J. Mech. Phys. Solids, 1974, Vol. 22, pp. 61 to 71. Pergamon Press. Printed in Great Britain.

BIFURCATION ANALYSIS OF THE ONSET OF NECKING IN AN ELASTIC/PLASTIC CYLINDER UNDER UNIAXIAL TENSION

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(*Received 27th February* 1973)

SUMMARY

THE BIFURCATION problem governing the onset of axisymmetric necking in a circular cylindrical specimen in uniaxial tension is analysed. The specimen is made of an incompressible elastic/plastic material. One end is subject to a prescribed uniform axial displacement relative to the other and both ends are shear free. The true stress at bifurcation is greater than the stress at which the maximum load is attained by an amount which depends on (i) the radius to length ratio of the specimen, (ii) the ratio of the elastic shear modulus to the tangent modulus, and (iii) the derivative of the tangent modulus with respect to the stress. Bifurcation takes place immediately following attainment of the maximum load when the specimen is sufficiently slender.

1. INTRODUCTION

CONSIDER a solid circular cylinder whose elastic/plastic properties are initially homogeneous and transversely anisotropic with respect to its axis. If the ends of the cylinder are subject to a uniform relative axial displacement in such a way that the ends remain free of tangential traction and the lateral surface traction-free, then one equilibrium solution for all values of relative displacement is the simple state of uniaxial tension. At some value of the elongation a loss of uniqueness of the uniaxial state can be expected signaling the onset of necking. According to the conventional engineering criterion, necking begins at the point at which the maximum support load of the specimen is attained. Recently, it has been proved that the state of uniaxial tension is unique prior to the attainment of the maximum load (MILES (1971)). In this paper we make a detailed study of the relationship between the lowest bifurcation point and the maximum load point for specimens of an incompressible elastic/plastic material undergoing axisymmetric bifurcations. Our study is based on HILL's (1958, 1961) theory of bifurcation and uniqueness. We continue and enlarge on work on this same problem by CHENG, ARIARATNAM and DUBEY (1971) and MILES (1971).

2. CONSTITUTIVE RELATION FOR AN ELASTIC/PLASTIC SOLID WITH A SMOOTH YIELD SURFACE

Introduce an embedded coordinate system and denote the metric associated with the current base vectors in the deformed body by G_{ij} and its inverse by G^{ij}. Let τ^{ij} be the contravariant components of the Kirchhoff stress tensor referred to the base vectors of the deformed body, and denote the convected rates of these components by $\dot{\tau}^{ij}$. Covariant components of the strain rate are denoted by ε_{ij}. If the yield condition is satisfied at any stage of the deformation history, the rate-constitutive relation is taken in the form

$$\dot{\tau}^{ij} = \mathscr{L}^{ijkl}\varepsilon_{kl} - \frac{\alpha}{g} m^{ij}m^{kl}\varepsilon_{kl}, \tag{2.1}$$

where

$$\alpha = \begin{cases} 0 & \text{if} \quad m^{kl}\varepsilon_{kl} < 0, \\ 1 & \text{if} \quad m^{kl}\varepsilon_{kl} \geq 0. \end{cases} \tag{2.2}$$

Here, $\mathscr{L}$ is the tensor of instantaneous elastic moduli (for this choice of stress-rate), $\mathbf{m}$ is the unit tensor normal to the elastic domain in strain-rate space, and g is a positive scalar which determines the instantaneous strain hardening characteristics of the material.

For future reference, we note that CHENG *et al.* (1971) and MILES (1971) used the Jaumann derivative of the Kirchhoff stress in formulating their rate-constitutive relation. Denote the contravariant components of the Jaumann derivative by $\mathscr{D}\tau^{ij}/\mathscr{D}t$. They are related to $\dot{\tau}^{ij}$ by

$$\frac{\mathscr{D}\tau^{ij}}{\mathscr{D}t} = \dot{\tau}^{ij} + G^{im}\tau^{kj}\varepsilon_{km} + G^{jm}\tau^{ki}\varepsilon_{km}. \tag{2.3}$$

If the Jaumann stress-rate is used, (2.1) is transformed to

$$\frac{\mathscr{D}\tau^{ij}}{\mathscr{D}t} = \hat{\mathscr{L}}^{ijkl}\varepsilon_{kl} - \frac{\alpha}{g} m^{ij}m^{kl}\varepsilon_{kl}, \tag{2.4}$$

where the tensors of elastic moduli in (2.1) and (2.4) are connected by

$$\mathscr{L}^{ijkl} = \hat{\mathscr{L}}^{ijkl} - \tfrac{1}{2}(G^{ik}\tau^{jl} + G^{il}\tau^{jk} + G^{jk}\tau^{il} + G^{jl}\tau^{ik}). \tag{2.5}$$

This is an example of the transformation of a rate-constitutive relation expressed in terms of one objective stress-rate to another as discussed by HILL (1967).

3. BIFURCATION CRITERION

In this section we state a specialized form of HILL's (1958, 1961) bifurcation criterion. We consider bodies subject to combinations of dead load surface tractions on S_T and displacements on S_u prescribed proportional to a single parameter λ. Quantities associated with the fundamental solution whose uniqueness is in question are labeled by a superscript or subscript o. At any stage of the deformation history corresponding to λ a quadratic functional for testing for bifurcation is

$$F(\lambda, \tilde{\mathbf{v}}) = \frac{1}{2}\int_V \{\tilde{\tau}^{ij}\tilde{\varepsilon}_{ij} + \tau_o^{ij}\tilde{v}^k{}_{,i}\tilde{v}_{k,j}\}dV, \tag{3.1}$$

where for *all* $\tilde{\varepsilon}$

$$\tilde{\tau}^{ij} = \left[\mathscr{L}^{ijkl} - \frac{1}{g} m^{ij}m^{kl}\right]\tilde{\varepsilon}_{kl}. \tag{3.2}$$

In this paper, application of (3.1) will be made with the reference configuration chosen to be that of the deformed body as characterized by the fundamental solution at the value of λ in question. The comma denotes covariant differentiation with respect to the base vectors of the reference configuration and the velocity field $\tilde{\mathbf{v}}$ is related to the strain rate $\tilde{\boldsymbol{\varepsilon}}$ by

$$\tilde{\varepsilon}_{ij} = \tfrac{1}{2}(\tilde{v}_{i,j} + \tilde{v}_{j,i}). \tag{3.3}$$

For the linear comparison solid with instantaneous moduli (3.2), F is the second variation of the potential energy with respect to the velocity field $\tilde{\mathbf{v}}$. As is well known, this functional plays the central role in the bifurcation analysis of elastic solids.

For an elastic/plastic body characterized by (2.1), suppose that $F(\lambda, \tilde{\mathbf{v}}) > 0$ for all admissible, nonzero velocity fields $\tilde{\mathbf{v}}$ (which vanish on S_u) in the range $0 \leq \lambda < \lambda_c$. Then it follows from Hill's general theory that the fundamental solution is unique in this range.

Now suppose that λ_c is the smallest value of λ such that there exists an admissible field $\tilde{\mathbf{v}}$ (which vanishes on S_u) that satisfies $F(\lambda_c, \tilde{\mathbf{v}}) = 0$. This field is termed the *eigenmode*; it also satisfies the associated variational equation $\delta F = 0$ for all admissible variations $\delta\mathbf{v}$. The eigenmode is taken to be normalized in some suitable fashion.

Suppose further that at λ_c the increment of the fundamental solution satisfies the condition that plastic loading takes place throughout the region where the yield condition is satisfied. More precisely, with $\boldsymbol{\varepsilon}^{\circ}$ denoting the strain-rate of the fundamental solution with respect to λ, suppose that

$$m^{kl}\varepsilon^{\circ}_{kl} \geq \Delta > 0 \tag{3.4}$$

throughout the currently yielded region. Under these circumstances, loss of uniqueness is possible at λ_c with a bifurcation mode of the form

$$\begin{Bmatrix} v_i \\ \varepsilon_{ij} \\ \dot{\tau}^{ij} \end{Bmatrix} = \begin{Bmatrix} v^{\circ}_i \\ \varepsilon^{\circ}_{ij} \\ \dot{\tau}^{ij}_{\circ} \end{Bmatrix} + \frac{d\xi}{d\lambda}\begin{Bmatrix} \tilde{v}_i \\ \tilde{\varepsilon}_{ij} \\ \tilde{\dot{\tau}}^{ij} \end{Bmatrix},$$

where ξ is the amplitude of the eigenmode and the rates of the fundamental solution are again taken with respect to λ. Condition (3.4) ensures that a range of $d\xi/d\lambda$ can be found so that no elastic unloading takes place at bifurcation, i.e. $m^{kl}\varepsilon_{kl} \geq 0$. In other words, bifurcation is possible at λ_c under continued plastic loading in the sense of Shanley (1947) as generalized by Hill (1958, 1961).

4. Eigenvalue Equation for Axisymmetric Bifurcation

The fundamental solution for the cylindrical solid discussed in the Introduction is the state of uniaxial tension with a true stress denoted by σ. The relative axial displacement of the ends is prescribed to be uniform and is identified with the parameter λ. The fundamental solution satisfies (3.4), and thus bifurcation is possible at the lowest eigenvalue λ_c associated with $F(\lambda_c, \tilde{\mathbf{v}}) = 0$. Let E_t be the tangent modulus relating the true stress-rate and natural strain-rate for a uniaxial increment of stress according to $\dot{\sigma} = E_t\,\varepsilon$. In the current state, characterized by the fundamental solution, introduce a cylindrical coordinate system (r, θ, z) and let R and L be the *current* radius and length of the specimen so that $0 \leq r \leq R$ and $0 \leq z \leq L$.

The rate-constitutive equations (2.1) for an incompressible solid with transverse anisotropy with respect to the z axis and subject to an axial stress σ can be written quite generally as

$$\left.\begin{aligned} \dot{t}_z &= 2\mu_1\varepsilon_z - g^{-1}\varepsilon_z + \dot{p}, \\ \dot{t}_r &= 2\mu_2\varepsilon_r + 2(\mu_1-\mu_2)\varepsilon_\theta + \tfrac{1}{2}g^{-1}\varepsilon_z + \dot{p}, \\ \dot{t}_\theta &= 2\mu_2\varepsilon_\theta + 2(\mu_1-\mu_2)\varepsilon_r + \tfrac{1}{2}g^{-1}\varepsilon_z + \dot{p}, \\ \dot{t}_{rz} &= 2\mu_3\varepsilon_{rz}, \end{aligned}\right\} \tag{4.1}$$

with $\dot{p} \equiv \frac{1}{3}(\dot{t}_r+\dot{t}_\theta+\dot{t}_z)$ and $\varepsilon_r+\varepsilon_\theta+\varepsilon_z = 0$. Here, and in the remainder of the paper, we have used physical components of the stress-rate and strain-rate components introduced in Sections 2 and 3 with the standard notation. In (4.1) only the quantities relevant to axisymmetric deformations are included. If

$$\mu_1 = \mu_2 = \mu_3, \tag{4.2}$$

then the relation between the stress-rate $\dot{t}^{ij}$ and the strain-rate is isotropic for elastic responses.

The true stress-rate for the incompressible material in uniaxial tension is given by $\dot{\sigma} = \dot{t}_z + 2\sigma\varepsilon_z$. From the definition of E_t and (4.1) it follows that

$$\frac{1}{g} = 2[\mu_1 + \tfrac{1}{3}(2\sigma - E_t)]. \tag{4.3}$$

If the Jaumann stress-rates are used as in (2.4) then (4.1) transforms to

$$\left.\begin{aligned} \frac{\mathscr{D}\tau_z}{\mathscr{D}t} &= 2\hat{\mu}_1\varepsilon_z - g^{-1}\varepsilon_z + \frac{\mathscr{D}p}{\mathscr{D}t}, \\ \frac{\mathscr{D}\tau_r}{\mathscr{D}t} &= 2\hat{\mu}_2\varepsilon_r + 2(\hat{\mu}_1-\hat{\mu}_2)\varepsilon_\theta + \tfrac{1}{2}g^{-1}\varepsilon_z + \frac{\mathscr{D}p}{\mathscr{D}t}, \\ \frac{\mathscr{D}\tau_\theta}{\mathscr{D}t} &= 2\hat{\mu}_2\varepsilon_\theta + 2(\hat{\mu}_1-\hat{\mu}_2)\varepsilon_r + \tfrac{1}{2}g^{-1}\varepsilon_z + \frac{\mathscr{D}p}{\mathscr{D}t}, \\ \frac{\mathscr{D}\tau_{rz}}{\mathscr{D}t} &= 2\hat{\mu}_3\varepsilon_{rz}, \end{aligned}\right\} \tag{4.4}$$

where

$$\frac{\mathscr{D}p}{\mathscr{D}t} \equiv \frac{1}{3}\left[\frac{\mathscr{D}\tau_r}{\mathscr{D}t} + \frac{\mathscr{D}\tau_\theta}{\mathscr{D}t} + \frac{\mathscr{D}\tau_z}{\mathscr{D}t}\right].$$

The connections between the moduli in (4.1) and (4.4) can be determined directly with (2.3) and are found to be

$$\mu_1 = \hat{\mu}_1 - \tfrac{2}{3}\sigma, \quad \mu_2 = \hat{\mu}_2 - \tfrac{1}{3}\sigma, \quad \mu_3 = \hat{\mu}_3 - \tfrac{1}{2}\sigma. \tag{4.5}$$

An isotropic relation between the Jaumann rate of the Kirchhoff stress and the strain-rate requires, for elastic responses,

$$\hat{\mu}_1 = \hat{\mu}_2 = \hat{\mu}_3. \tag{4.6}$$

For the class of materials considered here MILES (1971) has shown that the integrand of (3.1) is greater than zero for all non-uniform velocity fields $\tilde{\mathbf{v}}$ if the true stress is below the true stress associated with the maximum support load, σ_m. Therefore, bifurcation cannot take place prior to maximum load as has already been mentioned. This result holds for homogeneous cylindrical or prismatic bodies of arbitrary cross-section, whichever of the rate-constitutive equations (2.1) or (2.4) the material is assumed to satisfy.† Denote the value of the tangent modulus at the maximum load point by E_t^m. At maximum load the nominal (or engineering) stress-rate vanishes, i.e. $\dot{s}_z = \dot{t}_z + \sigma_m \dot{\varepsilon}_z = 0$. Using (4.1) and (4.3) this condition can be expressed as $(E_t^m - \sigma_m)\dot{\varepsilon}_z = 0$ which yields the maximum load condition

$$\sigma_m = E_t^m. \tag{4.7}$$

Equilibrium equations and boundary conditions associated with the variational equation $\delta F = 0$ for the bifurcation stress σ_c and the axisymmetric eigenmode are (these are determined with no restriction on compressibility)

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial(r\tilde{\tau}_r)}{\partial r} + \frac{\partial \tilde{\tau}_{rz}}{\partial z} - \frac{1}{r}\tilde{\tau}_\theta + \sigma_c \frac{\partial^2 \tilde{v}_r}{\partial z^2} = 0, \\ \frac{\partial \tilde{\tau}_z}{\partial z} + \frac{1}{r}\frac{\partial(r\tilde{\tau}_{rz})}{\partial r} + \sigma_c \frac{\partial^2 \tilde{v}_z}{\partial z^2} = 0, \end{aligned}\right\} \tag{4.8}$$

$$\left.\begin{aligned} \tilde{\tau}_r = 0 \\ \tilde{\tau}_{rz} = 0 \end{aligned}\right\} \text{ for } r = R, \text{ and } \left.\begin{aligned} \tilde{v}_z = 0 \\ \tilde{\tau}_{rz} = 0 \end{aligned}\right\} \text{ for } z = 0, L. \tag{4.9}$$

Eigenmodal strain-rates and stress-rates are related by (4.1), and the strain-rates are given by

$$\tilde{\varepsilon}_r = \frac{\partial \tilde{v}_r}{\partial r}, \quad \tilde{\varepsilon}_\theta = \frac{1}{r}\tilde{v}_r, \quad \tilde{\varepsilon}_z = \frac{\partial \tilde{v}_z}{\partial z}, \quad \tilde{\varepsilon}_{rz} = \frac{1}{2}\left(\frac{\partial \tilde{v}_r}{\partial z} + \frac{\partial \tilde{v}_z}{\partial r}\right). \tag{4.10}$$

Incompressibility requires $\tilde{\varepsilon}_r + \tilde{\varepsilon}_\theta + \tilde{\varepsilon}_z = 0$, which enables us to introduce a function $\Phi(r, z)$ such that with

$$\tilde{v}_r = -\frac{\partial \Phi}{\partial z} \quad \text{and} \quad \tilde{v}_z = \frac{1}{r}\frac{\partial(r\Phi)}{\partial r} \tag{4.11}$$

incompressibility is ensured.

Solutions to the equations governing the eigenvalue problem can be written in the separated form

$$\left.\begin{aligned} \Phi &= \phi(r)\sin(k\pi z/L), \\ \tilde{v}_r &= -\frac{k\pi}{L}\phi(r)\cos(k\pi z/L), \\ \tilde{\tau}_r &= \hat{\tau}_r(r)\cos(k\pi z/L), \\ \tilde{p} &= \hat{p}(r)\cos(k\pi z/L), \text{ etc.}, \end{aligned}\right\} k = 1, 2, 3, \ldots \tag{4.12}$$

such that the boundary conditions on $z = 0, L$ are satisfied and ordinary differential equations governing the r dependence of the quantities are obtained. Further manipulation reduces the eigenvalue problem to a single equation for ϕ:

$$L^2(\phi) + 2b\gamma^2 L(\phi) + c\gamma^4\phi = 0, \tag{4.13}$$

† It is therefore *not* possible that, as suggested by DUBEY and ARIARATNAM (1972), bifurcation in a rectangular elastic/plastic plate under uniaxial tension may occur before the maximum-load point.

where

$$\gamma = k\pi R/L, \qquad 2\mu_3 b = \mu_1 - 2\mu_2 + 2\mu_3 + \sigma_c - E_t^c, \qquad \mu_3 c = \mu_3 + \sigma_c, \tag{4.14}$$

and E_t^c is the value of E_t at bifurcation. The operator in (4.13) is defined by

$$L(\phi) = \frac{\partial}{\partial\zeta}\left[\zeta^{-1}\frac{\partial(\zeta\phi)}{\partial\zeta}\right]$$

where $\zeta = r/R$. Traction-free boundary conditions (4.9) on the lateral surface can be expressed as

$$\left.\begin{aligned} L(\phi)+\gamma^2\phi &= 0,\\ \gamma^2\alpha_1\frac{\partial}{\partial\zeta}(\zeta\phi)-\alpha_2\frac{\partial}{\partial\zeta}[\zeta L(\phi)]+\gamma^2\alpha_3\phi &= 0,\end{aligned}\right\} \quad \text{on} \quad \zeta = 1, \tag{4.15}$$

where

$$\mu_1\alpha_1 = 2\mu_2 - \mu_1 - \mu_3 + E_t^c - \sigma_c, \qquad \mu_1\alpha_2 = \mu_3, \qquad \mu_1\alpha_3 = 2(\mu_1 - 2\mu_2). \tag{4.16}$$

Equation (4.13) can be rewritten as

$$(L+\gamma^2\rho^2)(L+\gamma^2\bar{\rho}^2)\phi = 0, \tag{4.17}$$

where $\rho^2 = b + i(c-b^2)^{\frac{1}{2}}$ and the bar implies complex conjugation. Here we have anticipated that $c - b^2 > 0$.† The general solution to (4.17) which satisfies boundedness conditions at $\zeta = 0$ is

$$\phi = CJ_1(\gamma\rho\zeta) + \bar{C}J_1(\gamma\bar{\rho}\zeta), \tag{4.18}$$

where J_n is the Bessel function of order n of the first kind, here with complex argument $\gamma\rho\zeta$. Substitution of (4.18) into the boundary conditions (4.15) provides the eigenvalue equation for σ_c,

$$\mathscr{I}m\{(1-\bar{\rho}^2)J_1(\gamma\bar{\rho})[(\alpha_1+\rho^2\alpha_2)\gamma\rho J_o(\gamma\rho)+\alpha_3 J_1(\gamma\rho)]\} = 0. \tag{4.19}$$

The manipulations carried out in arriving at (4.17) are similar to those reported by CHENG *et al.* (1971). These workers assumed an isotropic tensor of elastic moduli at bifurcation in the sense of (4.6), and (4.17) reduces to their equivalent expression for this case.

5. ASYMPTOTIC AND NUMERICAL SOLUTIONS OF THE EIGENVALUE EQUATION

To obtain a relation between σ_c and $\gamma = k\pi R/L$ for small values of γ, we expand the eigenvalue equation in powers of $\gamma\rho$ using the series representation

$$J_n(z) = \sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2}z)^{2m+n}}{m!(m+n)!}.$$

Putting $B = (c-b^2)^{\frac{1}{2}}$, so that $\rho^2 = b + iB$, and using the fact that $|\rho|^4 = c$, we find that to order γ^6, (4.19) becomes

$$\tfrac{1}{2}\gamma^2 B|\rho|^2\left\{[\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_3]-\frac{\gamma^2}{8}[\alpha_1(1+2b)+\alpha_2(4b-c)+b\alpha_3]+\right.$$
$$\left.+\frac{\gamma^4}{192}[2\alpha_1(c+2b+2b^2)+2\alpha_2(c-2cb+6b^2)+\tfrac{1}{2}\alpha_3(c+4b^2)]\right\} = 0. \tag{5.1}$$

† For example, it is readily verified that for isotropic elastic moduli in the sense of either (4.2) or (4.6), $c-b^2 = \frac{3}{4}+O(\sigma_c/\mu,\ E_t/\mu)$.

By (4.16) the coefficient of the lowest order term in (5.1) is found to be $\alpha_1+\alpha_2+\frac{1}{2}\alpha_3=(E_t^c-\sigma_c)/\mu_1$.

We now consider two specializations of (5.1). First, take $\mu_1=\mu_2=\mu_3\equiv\mu$ at bifurcation as in (4.2). Then with the aid of (4.14) and (4.16), (5.1) reduces to

$$E_t^c-\sigma_c+\frac{\gamma^2}{8}\left[\sigma_c-(E_t^c-\sigma_c)+(E_t^c-\sigma_c)^2/\mu\right]+\frac{\gamma^4}{192}\left[\mu+O(\sigma_c,E_t^c)\right]=0. \tag{5.2}$$

This equation can be rewritten as

$$\sigma_c-E_t^c=\frac{\gamma^2}{8}\sigma_c+\frac{\gamma^4}{192}\mu+O(\gamma^4\sigma_c,\gamma^6\mu). \tag{5.3}$$

Secondly, take the elastic moduli relating the Jaumann rates to the strain rates to be isotropic at bifurcation in the sense of (4.6) following CHENG *et al.* (1971) and MILES (1971). Using the connections (4.5) between the two sets of moduli one again obtains (5.3). The difference between these two assumptions shows up only in terms of order $\gamma^4\sigma_c$ and also terms of order $\gamma^2\sigma_c^2/\mu$ which can be neglected compared to $\gamma^2\sigma_c$.

Now σ_c can be related to σ_m by expanding E_t about the maximum load point so that

$$E_t^c=E_t^m+\left.\frac{dE_t}{d\sigma}\right|_m(\sigma_c-\sigma_m)+\ \ldots. \tag{5.4}$$

Then by (4.7), (5.3) becomes

$$\sigma_c=\sigma_m+\left(1-\left.\frac{dE_t}{d\sigma}\right|_m\right)^{-1}\left(\frac{\gamma^2}{8}\sigma_m+\frac{\gamma^4\mu}{192}\right), \tag{5.5}$$

where terms of order $\gamma^4\sigma_m$ have again been neglected. The natural tensile strain at bifurcation, e_c, can be related to the natural strain at maximum load, e_m, in a similar way with the result

$$e_c=e_m+\left(1-\left.\frac{dE_t}{d\sigma}\right|_m\right)^{-1}\left(\frac{\gamma^2}{8}+\frac{\gamma^4\mu}{192\sigma_m}\right). \tag{5.6}$$

Recall that $\gamma=k\pi R/L$ where k is the number of half-wavelengths in the eigenmode introduced in (4.12) and R and L are the radius and length of the specimen at bifurcation. Equations (5.5) and (5.6) can be rendered explicit by replacing γ by its value evaluated at the maximum load point (i.e. $\gamma_m=k\pi R_m/L_m$). This substitution involves errors of the order of terms already omitted. The lowest bifurcation stress for a given ratio of radius to length is clearly given with $k=1$. From (4.12) it is seen that the associated eigenmode has a positive radial deflection at one end and an equal inward deflection at the other so that necking will start at one of the ends rather than at the center. This possibility is due to the assumed end conditions. If the bifurcation mode is restricted to be symmetric with respect to the mid-point of the specimen, then the lowest bifurcation stress is obtained with $k=2$.

Some numerical calculations based on the exact eigenvalue expression (4.19) have been carried out to assess the range of accuracy of expansions (5.5) and (5.6) and to show graphically the extent to which the bifurcation stress and strain exceed the maximum load values. Numerical results for σ_c/σ_m as a function of γ for a hypothetical

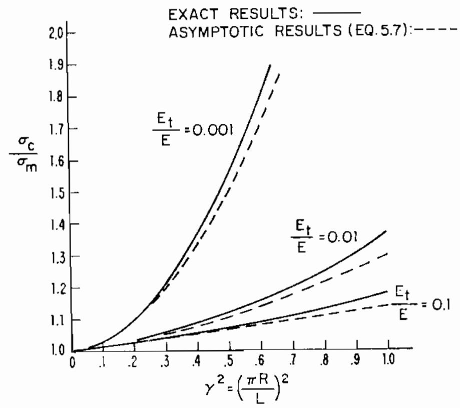

FIG. 1. Comparison of exact and asymptotic results for bifurcation in tension. Ratio of true stress at bifurcation to true stress at maximum load for an incompressible material with constant E_t.

material whose tangent modulus is independent of σ are shown in Fig. 1 and in Table 1. Here the elastic moduli at bifurcation were assumed to satisfy $\hat{\mu}_1 = \hat{\mu}_2 = \hat{\mu}_3 \equiv \mu$. The results given in Table 1 are accurate to the five significant figures shown. The calculations were repeated for the other case mentioned above where the moduli are related by (4.2); over the range of γ considered, the numerical results for the two cases agree to at least four significant figures. The solid line curves in Fig. 1 are the numerical results and the dashed line curves are computed from the appropriate specialization of (5.5), i.e.

$$\frac{\sigma_c}{\sigma_m} = 1 + \frac{\gamma^2}{8} + \frac{\mu}{E_t}\frac{\gamma^4}{192}. \tag{5.7}$$

Curves for three different values of E_t/E have been shown where $E = 3\mu$ is Young's modulus for the incompressible material.

TABLE 1. *Values of σ_c/σ_m for constant E_t/E*

γ	E_t/E	Asymptotic, equation (5.7)	Numerical, equation (4.19)
0·2	0·1	1·0050	1·0051
0·4		1·0204	1·0211
0·6		1·0473	1·0510
0·8		1·0871	1·0998
1·0		1·1424	1·1766
0·2	0·01	1·0053	1·0053
0·4		1·0244	1·0253
0·6		1·0675	1·0727
0·8		1·1511	1·1724
1·0		1·2986	1·3678
0·2	0·001	1·0078	1·0078
0·4		1·0644	1·0665
0·6		1·2700	1·2897
0·8		1·7911	1·8976
1·0		2·8611	3·2765

The large effect of a variable tangent modulus is brought out by considering a Ramberg-Osgood tensile relation (RAMBERG and OSGOOD, 1943) between true stress and natural strain according to

$$e/e_y = \sigma/\sigma_y + \tfrac{3}{7}(\sigma/\sigma_y)^n, \tag{5.8}$$

where e_y and $\sigma_y = Ee_y$ are the effective yield strain and yield stress and n is the hardening exponent. The derivative of the tangent modulus at maximum load is found to be

$$\left.\frac{dE_t}{d\sigma}\right|_m = -(n-1)(1-\sigma_m/E) \approx -(n-1).$$

Using the latter approximation, (5.5) and (5.6) can be written as

$$\sigma_c/\sigma_m = 1+(1/n)(\gamma_m^2/8+\gamma_m^4\mu/192\sigma_m) \quad \text{and} \quad e_c/e_m = 1+(1/ne_m)(\gamma_m^2/8+\gamma_m^4\mu/192\sigma_m), \tag{5.9}$$

where, as discussed, $\gamma_m = \pi R_m/L_m$.

Numerical calculations were made using the exact eigenvalue expression (4.19) and the instantaneous values of γ, E_t, etc. derived from (5.8). The elastic moduli were taken to satisfy (4.6). Predictions of the asymptotic formulae (5.9) are compared with full numerical calculations in Fig. 2 for the two values of the hardening exponent, $n = 5$ and 15, with $e_y = 0{\cdot}001$. Both the asymptotic results and the numerical results

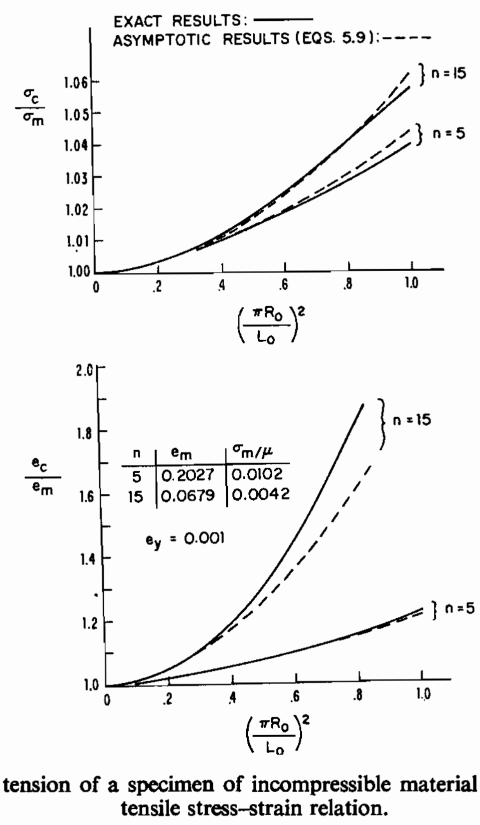

FIG. 2. Bifurcation in tension of a specimen of incompressible material with a Ramberg–Osgood tensile stress–strain relation.

have been plotted as a function of the *initial* ratio of radius to length, $\pi R_o/L_o$; γ_m is given by

$$\gamma_m = \frac{\pi R_o}{L_o}(L_o/L_m)^{3/2} = \frac{\pi R_o}{L_o}\exp(-\tfrac{3}{2}e_m),$$

which is evaluated numerically using (5.8).

6. DISCUSSION

The Figures indicate that the simple formulae retain reasonable accuracy for values of $\gamma = \pi R/L$ as large as unity, corresponding to a rather stubby specimen. The true stress at bifurcation for a slender specimen ($R/L < 1/10$, say) is at most a few per cent above the stress at the maximum load point even when the tangent modulus is independent of stress, as long as E_t/E is not much smaller than 0·001. It is clear from Fig. 1 and the asymptotic formulae themselves that it is essential to retain the term of order $\gamma^4\mu$. For all but slender specimens this term is the dominant one, and for any nonzero value of γ, $\sigma_c/\sigma_m \to \infty$ as $\mu/E_t \to \infty$ as illustrated by Fig. 1. This limiting behaviour is in keeping with the fact that a specimen of rigid/plastic material with a smooth yield surface cannot undergo a tensile bifurcation, at least not of the type found here (HILL, 1957). The analogous problem for a rectangular plane strain specimen of rigid/plastic material does admit a tensile bifurcation mode as shown by COWPER and ONAT (1962). They also found a delay in bifurcation beyond the maximum load point which depends on the ratio of width to length of the specimen.

The effect of a variable tangent modulus can be appreciable. Thus, for example, the bifurcation stress of even a stubby specimen of material with a hardening exponent of a typical metal will be at most a few per cent above the stress at maximum load. The same is true for the bifurcation strain if the specimen is relatively slender as typified by the dimensions of a standard tensile test specimen. Thus, the results of the bifurcation analysis of the specimen with the idealized end conditions is in accord with the accepted experimental observation that necking in a tensile specimen begins almost simultaneously with attainment of the maximum load. On the other hand, the bifurcation strain of a stubby specimen may exceed the strain at maximum load by as much as a factor of two, say, as in Fig. 2. Finite element calculations by NEEDLEMAN (1972) for specimens whose ends are taken to be cemented to rigid grips also indicate that the rapid growth of the neck is delayed to larger overall strain values when the specimen is stubby.†

The increasing separation between the bifurcation point and the maximum load point with increasing μ/E_t is intimately connected with the assumption of a material with a smooth yield surface. There are a number of well known examples in the bifurcation analysis of compressive buckling where use of a flow theory of plasticity with a smooth yield surface leads to theoretical predictions which consistently overestimate experimentally determined buckling loads. We mention in passing that the eigenvalue equation (4.19) governing bifurcation was derived for general transverse anisotropy. Thus the effect of reduced effective moduli (appropriate, for example,

† NEEDLEMAN (1972, Fig. 4) reported results for a specimen with a length to diameter ratio of 4. He subsequently repeated these calculations for a specimen which was identical in all respects except that its length to diameter ratio was 2. He found the above mentioned delay in necking (private communication).

under total loading conditions in the sense of the slip theory of plasticity) can be studied with only a slight reinterpretation of (4.19).

CHENG, ARIARATNAM and DUBEY (1971) arrived at the eigenvalue problem as posed by (4.15) and (4.17) for elastic moduli satisfying (4.6). Instead of the eigenvalue equation (4.19), they list six real eigenvalue equations distinguished from one another by the assumed complex character of the quantity ρ^2 in (4.17). They obtain an asymptotic relation which coincides with (5.3) to the lowest order term but they do not obtain the essential term $\mu\gamma^4$. Evidently as a consequence, they incorrectly conclude that a finite bifurcation stress is obtained in the limit of an infinite shear modulus. A most confusing aspect of their analysis is their incorrect identification of the complex character of the quantity ρ^2 and subsequent use of an erroneous eigenvalue equation. The lowest order term in the asymptotic formula is not affected by this mistake. However, the essential term, $\mu\gamma^4$, would not be obtained from the eigenvalue equation used.

ACKNOWLEDGMENT

The work of one of us (J. W. H.) was supported in part by the National Aeronautics and Space Administration under Grant NGL 22-007-012, in part by the National Science Foundation under Grant GP-34723, and by the Division of Engineering and Applied Physics, Harvard University.

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