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ABSTRACT

Aspects of crack growth in an elastic-plastic material under quasi-static, steady conditions are investigated for plane problems. A path-independent line integral is identified for the steady problem which generalizes the J-integral of deformation theory plasticity to incremental theories of plasticity. Implications of the integral for small scale yielding crack growth are discussed. A model study is made of the effect on the growth process of the residual plastic wake left behind the advancing crack. The influence of the wake on the stabilization of crack growth is discussed.

1. INTRODUCTION

Irreversibility effects which tend to stabilize crack growth arise from two related sources: nonproportional plastic deformation in the active plastic zone and elastic unloading which leaves a wake of residual plastic deformation behind the advancing crack tip. McClintock [1, 2] studied quasi-static crack growth under anti-plane shear in an elastic-perfectly plastic material. Of particular significance is his discovery that the strains in the active plastic zone depend logarithmically on the distance, r, from the crack tip as the tip is approached. The corresponding behavior for a stationary crack in a perfectly plastic material

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is far more singular with strain varying like $1/r$. McClintock was able to estimate the strain field in the region ahead of the crack for small scale yielding. Then employing a fracture criterion based on the attainment of a critical strain at some characteristic ahead of the crack, he showed that stable crack growth should be expected for any reasonable choice of critical strain and characteristic distance. Later, Chitaley and McClintock [3] carried out a detailed numerical analysis of the small scale yielding problem under steady growth conditions and verified the major features of the more approximate analysis given previously in [1, 2].

Rice [4, 5] has shown that a logarithmic dependence of the strains on r can also be expected for a growing crack in an elastic-perfectly plastic material for both plane stress and plane strain. More complete analyses of the plane problems, analogous to those cited above for anti-plane shear, have not been achieved. Nonetheless, because of the weak logarithmic singularity in the strains, it seems probable that such an analysis would again predict stable crack growth for the plane problems. If true, this would be cause for some concern since under conditions approximating plane strain little or no stable crack growth is frequently observed in small scale yielding tests. Since the stress history of each material point swept by the advancing plastic zone is distinctly nonproportional [3, 5], it is conceivable that the smooth yield surface of the plasticity theory used in the singularity analysis overly restricts the plastic flow and leads to a weaker singularity in the strains than might otherwise be found. For similar reasons, a smooth yield surface is thought to be inadequate for describing plastic flow in the bifurcation analysis of plastic buckling.

In this paper the influence of the residual plastic wake on the extending crack will be modeled for steady conditions. The model is similar in spirit to the

Dugdale-Barenblatt model for a stationary crack in a thin sheet. The model is not intended to be complete since irreversibility effects associated with nonproportional loading, such as those discussed above, are not included. Instead, the purpose of the model is to complement the above mentioned studies to illustrate that the wake by itself can have a significant effect on stabilizing crack propagation. The model is formulated for plane stress but some difference between plane stress and plane strain can be inferred with respect to the effect of the wake on growth.

Before introducing the model, a general derivation of an energy balance relation is made for steady propagation of cracks in elastic-plastic materials. A path-independent integral is identified which generalizes the J-integral of Rice [4] to hold for arbitrary material behavior. Our development is closely related to the work of Cherepanov [6, 7]. The integral is used to show the relation between the residual internal energy in the wake and the elastic stress intensity factor for small scale yielding. This information is then used in constructing the model.

2. A PATH-INDEPENDENT INTEGRAL FOR STEADY CRACK GROWTH IN THE PLANE

Let the coordinates x_1 and x_2 translate with the crack tip and be centered as in Fig. 1. Crack growth occurs in the x_1 direction; let $(X_1 = a, X_2 = 0)$ be the position of the crack tip with respect to some fixed coordinate system (X_1, X_2). The crack-tip coordinate a is assumed to increase monotonically and can be taken as the time-like variable in the quasi-static growth process. If steady conditions hold, either globally or in some vicinity of the crack tip, then the stresses, strains and displacements can be expressed as functions of the translating coordinates x_1 and x_2 independent of a. The following relations among the derivatives hold:

$$\left(\frac{\partial}{\partial a}\right)_{\underset{\sim}{X}} = -\left(\frac{\partial}{\partial X_1}\right)_a = -\frac{\partial}{\partial x_1} \tag{1}$$

Small strain plasticity is considered in which the equilibrium equations and the strain-displacement relations are taken to be linear. Thermal effects and inertia are neglected. Let W be the internal energy density at any material point, i.e., the stress work density

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \tag{2}$$

Here the notation of Rice [4] is used; however, we assume only that the behavior of the material is time-independent and we do not limit consideration to a deformation theory of plasticity. The integral in (2) will depend in general on the stress history at the material point in question.

Let Γ be any closed curve in the plane (not enclosing the tip) with enclosed area A_Γ which translates with the crack tip without deforming. Let Φ_Γ be the internal energy per unit thickness enclosed within Γ so that

$$\Phi_\Gamma = \int_{A_\Gamma} W \, dA \tag{3}$$

Under steady conditions Φ_Γ is constant and thus

$$0 = \frac{d\Phi_\Gamma}{da} = \int_{A_\Gamma} \left(\frac{\partial W}{\partial a}\right)_{\underset{\sim}{X}} dA + \int_\Gamma W \, n_1 \, ds \tag{4}$$

where n_1 is the outward unit normal to Γ and ds is the length element. Next note that

$$\int_{A_\Gamma} \left(\frac{\partial W}{\partial a}\right)_{\underset{\sim}{X}} dA = \int_{A_\Gamma} \sigma_{ij} \left(\frac{\partial \varepsilon_{ij}}{\partial a}\right)_{\underset{\sim}{X}} dA = -\int_{A_\Gamma} \sigma_{ij}\varepsilon_{ij,1} \, dA$$

$$= -\int_{A_\Gamma} (\sigma_{ij}u_{i,1})_{,j} \, dA = -\int_\Gamma \sigma_{ij}n_j u_{i,1} \, ds \tag{5}$$

where $\sigma_{ij,j} = 0$ and $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ have also been used, and where $(\ )_{,i} = \partial(\ )/\partial x_i$. Combining (4) and (5) gives, for any closed contour,

$$\int_\Gamma (W\, n_1 - T_i u_{i,1}) ds = 0 \tag{6}$$

where $T_i = \sigma_{ij} n_j$.

The path-independence of this integral can be demonstrated directly by applying Green's theorem to the first term in (6),

$$\int_\Gamma W\, n_1\, ds = \int_{A_\Gamma} W_{,1}\, dA = - \int_{A_\Gamma} \left(\frac{\partial W}{\partial a}\right)_{\underset{\sim}{X}} dA \ , \tag{7}$$

and then noting that (5) applies. The derivation assumes there are no singularities inside Γ . But it still holds when discontinuities in the second derivatives of the displacements and in the first derivatives of the stresses occur across contours contained within Γ , as can be expected across the instantaneous boundary between the plastically loading and elastically unloaded regions.

Each of the two terms in (6) has a simple physical interpretation. The first is the rate per unit crack extension at which stress work passes through the contour Γ which is translating with the tip. The second is the rate of work per unit crack extension done by the tractions acting on Γ on the material within. Since the total stress work within Γ is constant in the steady state, the sum of these two terms must vanish.

Now consider a closed contour such as that shown in Fig. 1 where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 - \Gamma_4$. The contributions along the crack face, Γ_2 and Γ_3 , vanish identically. Thus the integral on Γ_4 equals the integral on Γ_1 . That is, the integral on any contour Γ_c, such as Γ_1 or Γ_4, which encloses the crack tip is the same. Let

$$J^s = \int_{\Gamma_c} (W\, n_1 - T_i u_{i,1}) ds \tag{8}$$

where the superscript s is to emphasize that it is restricted to steady conditions.

The similarity in notation between (8) and the integrals discussed by Eshelby [8] and Rice [4] is deceptive since (8) is path-independent for any material as long as steady conditions pertain. In general, (8) is not path-independent when crack propagation is not steady. In contrast, the integral discussed by Eshelby and Rice is restricted to nonlinear elastic solids (e.g., a deformation theory of plasticity) but is not restricted to steady conditions.

We now argue that $J^s = 0$ for steady growth of a line crack in an elastic-plastic material under the small strain assumptions for which (8) was derived. For elastic-perfectly plastic problems involving a line crack in anti-plane shear, plane strain or plane stress it has been noted that the singularity in the strains (and more generally the displacement gradients) at the crack tip is only logarithmic. It follows that the singularity of the integrand in (8) is also logarithmic. Consequently, if Γ_c is taken as a small circle or radius r_c centered at the tip so that $ds = r_c\, d\theta$, then by letting $r_c \to 0$ it is immediately seen that $J^s = 0$. Physically, $J^s = 0$ implies that no energy feeds out at the advancing tip of the line crack in the elastic-plastic material as it does in a linear or nonlinear elastic material. This same procedure has quite a different conclusion when the material is nonlinear elastic. Then, it is known that $J \neq 0$ and the above procedure implies that the integrand in (8) must have a $1/r$ singularity, which has been verified for a class of nonlinear materials in [9, 10].

No information is available on the singular behavior of the stress and strain at the crack tip in a strain hardening material for nonstationary cracks. However it would not be expected that a small degree of strain hardening could lead to a dramatically stronger singularity in W than that occurring in the perfectly plastic case. In fact, we suspect that any amount of irreversible plastic flow in the steady problem will lead to a singularity in W which is weaker than $1/r$ so that $J^s = 0$.*

The path-independent integral (8) is closely related to an integral derived by Cherepanov [6], although in several respects our treatment differs from his. Most importantly, our conclusion that J^s vanishes for steady growth is in direct contradiction to his conclusion. As in this paper, Cherepanov considers general material behavior. However, he does not restrict consideration to steady crack growth. Following a series of physical and mathematical arguments, he arrives at the propagation criterion (in the present notation and neglecting thermal and inertial effects which are also considered in [6]):

$$\int_{\Gamma_c} (W\, n_1 - T_i u_{i,1}) ds = 2\gamma \tag{9}$$

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* Consider the following argument. Suppose irreversible plastic flow did occur such that $W \sim 1/r$ as $r \to 0$. If material having undergone this level of irreversible straining was left behind in the wake, this would imply that at any <u>fixed</u> value of x_1 behind the tip (i.e., $x_1 < 0$) $W \sim 1/|x_2|$ as $x_2 \to 0$. But this is clearly not possible since it implies infinite energy in any finite length of the wake. The argument is not rigorous since it is conceivable, although it seems unlikely, that, once <u>plastic</u> straining occurs such that $W \sim 1/r$, this level of plastic straining could be reversed so that the residual plastic strains in the wake were of smaller order of magnitude so that $|x_2|W \to 0$ as $x_2 \to 0$ for fixed x_1 .

where γ is the surface energy per unit area required to create a new surface. The surface energy γ is introduced into (9) on physical and not mathematical grounds. Furthermore, Cherepanov goes on to conclude that, since $\gamma \neq 0$, W must have a $1/r$ singularity in all cases. This is correct for nonlinear elastic materials as corroborated in [9, 10]. But for steady growth of a line crack in an elastic-perfectly plastic material, at least, the integral in (9) is zero; and, therefore, the proposed fracture criterion can never be satisfied within the context of this theory. From a different approach, Rice [11] has also shown that, for the small strain theory of a line crack in an elastic-perfectly plastic material, the surface energy cannot be accounted for in the energy balance, whether the motion is steady or not.

The derivation of [6] suggests that this equation holds for nonsteady growth. As already mentioned, the integral in (9) is not path-independent, in general, unless steady conditions prevail or unless the material is nonlinear elastic. Evidently, Cherepanov has tacitly invoked one or the other of these restrictions in his derivation.

With the proper interpretation of W and $\underset{\sim}{T}$, the integral (8) is path-independent for a full finite strain theory analogous to the nonlinear elastic results of Eshelby [8] and Knowles and Sternberg [12].

3. IMPLICATIONS OF THE PATH-INDEPENDENT INTEGRAL FOR SMALL SCALE YIELDING

If stable crack growth occurs, steady conditions in the vicinity of the crack tip will be approached once a crack has propagated a distance on the order of several times the active plastic zone size. If the plastic zone size and the width of the wake of residual plastic strains are sufficiently small they will have a negligible perturbating effect on the elastic solution away from the tip and shank of the crack. In some region surrounding the tip, which is small compared to the crack length and other relevant geometric lengths but large compared to the active plastic zone size, the field of the dominant elastic singularity governs. Under steady conditions the size of the active plastic zone and the width of the wake remain constant and the residual plastic strains outside the active plastic zone in the wake are necessarily independent of x_1 . We consider the limiting problem for steady, small scale yielding for a semi-infinite crack with a semi-infinite wake shown in Fig. 2a. A similar problem formulation is discussed in [3] for the anti-plane shear problem.

For fixed θ in the open range $|\theta| < \pi$

$$\sigma_{ij} \to \frac{K}{\sqrt{2\pi r}} \tilde{\sigma}_{ij}(\theta) \qquad \text{as} \quad r \to \infty \tag{10}$$

where K is the elastic stress intensity factor and $\tilde{\sigma}_{ij}(\theta)$ are the θ-variations of the stress components associated with the dominant singularity of the elastic solution. In the wake (i.e., $\theta = \pm\pi$) as $r \to \infty$ finite residual stresses and strains occur which are not known a priori but depend on the deformation history in the active plastic zone. In addition to (10), it can also be asserted that as $x_1 \to -\infty$ for any fixed x_2 outside the wake (i.e., $|x_2| > 2h$) the stress field approaches zero. This follows from the fact that far from the crack tip either half of the wake can be viewed as an infinite strip of width $2h$ which has undergone x_1-independent plastic straining and which is attached to an elastic foundation of infinite depth. In such a problem the stresses and strains outside the strip are not influenced by the residual stresses and strains in the strip, since for such x_1-independent deformations the stiffness of the strip is zero compared to that of the underlying elastic foundation. By viewing the wake far from the tip in this same way, it also follows that $u_{1,1} \to 0$ and $u_{2,1} \to 0$ in the wake as $x_1 \to -\infty$.

Let Γ_c be a circular contour of radius r_c centered at the tip as in Fig. 2a. Consider evaluating J^s using this contour and then let $r_c \to \infty$. The elastic field (10) makes a finite contribution to J^s since for large r_c, $W \sim 1/r_c$, for $|\theta| < \pi$, and $ds = r_c\, d\theta$. Furthermore, since the wake does not influence the elastic field outside it as $r_c \to \infty$ and since a $1/r_c$ contribution to W in the wake portion of the line integral (8) makes no contribution as $r_c \to \infty$, the contribution of (10) to the line integral decouples from the contribution of the wake. The contribution of (10) to J^s is K^2/E for plane stress and $K^2/[(1-\nu^2)E]$ for plane strain, where E is Young's modulus and ν is Poisson's ratio; i.e., the well known energy release rates per unit thickness for the elastic problem. The contribution from the wake is

$$-2 \lim_{x_1 \to -\infty} \int_0^{2h} (W - T_i u_{i,1}) dx_2 \tag{11}$$

The second term in the integrand vanishes since $u_{i,1} \to 0$ as discussed above. Let W_{AVE} denote the average residual internal energy density in the wake far from the tip, i.e.,

$$W_{AVE} \equiv \frac{1}{2h} \lim_{x_1 \to -\infty} \int_0^{2h} W\, dx_2 \tag{12}$$

Combining K^2/E (for plane stress) and (11) using (12) and the requirement that $J^s = 0$ gives

$$K^2/E = 4h\, W_{AVE} \tag{13}$$

For the problem posed, (13) implies that the elastic energy release rate, K^2/E, must equal the residual internal energy per unit length left behind in the wake. Equation (13) should not be regarded as a fracture criterion (unless the value of $4h\, W_{AVE}$ associated with steady fracture conditions happens to be known from other considerations). It is a necessary condition for steady problems which must hold for any value of K ; it is the simple consequence of the fact that all the elastic energy released by the extending crack is deposited in the wake. Equation (13) is a direct demonstration of the Irwin-Orowan modification of Griffith's original energy balance arguments for perfectly brittle materials to include plastic deformation. The small strain, elastic-plastic theory of a perfectly sharp line crack permits no energy to be lost at the crack tip, as already discussed, and thus (13) cannot include a surface energy contribution. Under most conditions the surface energy per unit length of propagation of a metal is an extremely small fraction of the residual internal energy per unit length associated with plastic deformation so that its absence from (13) is of little consequence, at least as far as the overall energy balance is concerned. It is not clear, in general, what importance the surface energy plays in establishing a fracture criterion.

Cherepanov [7] has derived a relation similar to (13) using a Dugdale model of the plastic zone with no consideration of the wake. However, he includes the work done by the tractions in the strip zone in with the surface energy to give an effective surface energy per unit of surface area formed.

4. PRELIMINARY SOLUTIONS FOR MODELING WAKE OF GROWING CRACK

In Section 5 a model of steady crack growth in thin sheets will be formulated. A wake of residual plastic strains and stresses will be introduced which extends from the crack tip at $x_1 = -d$ to $x_1 = -\infty$ as depicted in Fig. 2b. (For convenience, the origin is now chosen to coincide with the leading edge of the plastic zone.) Solutions used in representing this wake are given in this section. Since these solutions can be obtained using complex variable methods which are now reasonably well known, they will be given without derivation.

Introduce the complex variable $z = x_1 + ix_2$ and let $\phi(z)$ and $\psi(z)$ be the

Muskhelishvili functions for isotropic plane elasticity theory as usually defined so that

$$\begin{aligned}
\sigma_{11} + \sigma_{22} &= 2(\overline{\phi'} + \phi') \\
\sigma_{22} - \sigma_{11} + 2i\,\sigma_{12} &= 2(\bar{z}\phi'' + \psi') \\
u_1 + iu_2 &= [(1+\nu)/E][\kappa\phi - z\overline{\phi'} - \bar{\psi}]
\end{aligned} \tag{14}$$

where $\kappa = (3-\nu)/(1+\nu)$ for plane stress and the bar denotes complex conjugation.

4.1 Elastic field surrounding a circular plastic spot

Let a circular region of radius R and center $z_o = x_1^o + ix_2^o$ in an isotropically elastic infinite sheet undergo a uniform plastic straining $\varepsilon^p_{\alpha\beta}$. Continuity of tractions and displacements across the circular boundary is maintained. This is a plane stress version of the Eshelby problem [13]. The complex functions for the elastic field outside the circular plastic spot, $|z-z_o| > R$, are

$$\begin{aligned}
\phi_o(z) &= \tfrac{1}{2}EAc_2(z-z_o)^{-1} \\
\psi_o(z) &= EA[-(c_1+c_2)(z-z_o)^{-1} + \tfrac{1}{2}c_2\bar{z}_o(z-z_o)^{-2} + \tfrac{1}{2}c_2R^2(z-z_o)^{-3}]
\end{aligned} \tag{15}$$

where

$$A = \pi R^2 \quad , \quad c_1 = \frac{1}{2\pi}(\varepsilon^p_{22} - i\varepsilon^p_{12}) \quad , \quad c_2 = \frac{1}{4\pi}(\varepsilon^p_{11} - \varepsilon^p_{22} + 2i\varepsilon^p_{12}) \tag{16}$$

4.2 Elastic field due to two circular plastic spots which are symmetrically placed on either side of a crack

Consider two circular regions of equal radius R. The center of the upper spot is at $z_o = x_1^o + ix_2^o$ and the center of the lower one is at $\bar{z}_o$; the restriction $x_2^o \geq R$ is assumed. Along $x_2 = 0$ for $x_1 < 0$ the tractions are zero on the crack faces. Let the upper spot undergo a uniform plastic straining $(\varepsilon^p_{11}, \varepsilon^p_{22}, \varepsilon^p_{12})$ and let the lower spot undergo $(\varepsilon^p_{11}, \varepsilon^p_{22}, -\varepsilon^p_{12})$. The resulting stress distribution is symmetric with respect to $x_2 = 0$. The solution for the elastic field outside the two spots is given by

$$\begin{aligned} \phi(z) &= \phi_o(z) + \bar{\phi}_o(z) + \phi_1(z) + \bar{\phi}_1(z) \\ \psi(z) &= \psi_o(z) + \bar{\psi}_o(z) + \psi_1(z) + \bar{\psi}_1(z) \end{aligned} \tag{17}$$

where ϕ_o and ψ_o are the functions in (15). Here the standard notation $\bar{\phi}_o(z) \equiv \overline{\phi_o(\bar{z})}$, etc., is used. The functions ϕ_1 and ψ_1 are given by $\psi_1' = -z\phi_1''$ and

$$\phi_1'(z) = -\frac{1}{2}[2\phi_o'(z) + z\phi_o''(z) + \psi_o'(z)] - \frac{1}{2}\, EAf(z)/\sqrt{z} \tag{18}$$

where

$$f(z) = \frac{\sqrt{z_o}}{2z_o(z-z_o)^2}\left\{-c_1(z+z_o) + \frac{c_2(-z_o^3 + 6zz_o^2 + 3z^2 z_o - 3z_o^2\bar{z}_o - 6zz_o\bar{z}_o + z^2\bar{z}_o)}{4z_o(z_o-z)} - \frac{3c_2R^2(-5z_o^3 - 15z_o^2 z + 5z_o z^2 - z^3)}{16(z_o-z)^2 z_o^2}\right\} \tag{19}$$

The square root is defined such that $\sqrt{1} = 1$ with its branch line chosen to lie along the negative real axis.

Two quantities which will be of special interest are the stress intensity factor and the crack opening displacement. Using the standard definition for the stress intensity factor, one finds

$$K = \lim_{x_1 \to 0^+} (2\pi x_1)^{1/2}\, \sigma_{22}(x_1,0) = -2\sqrt{2\pi}\, EA\,\mathscr{Re}\,[f(0)] \tag{20}$$

where $\mathscr{Re}\,[z]$ stands for the real part of z . For a pair of spots whose distances from the free crack surface are small compared to the distances from the tip, i.e., $x_2^o/|x_1^o| \ll 1$, (20) reduces to

$$K = -\frac{3}{2\sqrt{2\pi}}\,|x_1^o|^{-3/2}\, EA\left\{\varepsilon_{11}^p \frac{x_2^o}{|x_1^o|} + \mathcal{O}\left[(\varepsilon_{22}^p, \varepsilon_{12}^p)\left(\frac{x_2^o}{x_1^o}\right)^2\right]\right\} \tag{21}$$

The plastic strain components ε_{12}^p and ε_{22}^p have an inherently smaller influence on the stresses at the tip than does ε_{11}^p when $x_2^o/|x_1^o|$ is small.

With $\delta(x_1) \equiv u_2(x_1,0^+) - u_2(x_1,0^-)$ for $x_1 \leq 0$, the crack opening displacement is given by

$$\delta(x_1) = 8A \int_0^{x_1} (-\eta)^{-1/2} \mathscr{R}e\ [f(\eta)]d\eta \tag{22}$$

where η is a real integration variable. Let ε_o be an arbitrary reference strain. It is a simple matter to reduce (22) to a nondimensional form for $\delta/(R\varepsilon_o)$ involving only the following quantities: $x_1/|x_1^o|$, $x_2^o/|x_1^o|$, R/x_2^o , $\varepsilon_{11}^p/\varepsilon_o$, $\varepsilon_{12}^p/\varepsilon_o$ and $\varepsilon_{22}^p/\varepsilon_o$. Curves of $\delta(x_1)/(R\varepsilon_o)$ calculated numerically using (22) are shown as solid line curves in Fig. 3. In Fig. 3a $x_2^o/|x_1^o|$ is taken to be 1/10 while in Fig. 3b it is 3/10 ; in both cases R is chosen to be equal to x_2^o so that the spots just touch the crack faces as indicated in the insert. The curves identified by ($\varepsilon_{11}^p/\varepsilon_o = 1$, $\varepsilon_{12}^p = \varepsilon_{22}^p = 0$) were computed using those values for the parameters, with similar identifications for the other cases.

When $x_2^o/|x_1^o|$ is small one would expect the displacements in the vicinity of the spot to be essentially the same as those predicted for a plastic spot situated near the free edge of a semi-infinite sheet (i.e., where the tractions vanish along $x_2 = 0$ for <u>all</u> x_1). By taking the appropriate limit of (22) as $x_2^o \rightarrow 0$ one can show that this is the case. The dashed line curves in Fig. 3 are the predictions of this simpler solution for the spot of radius R centered at z_o in a semi-infinite sheet ($x_2 \geq 0$) . The identification $\delta(x_1) = 2u_2(x_1,0)$ has been made. Note that for $x_2^o/|x_1^o| = 1/10$ the two sets of predictions are essentially indistinguishable over the range of x_1 shown, and even for $x_2^o/|x_1^o| = 3/10$ the approximation is reasonably good except right at the crack tip.

5. SMALL SCALE YIELDING MODEL OF THE EFFECT OF THE WAKE ON STEADY CRACK GROWTH IN THIN SHEETS

As indicated in Fig. 2b, the active plastic zone will be modeled by a Dugdale strip zone extending from the crack tip at $x_1 = -d$ to $x_1 = 0$. The sheet is slit along $x_2 = 0$ in this zone and the tractions are required to satisfy $\sigma_{22} = \sigma_o$ and $\sigma_{12} = 0$ on each face. Identify σ_o with an effective yield stress of the material in tension. The length d of the zone will be adjusted so that the stresses are bounded ahead of the plastic zone.

Small scale yielding is assumed and thus far from the tip on any radial line except $\theta = \pm\pi$ the dominant singularity of the elastic solution (10) is approached. The stress intensity factor K is regarded as prescribed. In terms of the complex Muskhelishvili functions the far field is

$$\phi' \cong 2\psi' \to K/[2(2\pi z)^{1/2}] \quad \text{as} \quad |z| \to \infty \tag{23}$$

No attempt will be made to relate the details of the distribution of the plastic strains in the wake to the deformation field in the active plastic zone, but it will be assumed that <u>all</u> the elastic energy released goes into the internal energy of the wake consistent with (13). In other words, the work done by the tractions $\sigma_{22} = \sigma_o$ in the strip zone is considered to appear as part of the internal energy left in the wake.

Let $\varepsilon^p_{\alpha\beta}$ represent the inplane plastic strains averaged with respect to x_2 across the wake; these are independent of x_1. The wake is modeled by integrating contributions of the plastic spot solution (17) from $x_1^o = -\infty$ to $x_1^o = -d$. Identify the uniform plastic strains in an elemental spot with the average plastic strains in the wake $\varepsilon^p_{\alpha\beta}$. At a typical point x_1^o in the wake identify the area element A in (16) and (18) with $2h\, dx_1^o$ and take the element to be centered at $z_o = x_1^o + ih$

with its partner centered at $\bar{z}_o$. Note that (17) makes no contribution to the tractions on the surfaces in the strip zone ahead of the crack.

The energy relationship, $K^2/E = 4h\, W_{AVE}$, discussed in §3 will be imposed. It is convenient to introduce a set of dimensionless quantities $g_{\alpha\beta}$ defined by the equation

$$\sigma_o \varepsilon^p_{\alpha\beta} \equiv g_{\alpha\beta}\, W_{AVE} \qquad (24)$$

These quantities reflect the relative amounts of each component of the average strain in the wake. For example, if ε^p_{22} is the only nonzero inplane component generated in the active zone, then the average residual internal energy in the wake, W_{AVE} , is equal to $\sigma_o \varepsilon^p_{22}$ if the material is elastic-perfectly plastic and, thus, $g_{22} = 1$ and $g_{11} = g_{12} = 0$. In general, the elastic residual energy in the wake will comprise only a relatively small fraction of W_{AVE} so that the quantities $g_{\alpha\beta}$ will range in magnitude from zero to approximately unity.

At this stage the width of the wake, $4h$, will be left unspecified except that it will be assumed that the ratio h/d is small compared to unity, not exceeding about $4/10$, consistent with theoretical and experimental results on the shape of plane stress plastic zones. Later it will be seen that the main predictions of the model are essentially independent of the choice of h .

The amplitude of the inverse square root singularity of $\sigma_{22}(x_1,0)$ ahead of the active plastic zone is the sum of the contributions from (23), from the traction $\sigma_{22} = \sigma_o$ applied along the strip zone and from the wake. The first two are the contributions which enter into the Dugdale-Barenblatt model and the third is obtained by integrating the contributions in (20) over the entire wake. Thus,

$$\lim_{x_1 \to 0^+} \sqrt{2\pi x_1}\, \sigma_{22}(x_1,0) = K - \sigma_o (8d/\pi)^{1/2} - 4\sqrt{2\pi}\, Eh \int_{-\infty}^{-d} dx_1^o\, \mathscr{R}e\, [f(0)] \qquad (25)$$

The radius R of the spot appears explicitly only in the third term in the brackets in the expression (19) for $f(z)$. For the time being we will retain the term in (19) which explicitly involves R without identifying it with a specific length, but it will be assumed that R is not greater than h . Later it will be seen that this term has very small influence on the predictions.

For the stationary crack the third term in (25) is absent and d is chosen such that there is no inverse square root singularity in the stress σ_{22} at the leading edge of the plastic zone; i.e., for the Dugdale-Barenblatt model,

$$d_D = (\pi/8)(K/\sigma_o)^2 \tag{26}$$

The third term in (25) can be integrated with the result that this term is (letting $b \equiv -d + ih$)

$$-4\sqrt{2\pi}\ Eh\mathscr{R}e\ [(c_1+c_2)b^{-1/2} - \frac{1}{2}\ ih\ c_2 b^{-3/2} - \frac{3}{16}\ c_2 R^2 b^{-5/2}]$$

Expanding this expression in a Taylor series in small values of h/d gives

$$-2\sqrt{2\pi}\ Eh\ d^{-1/2}\mathscr{R}e\ [(h/d)(c_1+2c_2) + \mathscr{O}(h/d)^2]$$

$$= -E(2d/\pi)^{1/2}(h/d)^2[\varepsilon^p_{11} + \mathscr{O}\{(h/d)\varepsilon^p_{12}, (h/d)\varepsilon^p_{22}\}]$$

Now, use (13), (24) and (26) to rewrite the above expression for the contribution of the wake to (25) in terms of the $g_{\alpha\beta}$'s with the result

$$-\sigma_o\left(\frac{8d}{\pi}\right)^{1/2} \frac{1}{\pi}\left(\frac{h}{d}\right)\left(\frac{d_D}{d}\right)\underline{\left[g_{11} + \mathscr{O}\left\{\left(\frac{h}{d}\right)g_{12}, \left(\frac{h}{d}\right)g_{22}\right\}\right]} \tag{27}$$

For the steady growing crack d will also be chosen such that (25) vanishes. The underlined term in (27) will be only a few percent if h/d is not larger than about 3/10 and if ε^p_{11} is not the predominant plastic strain component so that the magnitude of g_{11} will not exceed, say, 0.3 to 0.4 . Thus, the contribution of the wake (27) will not exceed a few percent of the second term in (25). Therefore, the length of the active plastic zone for the steadily growing crack is almost identical to that of the stationary crack at the same value of K, i.e., $d \cong d_D$. This result is consistent with the similar finding of Chitaley and McClintock [3] in their complete analysis of the anti-plane shear problem.

The total opening displacement in the active plastic zone is the sum of the same three contributions mentioned above. Using (22) for the contribution from the wake plus the two contributions present in the stationary strip model gives, for $-d \le x_1 \le 0$,

$$\delta(x_1) = 4\frac{K}{E}\left(-\frac{2x_1}{\pi}\right)^{1/2} - \frac{4}{\pi}\frac{\sigma_o}{E}d\left\{2\left(-\frac{x_1}{d}\right)^{1/2} - \left(1+\frac{x_1}{d}\right)\ln\left[\frac{1+(-x_1/d)^{1/2}}{1-(-x_1/d)^{1/2}}\right]\right\}$$

$$+ 16h\int_{-\infty}^{-d} dx_1^o \int_0^{x_1} d\eta(-\eta)^{-1/2}\mathscr{R}e\,[f(\eta)] \tag{28}$$

To calculate $\delta(x_1)$ for a prescribed value of K it is necessary to first choose d such that (25) vanishes. We will simplify this procedure by taking $d = d_D$, which as already discussed is an excellent approximation if h/d is not too large and is exact for $h/d \to 0$. The integrations in (28) cannot be carried out in closed form, but an excellent approximation can be obtained for small h/d using the simpler solution alluded to in §4.2 for the spot in a semi-infinite sheet. The approximate expression for $\delta(x_1)$ is (with $d = d_D$ and for $-d \le x_1 \le 0$)

$$\delta(x_1) = \frac{8}{\pi}\frac{\sigma_o d}{E}\left\{\left(-\frac{x_1}{d}\right)^{1/2} - \frac{1}{2}\left(1+\frac{x_1}{d}\right)\ln\left[\frac{1+(-x_1/d)^{1/2}}{1-(-x_1/d)^{1/2}}\right]\right\}$$

$$+ 16h\,\mathscr{I}m\left\{(c_1+c_2)\ln(1-x_1/b) + ih\,c_2\left[(b-x_1)^{-1}-b^{-1}\right] + \frac{1}{4}c_2R^2\left[(b-x_1)^{2}-b^{-2}\right]\right\} \tag{29}$$

where $\mathscr{I}m\,[z]$ denotes the imaginary part of z , $b \equiv -d + ih$, and $\ell n[z]$ is the natural logarithm of complex argument defined such that $\ell n[1] = 0$ with the branch line taken along the negative real axis.

Let $\delta_t \equiv \delta(x_1 = -d)$ denote the crack tip opening displacement. By setting $x_1 = -d$ in (29) and rearranging the result, one can obtain

$$\delta_t = (8/\pi)\sigma_o d/E - 8\pi h(c_1+c_2) + 4h[4 - (R/h)^2]\,\mathscr{I}m\,[c_2]$$
$$+ 8h\,\mathscr{I}m\left\{-2(c_1+c_2)\ell n(-b/d) - 2ih\,c_2 b^{-1} - \tfrac{1}{2}(R/h)^2 c_2 h^2 b^{-2}\right\} \tag{30}$$

Denote by δ_t^D the crack tip opening displacement for the stationary crack; i.e., the Dugdale-Barenblatt value

$$\delta_t^D = (8/\pi)\sigma_o d/E = K^2/(\sigma_o E) \tag{31}$$

Here again, use the relation (13) between K and the residual internal energy in the wake together with (24), and (26) to express δ_t in terms of the $g_{\alpha\beta}$'s with the result

$$\delta_t/\delta_t^D = 1 - \tfrac{1}{2}(g_{11}+g_{22}) + \tfrac{1}{2\pi}[4 - (R/h)^2]g_{12} + \mathscr{O}(h/d) \tag{32}$$

The terms indicated by $\mathscr{O}(h/d)$ are the terms from the second line of (30); they go to zero linearly with h/d as $h/d \to 0$. For the mathematical limit $h/d \to 0$, (32) gives the exact expression for δ_t as based on the unapproximated equations (25) and (28).

For small h/d , δ_t as given by (32) is essentially <u>independent</u> of the choice of the width $4h$ of the wake by virtue of the fact that the quantities $g_{\alpha\beta}$ introduced in (24) are themselves independent of h , except to the extent that the relative proportions of the average plastic strain components in the wake may be a function of h/d . Note that the term involving R , which has been retained in the analysis, enters only in the one term. Its influence is clearly small since over its meaningful range, $0 \leq R \leq h$, the coefficient of g_{12} in (32) changes by only twenty-five percent.

The values of $g_{\alpha\beta}$ depend on the details of the plastic deformation occurring in the active plastic zone. For the purpose of discussion suppose that ε_{22}^p is the predominant inplane plastic strain component so that $g_{22} \cong 1$ and $g_{11} \cong g_{22} \cong 0$. In this case, according to (32), the crack tip opening displacement of the steadily growing crack is about one-half the value of the stationary crack at the same value of K . If δ_t is used as a measure of the intensity of deformation at the crack tip and if a fracture criterion is used which involves the attainment of a critical value of δ_t , then the model indicates that, due to the wake alone, the critical value of K for steady growth should be approximately $\sqrt{2}$ times the value for initiation.

The fact that the wake reduces the crack tip displacement by an amount $\delta_t^D/2$ when ε_{22}^p is predominant can be seen very simply in the following way. Neglect the interaction with the end of the zone at $x_1 = 0$ and consider a semi-infinite half sheet with a traction-free edge on $x_2 = 0$. Let a uniform plastic straining ε_{22}^p be induced in the wake region, $0 \leq x_2 \leq 2h$ and $x_1 \leq -d$. The displacement u_2 for this problem at $(x_1 = -d, x_2 = 0)$ is <u>exactly</u> $-h\varepsilon_{22}^p$. Using $\sigma_o\varepsilon_{22}^p = W_{AVE}$ and (13) to express $-h\varepsilon_{22}^p$ in terms of K gives

$$u_2(-d,0) = -K^2/(4\sigma_o E) \tag{33}$$

Since the total contribution to δ_t from both halves of the wake is twice the above amount, it is immediately seen from (31) that this is exactly $-\delta_t^D/2$. This same result (33) holds for <u>any</u> distribution of $\varepsilon_{22}^p(x_2)$ in the wake region as long as it is independent of x_1 . This observation lends some confidence in our representation of the plastic strains in the wake as being essentially uniform.

Generally ε_{11}^{p} and ε_{12}^{p} will also be present in the wake and their contributions can also be expected to diminish δ_t relative to δ_t^D since various evidence points to positive g_{11} and negative g_{12}. In particular, the trailing edges of an advancing crack in a thin sheet are often observed to buckle out of the plane which is most likely due to a residual compressive stress σ_{11} associated with positive ε_{11}^{p}. If lines are lightly scored perpendicular to the line of the advancing crack in the face of the sheet ahead of the tip, following passage of the crack the lines indicate negative values of ε_{12}^{p} above the crack and positive values below, consistent with negative g_{12}. The value $\delta_t/\delta_t^D \cong 1/2$ seems to be more-or-less representative for any such combination of $g_{\alpha\beta}$'s constrained by (24).

The result (32) holds for plane strain as well as plane stress, although the strip model of the plastic zone is not realistic in plane strain. Nevertheless, (32) is very suggestive as far as the effect of the wake is concerned. In plane strain ε_{33}^{p} is small and, since the plastic volume change is zero, this implies that $\varepsilon_{11}^{p} + \varepsilon_{22}^{p} \cong 0$ or $g_{11} + g_{22} \cong 0$. Thus, the term $(g_{11}+g_{22})/2$ in (32), which gives the major contribution to reducing the crack tip opening displacement in plane stress, is negligible in plane strain. This suggests that the effect of the wake on stabilizing crack growth is less in plane strain than in plane stress.

Figure 4 shows curves of $\delta(x_1)$ in the strip zone for three combinations of $g_{\alpha\beta}$'s from which any other combination can be computed at the same value of h/d by linear superposition. The curves were calculated using (29) for $h/d = 0.2$ and $R = h$. The smaller is h/d, the more the effect of the wake on $\delta(x_1)$ in the strip zone becomes concentrated near $x_1 = -d$. Thus, for example, for $g_{22} = 1$ and $g_{11} = g_{12} = 0$, δ_t is no longer the maximum value of $\delta(x_1)$ on the interval $-d \le x_1 \le 0$ when h/d is less than about 0.2. The model as formulated can no longer be considered physically relevant when $\delta(x_1)$ does not decrease monotonically from the crack tip to the leading edge of the strip zone. The cutoff value of h/d ranges from approximately .1 to .2 depending on the combination of $g_{\alpha\beta}$'s. Thus, while (32) is exact for the mathematical limit $h/d \to 0$, the model loses its significance before this limit is reached. In particular, its relevance to stable propagation in very thin sheets becomes questionable when the plastic deformation is largely confined to a thin zone of width comparable to the sheet thickness in which necking occurs. Then, if the sheet is very thin the length of the active zone may be many times the sheet thickness such that h/d falls outside the range of validity of the model.

We reiterate that irreversibility associated with nonproportional loading in the active plastic zone, which also contributes to a stabilization of crack growth, has not been taken into account in this study. A complete analysis would necessarily involve an accurate modeling of both the active plastic zone and the wake such as that carried out in [3] (but, perhaps, based on a stress-strain relation with a non-smooth yield surface). The main conclusion of this study is that the wake of residual stresses and strains contributes a significant stabilizing effect to crack growth.

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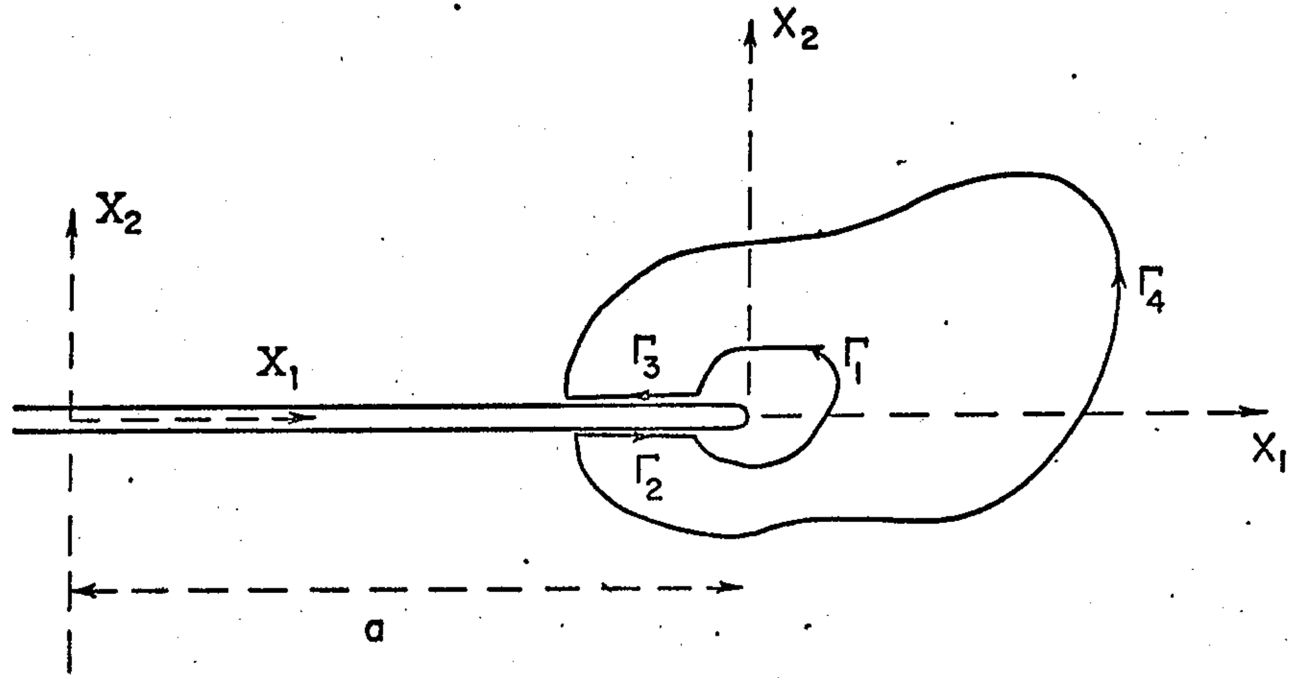

FIG. 1 CONVENTIONS AT THE CRACK TIP

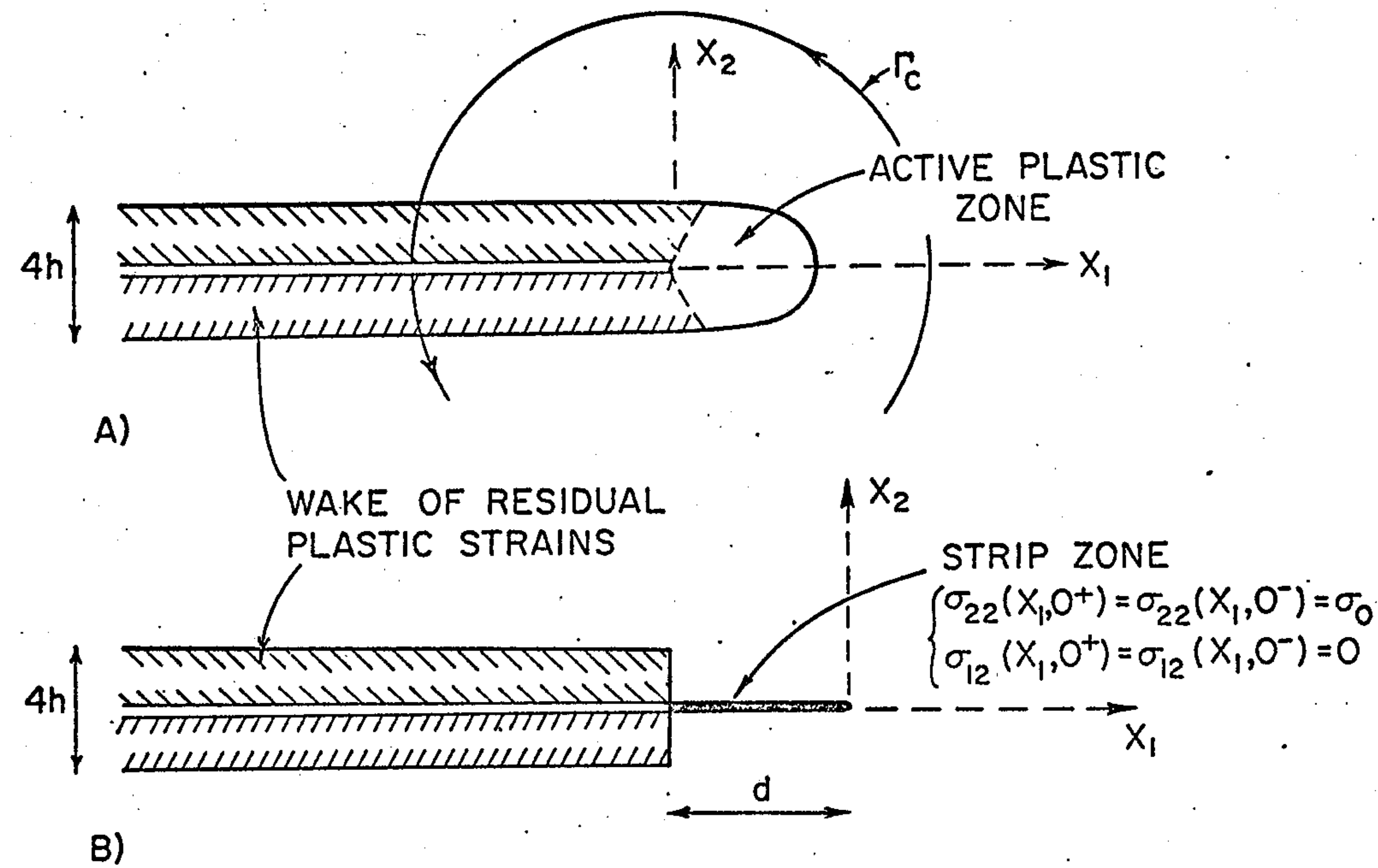

FIG. 2 A) STEADILY GROWING CRACK IN SMALL SCALE YIELDING
B) STRIP MODEL INCLUDING WAKE

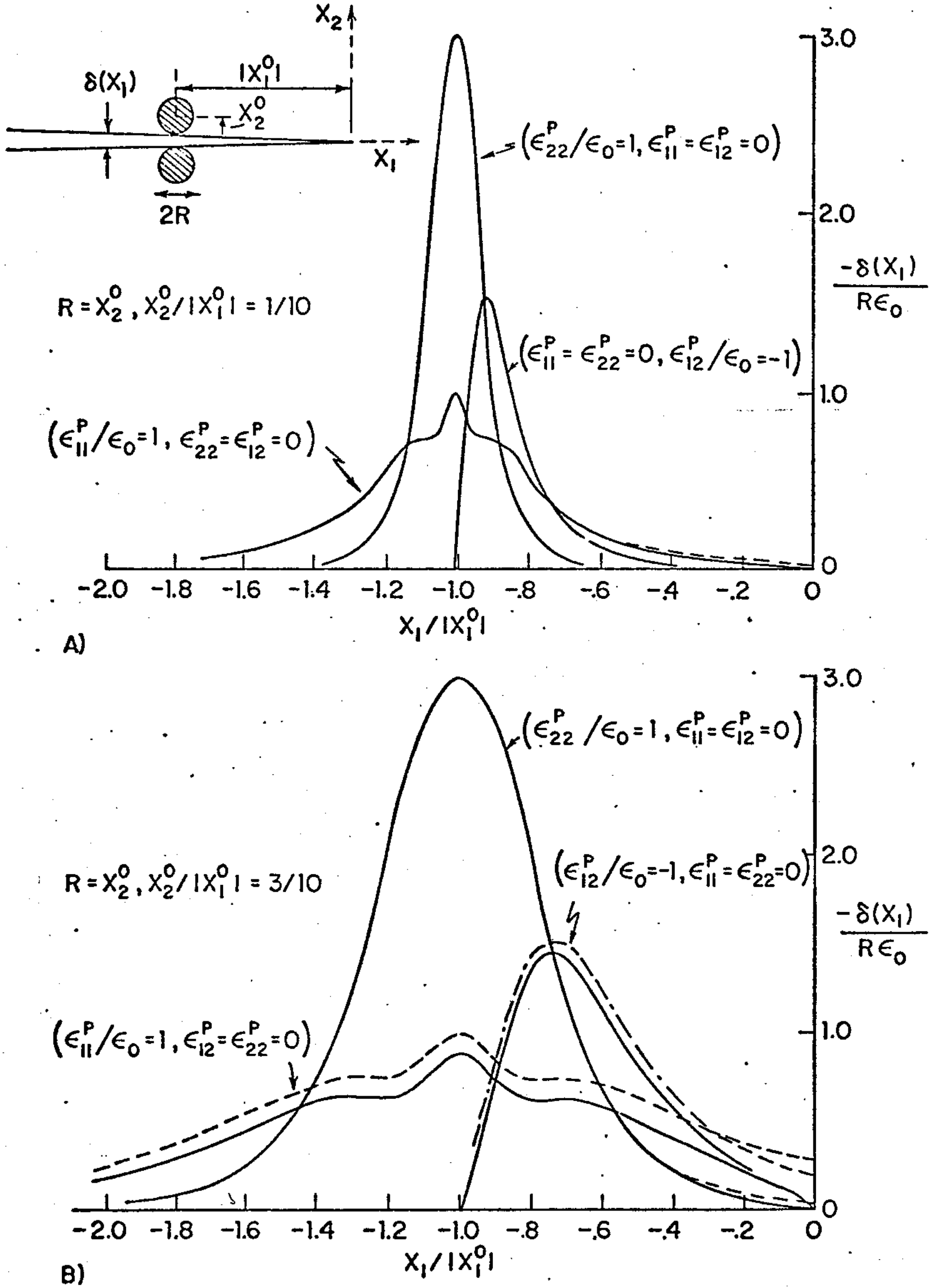

FIG. 3 CRACK OPENING DISPLACEMENT RESULTING FROM A PAIR OF CIRCULAR SPOTS WHICH HAVE UNDERGONE UNIFORM PLASTIC STRAINING

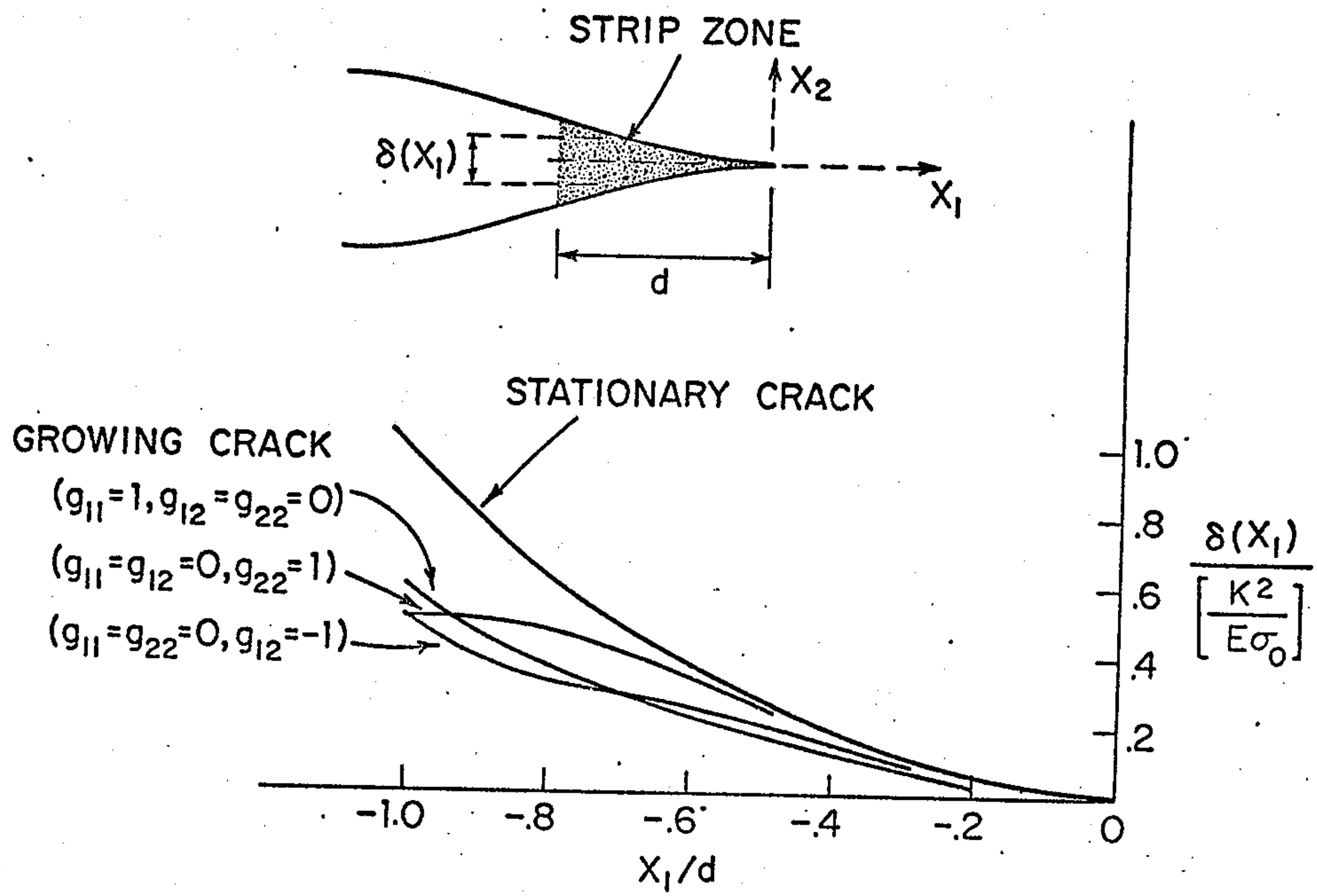

FIG. 4 OPENING DISPLACEMENT IN STRIP ZONE FOR STEADILY GROWING CRACK COMPARED WITH THAT OF STATIONARY CRACK (h/d = 2/10).