{\nu} = -d\varepsilon_2/d\varepsilon$, where ε_2 is the logarithmic strain transverse to the tensile direction. For an elastically incompressible material, $\nu = \bar{\nu} = 1/2$ and $G/g = 1$ so that (32) reduces to the small strain expression (20) with the proper interpretation of E_t .

As has been discussed by many authors, two conditions are required for the small strain relation to provide an accurate approximation to a full finite strain version. For the purpose of discussion choose a Cartesian system in the undeformed body. If the strains are sufficiently small the distinction between the deformed and undeformed metric tensors in (31) can be ignored. Secondly, if the stresses are small compared to the instantaneous moduli then the second set of bracketed terms in (31) can be neglected compared to the first. In addition, (32) becomes (20), and the rate-constitutive relation becomes indistinguishable from the small strain version.

The above relation is due essentially to Budiansky (14). His original suggestion is identical in all respects except that the contravariant components of the Cauchy stress (5) σ^{ij} are used everywhere in place of the contravariant components of Kirchhoff stress τ^{ij} in equations (21) through (31). In particular, the deviator components (21) are formed from the Cauchy stress components and J_2 is based on these deviator components. Similarly, $\overset{*}{\tau}{}^{ij}$ and $\dot{\tau}^{ij}$ are replaced in (24) by $\overset{*}{\sigma}{}^{ij}$ and $\dot{\sigma}^{ij}$, which are the Jaumann and convected rates, respectively, of the contravariant components of the Cauchy stress. In this alternative formulation equations (21) through (31) remain unchanged although (27) is now interpreted as the stress work per unit deformed volume.

One feature which is particularly attractive about this second form is that, instead of (32), f and g are given by the formulas for the classical small strain version (20) where E_t is the tangent modulus of the true stress-natural strain curve in uniaxial tension. The one drawback of the version formulated in the form of the Cauchy stress is that when the moduli are converted to the form (9) involving $\dot{\tau}^{ij}$, the moduli do not satisfy the symmetry $L^{ijk\ell} = L^{k\ell ij}$ required for the variational principle (13) to hold. This can be noted directly using the relation

$$\dot{\tau}^{ij} = (G/g)^{1/2}\dot{\sigma}^{ij} + \tau^{ij}\bar{G}^{k\ell}\dot{\eta}_{k\ell} \ . \tag{33}$$

For elastically incompressible solids the two versions are obviously identical. Numerically the difference between the two formulations will be inconsequential as long as the pressure is small compared to the bulk modulus.

APPLICATIONS TO NECKING ANALYSIS

In the analysis of necking in tension it is essential to use a bona fide finite strain formulation. The above formulation of the field equations has been used in the analysis of two separate aspects of necking of a solid circular cylindrical bar in tension (6, 8, 15).

First consider a bar whose ends are subject to a prescribed uniform relative axial displacement in such a way that the ends remain free of tangential traction and the lateral surface is traction-free. For these ideal boundary conditions, the uniform state of uniaxial tension is an exact solution at all values of the relative end displacement. Necking will start as a bifurcation from the uniform state. Bifurcation first becomes possible at the value of the elongation where there first exists a nonzero displacement-rate field $\dot{u}_i$ such that

$$\int_V \{L^{ijk\ell}\dot{\eta}_{ij}\dot{\eta}_{k\ell} + \tau^{ij}\dot{u}^k{}_{,i}\dot{u}_{k,j}\}dV = 0 \ , \tag{34}$$

where the moduli $\underset{\sim}{L}$ are given by (31) with $\alpha = 1$. Here the strain-rate is given by (11) and the axial component of the eigenmodal displacement-rate must vanish on the ends of the specimen.

Denote the true stress and natural strain associated with the state at which the maximum total load of the cylindrical bar is attained by σ_m and ε_m, respectively. Miles (16) has proved that bifurcation cannot occur before the maximum load is attained. The axisymmetric bifurcation problem for an incompressible bar has been studied within the context of the full three-dimensional formulation in (15, 16, 17). Let R_m and L_m denote the radius and length of the specimen when the maximum load is attained and let $\gamma = \pi R_m/L_m$. For an incompressible material characterized by (31), the true stress and natural strain at bifurcation, σ_c and ε_c respectively, are given by the expressions (15)

$$\sigma_c = \sigma_m + \left(1 - \left.\frac{dE_t}{d\sigma}\right|_m\right)^{-1}\left(\frac{\gamma^2}{8}\sigma_m + \frac{\gamma^4}{192}\mu\right) + \ldots \tag{35a}$$

and

$$\varepsilon_c = \varepsilon_m + \left(1 - \left.\frac{dE_t}{d\sigma}\right|_m\right)^{-1}\left(\frac{\gamma^2}{8}\sigma_m + \frac{\gamma^4}{192}\mu\right) + \ldots \ , \tag{35b}$$

which are asymptotically exact for small γ. In these formulas $\mu = E/3$ is the elastic shear modulus of the incompressible material, and $(dE_t/d\sigma)_m$ denotes the derivative of the tangent modulus of the true stress-natural strain curve with respect to the true stress and evaluated at the maximum load.

An example presented in (15) uses the Ramberg-Osgood tensile relation

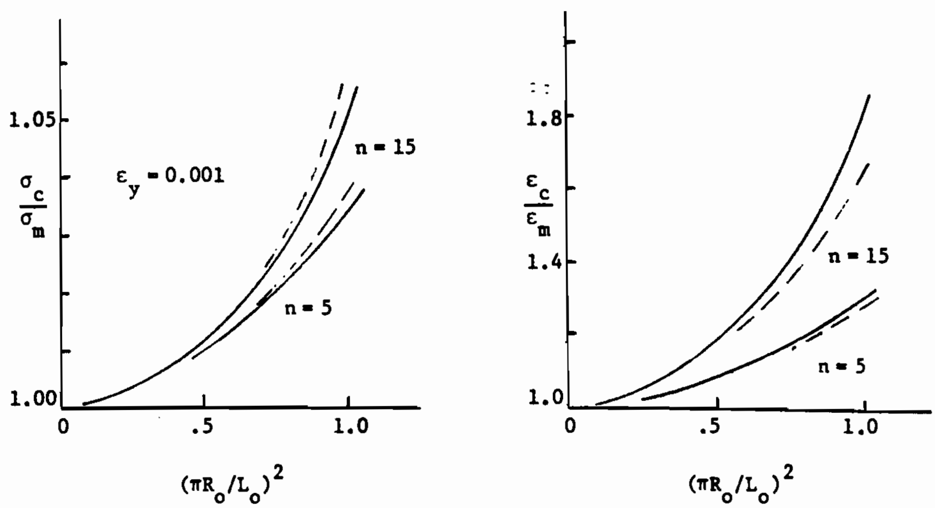

Fig. 1 Tensile bifurcation of a solid cylindrical specimen of an incompressible material with a Ramberg-Osgood tensile stress-strain relation. See (15) for an accurate plot. (Asymptotic results ----------; Exact results ————)

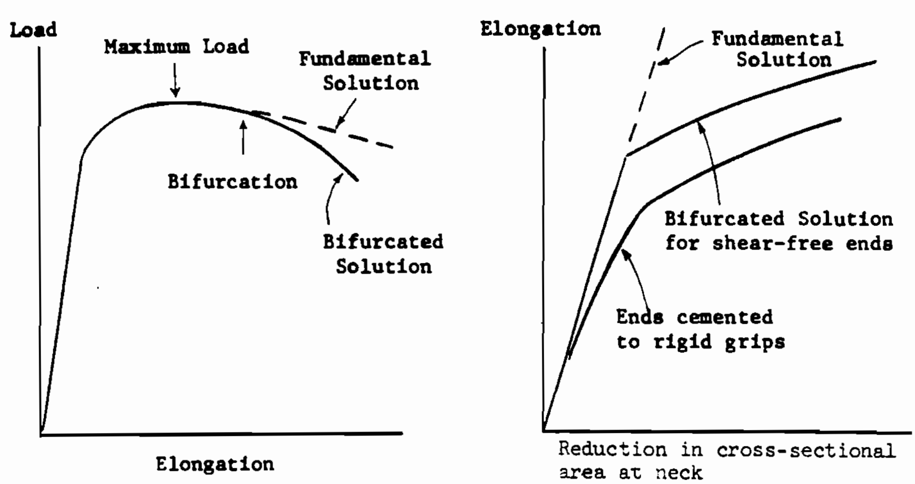

Fig. 2 Schematic of results of a numerical analysis of necking of a solid cylindrical bar from (8).

between the true stress and natural strain, i.e.,

$$\frac{\varepsilon}{\varepsilon_y} = \frac{\sigma}{\sigma_y} + \frac{3}{7}\left(\frac{\sigma}{\sigma_y}\right)^n , \tag{36}$$

where ε_y and $\sigma_y \equiv E\varepsilon_y$ are the effective yield strain and yield stress and n is the hardening exponent. For this case Eqs. (35) become

$$\frac{\sigma_c}{\sigma_m} \simeq 1 + \frac{1}{n}\left(\frac{\gamma^2}{8} + \frac{\gamma^4}{192}\frac{\mu}{\sigma_m}\right) \tag{37a}$$

and

$$\frac{\varepsilon_c}{\varepsilon_m} \simeq 1 + \frac{1}{n\varepsilon_m}\left(\frac{\gamma^2}{8} + \frac{\gamma^4}{192}\frac{\mu}{\sigma_m}\right) \tag{37b}$$

Figure 1 displays plots of σ_c/σ_m and $\varepsilon_c/\varepsilon_m$ as a function of $\pi R_o/L_o$ where where R_o and L_o are the undeformed radius and length of the bar. The dashed line curves are derived from the asymptotic formulas (37) and the solid line curves are the exact results for arbitrarily large $\pi R_o/L_o$ (which require some numerical analysis in their evaluation).

Needleman (8) used the variational principle of the previous section to formulate a finite element scheme and applied it to the necking problem. He considered elastically compressible solids and used the moduli (31) together with an inconsequential approximation in which the right-hand-side of (32) is replaced by E_t/E . The axisymmetric eigenvalue problem governing bifurcation was solved using a finite element method and the post-bifurcation calculation was carried until a point where the specimen had undergone significant necking down. Figure 2 depicts the character of his solution in a typical specimen with $R_o/L_o = 4$. Bifurcation occurs beyond the maximum load and from that point on the solution for the necking specimen turns down from the fundamental solution for the uniform specimen which undergoes no bifurcation. The second part of the plot shows that bifurcation marks the onset of the rapid contraction at the neck.

Included in the second plot are results for a calculation (8) for another set of boundary conditions where the ends of the bar are considered to be cemented to rigid grips. In this case no bifurcation occurs. Instead departure from the uniform state occurs with the first application of load. The maximum load was found to be essentially the same as in the other case; but as can be seen from the plot, significant necking starts at somewhat lower elongations.

As mentioned previously, Chen (6) used the same formulation together with a Kantorovich approximation method to study the same problem. He considered the shear-free end conditions case and initiated necking by introducing a small initial axisymmetric imperfection. This same technique was used by Osias (18) in his study of tensile necking under plane stress and plane strain conditions. However, Osias's approach was based on a Eulerian formulation and his numerical scheme derived from a discretization of the governing differential equations directly.

APPLICATIONS TO THIN PLATE AND SHELL PROBLEMS

As emphasized in the Introduction, the structure of the field equations as developed for the three-dimensional solid closely resembles the structure of the equations for the most widely used nonlinear theories of plates and shells. In most applications involving structural materials, whether the response is elastic or elastic-plastic, the strains are small and the significant geometric nonlinearity is due to rotations. In a first order theory in which the strains are assumed to vary linearly through the thickness the inplane Lagrangian strain tensor is often approximated by

$$\eta_{\alpha\beta} = E_{\alpha\beta} + zK_{\alpha\beta} \qquad (\alpha = 1,2 \; ; \; \beta = 1,2) \; , \tag{38}$$

where $E_{\alpha\beta}$ and $K_{\alpha\beta}$ are the stretching and bending strain tensors of the middle surface. The coordinate z is measured along the normal to middle surface in the undeformed shell. The stretching and bending strains are expressed in terms of the displacements of the middle surface in directions normal and tangential to the undeformed middle surface.

The internal virtual work is approximated by

$$\int_V \tau^{\alpha\beta}\delta\eta_{\alpha\beta}dV = \int_A \{M^{\alpha\beta}\delta K_{\alpha\beta} + N^{\alpha\beta}\delta E_{\alpha\beta}\}dA \; , \tag{39}$$

where dA is the element of the undeformed middle surface. The bending moment and resultant stress tensors are related to the Kirchhoff stress tensor by

$$N^{\alpha\beta} = \int_{-t/2}^{t/2} \tau^{\alpha\beta}dz \quad \text{and} \quad M^{\alpha\beta} = \int_{-t/2}^{t/2} \tau^{\alpha\beta}z dz \; , \tag{40}$$

where t is the thickness of the undeformed shell. The contravariant components of the Kirchhoff stress enter into these expressions because the Lagrangian strain tensor has been used along with the choice of the undeformed body as the reference configuration.

Suppose the three-dimensional rate-constitutive relation is of the form discussed in the previous sections for the finite strain formulation, i.e.,

$$\dot{\tau}^{ij} = L^{ijk\ell}\dot{\eta}_{k\ell} \tag{41}$$

The assumption of approximate plane stress in a first order plate or shell theory requires $\eta_{\alpha 3} = 0$ for $\alpha = 1,2$ and $\tau^{33}\delta\eta_{33} = 0$, i.e., $\tau^{33} = 0$ Thus from (41)

$$\dot{\eta}_{33} = -(L^{\alpha\beta 33}/L^{3333})\eta_{\alpha\beta} \tag{42}$$

The plane stress moduli $\bar{L}$ relating the inplane stress-rates and strain-rates, i.e.,

$$\dot{\tau}^{\alpha\beta} = \bar{L}^{\alpha\beta\kappa\gamma}\dot{\eta}_{\kappa\gamma} \tag{43}$$

are given by

$$\bar{L}^{\alpha\beta\kappa\gamma} = L^{\alpha\beta\kappa\gamma} - L^{\alpha\beta 33}L^{33\kappa\gamma}/L^{3333} \tag{44}$$

From (40) the rate-constitutive relations written in terms of the stress-rate and strain-rate quantities of the plate or shell theory are

$$\dot{N}^{\alpha\beta} = H_{(1)}^{\alpha\beta\kappa\gamma}\dot{E}_{\kappa\gamma} + H_{(2)}^{\alpha\beta\kappa\gamma}\dot{K}_{\kappa\gamma} \tag{45}$$

$$\dot{M}^{\alpha\beta} = H_{(2)}^{\alpha\beta\kappa\gamma}\dot{E}_{\kappa\gamma} + H_{(3)}^{\alpha\beta\kappa\gamma}\dot{K}_{\kappa\gamma} \tag{46}$$

where

$$H_{(i)}^{\alpha\beta\kappa\gamma} = \int_{-t/2}^{t/2} \bar{L}^{\alpha\beta\kappa\gamma} z^{i-1} dz \ . \tag{47}$$

In particular, note that for the case of a flat plate with $K_{\alpha\beta} = 0$ and $E_{\alpha\beta}$ uniform through the thickness, Eq. (45) gives exactly the same relation between $\dot{N}^{\alpha\beta}$ and $\dot{E}_{\alpha\beta}$ as would be obtained from the full finite strain formulation by integrating through the thickness.

Equations (38) through (47) constitute a full complement of equations for first order plate and shell theories including finite strain effects. If the moduli $\underset{\sim}{L}$ given by (31) or some similar prescription are used, then the quantities needed for updating $\underset{\sim}{L}$ from one incremental step to another are contained in the above set of equations.

If the strains are small <u>and</u> the stress levels are low compared to the instantaneous moduli, then as discussed previously the finite strain formulation can be replaced by a small strain formulation in which it is not necessary to give a precise definition to the stress measure. Most plastic buckling problems in thin plates and shells fall into this category. Typically, the stresses at buckling are proportional to the product of an instantaneous modulus and some ratio of the thickness to a characteristic length much greater than the thickness. On the other hand, in problems involving the onset of necking or bulging, for example, it may be essential to use an appropriate finite strain formulation even when the strains are small. As long as the characteristic length of the deformation field is large compared to the thickness one can expect the first order theory to have approximate validity. Of course, once the characteristic deformation length becomes on the order of the thickness, as in the advanced stages of necking, the first order theory is no longer applicable.

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REFERENCES

1. Budiansky, B., "Remarks on Theories of Solid and Structural Mechanics", Problems of Hydrodynamics and Continuum Mechanics, SIAM, Philadelphia, 1969, pp. 77-83.

2. Bolotin, V. V., Nonconservative Problems of the Theory of Elastic Stability, English translation edited by Herrmann, G., Macmillan, New York, 1963.

3. Green, A. E. and Zerna, W., Theoretical Elasticity, 2nd Ed., Oxford University Press, Oxford, 1968.

4. Hill, R., "On the Classical Constitutive Relations for Elastic-Plastic Solids", Recent Progress in Applied Mechanics, the Folke Odqvist Volume, Almqvist and Wiksell, Stockholm, 1967, pp. 241-249.

5. Hill, R., "Bifurcation and Uniqueness in Nonlinear Mechanics of Continua", the Muskhelishvili Volume, SIAM, Philadelphia, 1961, pp. 155-164.

6. Chen, W. H., "Necking of a Bar", International Journal of Solids and Structures, Vol. 7, 1971, pp. 685-717.

7. Needleman, A., "Void Growth in an Elastic-Plastic Medium", Journal of Applied Mechanics, Vol. 39, 1972, pp. 964-970.

8. Needleman, A., "A Numerical Study of Necking in Circular Cylindrical Bars", Journal of the Mechanics and Physics of Solids, Vol. 20, 1972, pp. 111-127.

9. Oden, J. T., Finite Elements of Nonlinear Continua, McGraw-Hill, New York, 1972.

10. Hibbitt, H. D., Marcal, P. V. and Rice, J. R., "A Finite Element Formulation for Problems of Large Strain and Large Displacement", International Journal of Solids and Structures, Vol. 6, 1970, pp. 1069-1086.

11. Green, A. E. and Naghdi, P. M., "A General Theory of an Elastic-Plastic Continuum", Archive for Rational Mechanics and Analysis, Vol. 18, 1965, pp. 251-281.

12. Lee, E. H., "Elastic-Plastic Deformation at Finite Strains", Journal of Applied Mechanics, Vol. 36, 1969, pp. 1-6.

13. Willis, J. R., "Some Constitutive Equations Applicable to Problems of Large Dynamic Plastic Deformations", Journal of the Mechanics and Physics of Solids, Vol. 17, 1969, pp. 359.

14. Budiansky, B., (unpublished work) see References 6, 7, or 8.

15. Hutchinson, J. W. and Miles, J. P., "Bifurcation Analysis of the Onset of Necking in an Elastic-Plastic Cylinder under Uniaxial Tension", to be published in Journal of the Mechanics and Physics of Solids, Vol. 21, 1973.

16. Miles, J. P., "Bifurcation in Plastic Flow under Uniaxial Tension", Journal of the Mechanics and Physics of Solids, Vol. 19, 1971, pp. 89-102.

17. Cheng, S. Y., Ariaratnam, S. T. and Dubey, R. N., "Axisymmetric Bifurcation in an Elastic-Plastic Cylinder under Axial Load and Lateral Hydrostatic Pressure", Quarterly of Applied Mathematics, Vol. 29, 1971, pp. 41-51.

18. Osias, J. R., "Finite Deformation of Elasto-Plastic Solids", NASA Contractor Report CR-2199, March 1973.