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Analytical and Numerical Study of the Effects of Initial Imperfections on the Inelastic Buckling of a Cruciform Column[1]

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Abstract

The inelastic buckling of a cruciform column is investigated by a combination of analytical and numerical methods. An exact asymptotic analysis for the effect of small imperfections on the maximum load reveals clearly how it is possible for an exceedingly small imperfection to have a very large influence. As long as the strain hardening is sufficiently low, the numerical analysis confirms the Onat-Drucker conclusion that unavoidably small imperfections, together with the use of J_2 flow theory, give rise to a maximum load prediction which is approximated by the bifurcation load prediction based on a deformation theory of plasticity.

1. Introduction

The cruciform column is perhaps the best example for illustrating the discrepancy between the buckling predictions of the simplest flow and deformation theories of plasticity. According to *any* flow theory for an initially isotropic material in conjunction with a smooth yield surface, the critical compressive stress for twisting bifurcation is

$$\sigma_c = G(t/b)^2, \tag{1}$$

where G is the elastic shear modulus and t and b are shown in Fig. 1. According to *any* deformation theory of plasticity for an initially isotropic material the bifurcation stress is

$$\bar{\sigma}_c = \bar{G}(t/b)^2, \tag{2}$$

where $\bar{G}$ is a reduced effective shear modulus given by

$$\bar{G}/G = [1 + 3G(1/E_s - 1/E)]^{-1}, \tag{3}$$

where E is Young's modulus and $E_s = \sigma/\varepsilon$ is the secant modulus of the compressive stress-strain curve at $\bar{\sigma}_c$.

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The experimental results for the cruciform column cited by Stowell (1951) are in good agreement with the deformation theory prediction as seen in Fig. 1. Arguments justifying the use of deformation theory

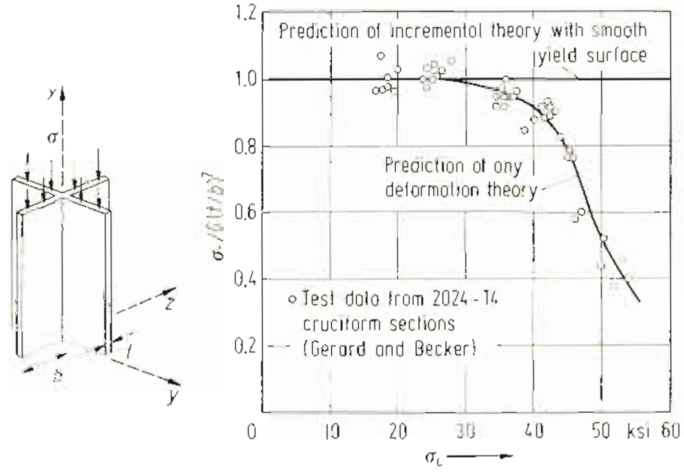

Fig. 1.
Comparison of tests and theory for torsional buckling of a cruciform column.

in bifurcation applications have appealed to the fact that the deformation theory predictions are identical to predictions of a more sophisticated flow theory which permits corners to develop on the yield surface. For example, in the case of the cruciform column, (2) is also the prediction of the slip theory of plasticity. Onat and Drucker (1953) have argued that the poor agreement between tests and the prediction (1) based on a smooth yield surface stems from the presence of unavoidably small imperfections which give rise to a maximum load which can be substantially below (1) and closely approximated by (2). The appropriateness of smooth or cornered yield surfaces in modeling elastic-plastic solids is still unresolved as discussed by Hutchinson (1974).

In this paper the imperfection-sensitivity associated with the simplest flow theory of plasticity is examined in somewhat more detail than has been previously reported. To set the stage, some accurate numerical results are given. Then an exact asymptotic analysis is carried out which reveals the source of the extreme imperfection-sensitivity associated with J_2 flow theory.

2. Governing Equations

The von Kármán nonlinear plate equations will be used together with the following geometric assumptions. Each cross section is assumed to rotate as a rigid body about the x-axis common to the four flange plates and to translate only in the x direction. The compressive strain ε of the axis and the rotation per unit length θ about the axis are taken to be independent of x. An initial imperfection is taken in the form of an x-independent initial rotation per unit length $\bar{\theta}$ so that the total twist per unit length is $\theta + \bar{\theta}$. The three middle surface displacements for the plate lying in the $x - y$ plane (see Fig. 1) are

$$U = -\varepsilon_0 x, \quad V = -(\theta^2 + 2\theta\bar{\theta})\, x^2 y/2, \quad W = \theta x y. \tag{4}$$

According to nonlinear plate theory the nonzero strains associated with the additional twist θ are given by

$$\varepsilon_{xx} = -\varepsilon_0 + y^2(\theta\bar{\theta} + \theta^2/2) \quad \text{and} \quad \gamma_{xy} = 2zK_{xy} = -2z\theta, \tag{5}$$

where $K_{xy} = -W_{,xy}$ is the only nonzero bending strain.

The only stresses considered are σ_{xx} and σ_{xy} and these are necessarily even and odd functions of z, respectively. The two pertinent resultant stress and bending moment quantities are

$$N_{xx} = \int_{-t/2}^{t/2} \sigma_{xx}\, dz \quad \text{and} \quad M_{xy} = \int_{-t/2}^{t/2} \sigma_{xy} z\, dz. \tag{6}$$

Equilibrium equations follow from the principle of virtual work which, in terms of work quantities per unit length in the x direction, is

$$\int_0^b \{2M_{xy}\, \delta K_{xy} + N_{xx}\, \delta\varepsilon_{xx}\}\, dy - P\, \delta\varepsilon_0 = 0 \tag{7}$$

for arbitrary $\delta\theta$ and $\delta\varepsilon_0$, where P is the compressive load carried by one flange plate. The two equilibrium equations resulting from (7) are

$$P = -\int_0^b N_{xx}\, dy \tag{8}$$

and

$$2\int_0^b M_{xy}\, dy - (\theta + \bar{\theta}) \int_0^b N_{xx} y^2\, dy = 0. \tag{9}$$

The rate-constitutive relation is specialized assuming σ_{xx} and σ_{xy} are the only nonzero stress components. (Thus by the constitutive relation, ε_{yy} will not be zero, in general, in contradiction to the consequence of the geometric assumptions. This is an unimportant inconsistency since σ_{yy} is indeed expected to be very small.) The inverted form of the stress-strain relation according to J_2 flow theory is

$$\begin{aligned} \dot{\sigma}_{xx} &= c_1 \dot{\varepsilon}_{xx} + c_2 \dot{\gamma}_{xy}, \\ \dot{\sigma}_{xy} &= c_2 \dot{\varepsilon}_{xx} + c_3 \dot{\gamma}_{xy}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} c_1 &= E[1 - (\varrho - 1)\, \sigma_{xx}^2 D^{-1}], \quad c_2 = -3G(\varrho - 1)\, \sigma_{xx}\sigma_{xy}\, D^{-1}, \\ c_3 &= G[1 - 9(\varrho - 1)\, \sigma_{xy}^2 \{2(1 + \nu)\, D\}^{-1}], \quad \varrho = E/E_t, \\ D &= \varrho(\sigma_{xx}^2 + 3\sigma_{xy}^2) + \alpha(\varrho - 1)\, \sigma_{xy}^2, \quad \alpha = 3(1 - 2\nu)/[2(1 + \nu)]. \end{aligned} \tag{11}$$

Here, ν is Poisson's ratio and E_t is the tangent modulus of the uniaxial stress-strain curve which is regarded as a funtion of the effective stress $(\sigma_{xx}^2 + 3\sigma_{xy}^2)^{1/2}$. Equations (11) hold as long as the effective stress is increasing. The elastic constants pertain for unloading.

The bifurcation predietion based on Eqs. (4)—(11) (with $\bar{\theta} = 0$) is precisely (1). It can be noted that a bifurcation analysis based on the unapproximated von Kármán plate equations for a flange plate of length L which is simply supported on three sides and free on the fourth gives (1) plus smaller terms of relative order $(b/L)^2$. Thus the present theory is restricted to the torsional buckling of relatively long cruciforms.

3. Numerical Predictions

The above equations were solved numerically for a wide range of cases and imperfection levels. An incremental method was used. The region, $0 < y/b < 1$ and $|z/t| < 1/2$, was divided into subareas in which the stress was taken to be uniform. Integrals over the region were evaluated in such a way that when the moduli are aetually uniform the integrals are evaluated exactly. Calculations were made using two, tensile stress-strain curves: The bilinear relation,

$$\dot{\sigma} = E\dot{\varepsilon} \quad \text{for} \quad \sigma < \sigma_y \quad \text{and} \quad \dot{\sigma} = E_t\dot{\varepsilon} \quad \text{for} \quad \sigma > \sigma_y, \tag{12}$$

where E_t is constant, and the Ramberg-Osgood relation

$$\varepsilon/\varepsilon_y = \sigma/\sigma_y + (3/7)\, (\sigma/\sigma_y)^n, \tag{13}$$

where $\varepsilon_y = \sigma_y/E$.

Imperfection-sensitivity curves in terms of the maximum load of the imperfect column, $P_{\max}$, normalized by the bifurcation load, $P_c = bt\sigma_c$, are plotted as a function of the imperfection parameter $\bar{\xi} = b^2\bar{\theta}/t$ in Fig. 2. The normalization $\bar{\xi} = b^2\bar{\theta}/t$ eliminates any explicit b/t dependence when results are plotted in the manner of Fig. 2. Note that $\bar{\xi}t$ is the initial relative displacement of any two points at the outer edge of the flange a distance b apart, i.e., $\bar{\xi}t = \overline{W}(x + b, b) - \overline{W}(x, b)$. Thus, for a typical cruciform an imperfection $\bar{\xi} = 0.1$ can be regarded as very large, while $\bar{\xi} = 0.01$ is small but not necessarily unavoidable. Imperfection levels less than about $\bar{\xi} = 0.001$ can probably be considered unavoidable.

The parameters for the examples in Fig. 2a were chosen so that the ratio of (1) to (2) is exactly 3/2. In Fig. 2b this ratio is 2; the deformation theory bifurcation prediction is indicated by a dashed line in both

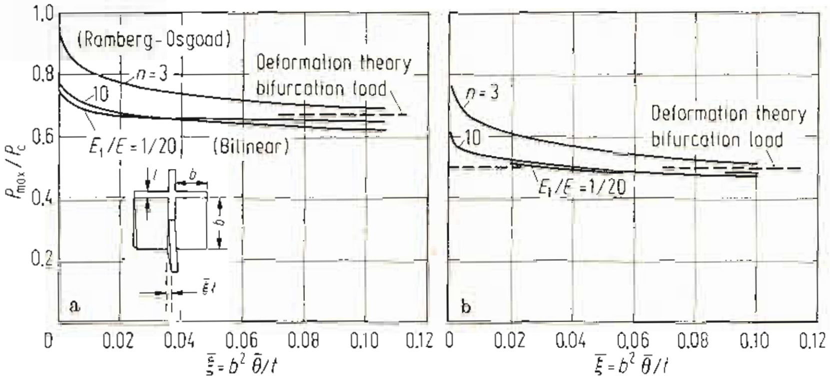

Fig. 2a and b. Numerical results for maximum load as a function of imperfection amplitude. a) $\sigma_c/\bar{\sigma}_c = 1.5$; b) $\sigma_c/\bar{\sigma}_c = 2$.

figures. The curves for the bilinear relation (12) with $E_t/E = 1/20$ apply to the example studied by Onat and Drucker. As they concluded, extremely small imperfections do indeed reduce the maximum load to about the deformation theory bifurcation load. The same is true for the case of a low strain hardening Ramberg-Osgood tensile relation ($n = 10$). However, when the strain hardening is high ($n = 3$) there is approximately a 20% variation in the maximum load over the range of imperfection levels which cannot necessarily be considered unavoidable.

These same calculations were repeated with J_2 deformation theory, and no elastic unloading. Over the imperfection range shown the maximum load differs very little from the deformation-theory bifurcation load of the perfect column.

4. Asymptotic Predictions

Torsional buckling of a cruciform column is one of the rare examples where bifurcation takes place in the plastic range under constant, as opposed to increasing, load to lowest order. In other words, the lowest bifurcation load is simultaneous with the so called reduced modulus load at which the straight configuration loses stability. For the *perfect* cruciform elastic unloading starts at bifurcation; but in most instances the numerical results for the *imperfect* cruciform showed that elastic unloading starts *after* the maximum load has been attained. We anti-

cipate the load-twist behavior depicted for Case II in Fig. 3 and consider only the plastic loading relations given by (11). This does not mean that the analysis becomes the same as that for a nonlinear elastic material

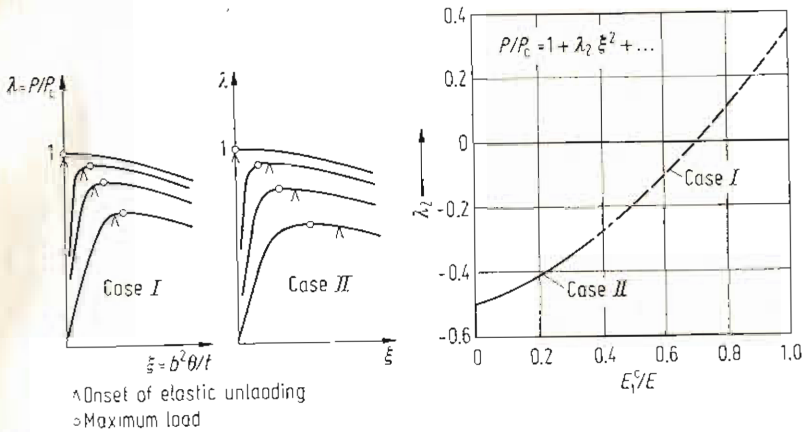

Fig. 3. Sketches of load-twist curves and initial post-bifurcation coefficient for perfect column calculated using J_2 theory with no elastic unloading.

Since the moduli in (11) cannot be derived from a strain energy density functional the solid characterized by J_2 flow theory with no unloading must be considered to be hypo-elastic. This distinction is essential as will be seen below. An a posteriori evaluation of the condition for attainment of the maximum load prior to the onset of unloading will be made.

In this section the results of an exact asymptotic analysis of the cruciform equations (4)–(11) are given. The details of the analysis are omitted. The analysis has three steps: a regular perturbation analysis giving the lowest order effect of $\bar{\xi}$ at loads below P_c, a singular perturbation analysis for the behavior in the vicinity of P_c, and, the final step, a matching of the regular and singular expansions.

Let $\lambda = P/P_c$ and define $f(\lambda)$ as

$$f(\lambda) = 3[E/E_t(\lambda) - 1]\,[2(1+\nu)]^{-1}, \tag{14}$$

where $E_t(\lambda)$ denotes the value of the tangent modulus at the uniaxial stress $\sigma = \lambda\sigma_c$. The solution to the regular perturbation problem gives the following relation between $\xi \equiv b^2\theta/t$, $\bar{\xi}$ and λ:

$$\xi = \bar{\xi}\{(1-\lambda)^{-\varkappa}\, e^{-h(\lambda)} - 1\} + O(\bar{\xi}^2), \tag{15}$$

where

$$\varkappa = 1 + f(1) = 1 + 3[E/E_t^c - 1]\,[2(1+\nu)]^{-1}, \tag{16}$$

and

$$h(\lambda) = \int_0^\lambda [f(1) - f(\eta)] [1 - \eta]^{-1} d\eta . \tag{17}$$

Equation (15) holds on the rising branch of the curve of λ as a function of ξ for $\lambda < 1$. Note that, through (17), ξ depends on the entire history of the loading up to λ. For linear or even nonlinear elastic solids $\varkappa$ in the exponent in (15) is always unity and there is no history dependence. However for a bypo-elastic solid that obeys (11), $\varkappa$ will be larg ecompared to unity if E_t^c/E is small. For λ near 1, (15) can be rewritten as

$$(1 - \lambda) \approx [e^{-h(1)} \bar{\xi}/\xi]^{1/\varkappa} . \tag{18}$$

The singular perturbation analysis in the neighborhood of $\lambda = 1$ gives

$$\lambda = 1 + \lambda_2 \xi^2 + c \bar{\xi}^\omega \xi^{-1/\varkappa} + \cdots , \tag{19}$$

where $\varkappa$ is the same as in (16). The exponent ω and the coefficient c are undetermined. The quantity λ_2 depends only on $\varrho_c = E/E_t^c$ and ν; it is

$$\lambda_2 = \varrho_c^{-1}\{4(1 + \nu)^2/15 - (\varrho_c - 1)^2/2 - 9(\varrho_c - 1)/10\}/\{\nu + \varrho_c\} . \tag{20}$$

For buckling in the elastic range, $\varrho_c = 1$ and $\lambda_2 = 4(1 + \nu)/15$, implying increasing load following bifurcation. For $\varrho_c \to \infty$, $\lambda_2 \to -1/2$. A plot of λ_2 as a function of E_t^c/E for $\nu = 1/3$ is shown in Fig. 3.

A matching of (15) and (19) determines ω and c. On the rising branch of the relation between λ and ξ, $\lambda_2 \xi^2$ is negligible compared to $c \bar{\xi}^\omega \xi^{-1/\varkappa}$ for fixed $\lambda < 1$ in the limit as $\bar{\xi} \to 0$. Note that the exponents of ξ in (18) and (19) are the same, and to complete the match we choose

$$c = -e^{-h(1)/\varkappa} \quad \text{and} \quad \omega = 1/\varkappa . \tag{21}$$

A maximum value of λ occurs in the relation (19) between λ and ξ if λ_2 is negative. Using the condition $d\lambda/d\xi = 0$ in connection with (19) gives the asymptotic dependence of the maximum load on $\bar{\xi}$, i.e.,

$$\lambda_{max} = P_{max}/P_c = 1 - \mu \bar{\xi}^{2/(2\varkappa+1)} , \tag{22}$$

where

$$\mu = -\lambda_2(2\varkappa + 1) (-2\varkappa\lambda_2)^{-2\varkappa/(2\varkappa+1)} e^{-2h(1)/(2\varkappa+1)} .$$

The condition ensuring that elastic unloading does not occur prior to attainment of the maximum load for sufficiently small imperfections is found to be simply that

$$E_t^c/E \leq 3/(7 + 4\nu) . \tag{23}$$

Curves of μ and the exponent in (22), $2/(2\varkappa + 1)$, are shown in Fig. 4 as a function of the bifurcation strain ε_c normalized by ε_y for the Ramberg-Osgood tensile stress-strain relation (13). The dashed portions, on which

(23) is not satisfied, apply only to a hypo-elastic (rather than an elastic-plastic) cruciform. But (23) is satisfied on the solid line portions and thus these results are rigorously applicable to J_2 flow theory.

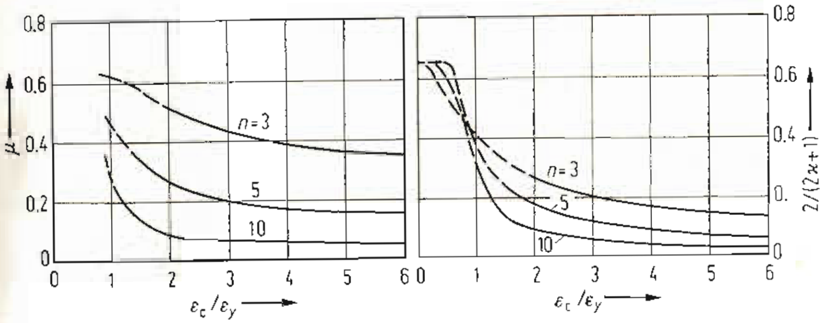

Fig. 4. Parameters for the asymptotic imperfection-sensitivity analysis calculated using the Ramberg-Osgood stress-strain relation and $\nu = 1/3$ (see Eq. (22)).

The essence of the extreme imperfection-sensitivity associated with the simple flow theory is the small exponent, $2/(2\varkappa + 1)$, in (22). As seen from (16) or in Fig. 4 this exponent may easily be as small as 1/10 for buckling in the plastic range, implying that imperfection levels on the order of $\bar{\xi} = 10^{-10}$ will have a non-negligible effect. Essentially this same strong dependence on the imperfection amplitude was found for the cruciform model of Onat and Drucker (1953). In contrast, for elastic buckling $\varkappa = 1$ and the exponent in (22) is always 2/3 for a symmetric bifurcation point (Koiter, 1945) (although (22) does not apply to the cruciform in the elastic range since $\lambda_2 > 0$ and there is no maximum in the vicinity of the bifurcation load). Even for nonlinear elastic solids (e.g., deformation theory) this exponent is 2/3. Thus, the extreme imperfection-sensitivity typified by (22) is really a consequence of the hypo-elastic character of J_2 flow theory for loading.

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