RECENT DEVELOPMENTS IN NONLINEAR FRACTURE MECHANICS

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INTRODUCTION

Within the past few years research into the nonlinear mechanics of fracture has started to have a practical payoff. I would like to use the opportunity of this CANCAM lecture to describe some of these recent developments. I will start by reviewing some of the fundamentals of nonlinear crack problems. Then the initiation of crack growth will be discussed, followed by a discussion of a new approach to the growth and stability analysis of small amounts of crack advance in the presence of large scale plastic yielding. The last part of the lecture deals with the limited success which has been achieved to date in employing a single basic near-tip fracture criterion in the analysis of both initiation and growth under general conditions of yielding. Other recent survey articles which cover some of the same ground reviewed here have been given by Carlsson [1], Paris [2] and Rice [3]. My coverage will emphasize the theoretical side of the subject, but I will try to bring out the vital interaction between theory and experiment which has been so characteristic of much of the development of nonlinear fracture mechanics.

THE J-INTEGRAL AND CRACK-TIP FIELDS

The unifying theoretical idea behind the extension of linear elastic fracture mechanics into the range of large scale plastic yielding is the J-integral introduced for crack problems by Rice [4] in 1968 and, independently, by Cherepanov [5] in Russia.

A small strain, nonlinear elastic (deformation theory of plasticity) material is assumed with strain energy density $W(\underset{\sim}{\varepsilon})$ such that the stress is

$$\sigma_{ij} = \partial W/\partial \varepsilon_{ij} \qquad (1)$$

The proto-type body shown in Fig. 1 is assumed to be in conditions of either plane strain or plane stress. The material is taken to be homogeneous and isotropic. Let P denote the generalized force per unit thickness acting on the body and let Δ be the generalized displacement quantity through which P works. For reasons which will be clear later, a linear spring with compliance (per unit thickness) C_M is placed in series with the cracked body such that the total generalized displacement of the system is

$$\Delta_T = \Delta + C_M P \qquad (2)$$

With PE defined as the potential energy of the system per unit thickness, J is defined as the energy release-rate per unit advance of the crack in its plane (per unit thickness) with Δ_T held fixed, i.e.

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$$J = -\left(\frac{\partial PE}{\partial a}\right)_{\Delta_T} \tag{3}$$

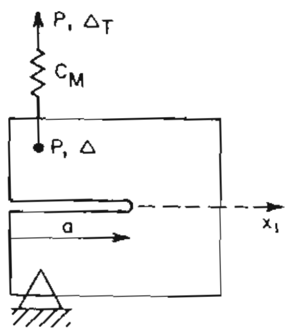

Fig. 1 Cracked body in series with a linear spring

The energy release rate defined above is easily shown to be independent of the compliance of the spring C_M. Since $C_M \to \infty$ corresponds to dead load with P prescribed and $C_M = 0$ corresponds to prescribed Δ, J is the same for these limiting cases, as well as all those in between. With P regarded as a function of Δ and a, (3) reduces to

$$J = -\int_0^{\Delta} \frac{\partial P}{\partial a}(\tilde{\Delta}, a)\, d\tilde{\Delta} \tag{4}$$

Or with Δ as a function of P and a, (3) becomes

$$J = \int_0^{P} \frac{\partial \Delta}{\partial a}(\tilde{P}, a)\, d\tilde{P} \tag{5}$$

These latter expressions are given by Rice [6]. His path-independent line integral expression for J is

$$J = \int_{\Gamma} (W n_1 - \sigma_{ij} n_j u_{i,1})\, ds \tag{6}$$

where Γ is any contour encircling the tip of the crack in a counter-clockwise direction, u_i is the displacement vector, n_i is the outward unit normal to Γ and ds is the length of the line element.

While J is the energy release-rate for the cracked deformation theory body, it has another role which is more pertinent to non-linear fracture mechanics. It can be regarded as the amplitude of the singularity fields at the tip of the crack. As an example, assume that the uniaxial stress-strain curve is represented by

$$\varepsilon/\varepsilon_o \sim \alpha(\sigma/\sigma_o)^n \tag{7}$$

for $\varepsilon \gg \varepsilon_o$, where σ_o is the yield stress and $\varepsilon_o = \sigma_o/E_o$ the yield strain. Furthermore, assume the J_2 deformation theory generalization of (7) to multi-axial states, i.e.

$$\varepsilon_{ij}/\varepsilon_o \sim \tfrac{3}{2}\alpha(\sigma_e/\sigma_o)^{n-1} s_{ij}/\sigma_o \, , \quad \sigma_e^2 = \tfrac{3}{2} s_{ij} s_{ij} \tag{8}$$

where s_{ij} is the deviator stress. Then the asymptotic crack-tip fields are [7, 8]

$$\sigma_{ij} \sim \sigma_o \left(\frac{J}{\alpha \sigma_o \varepsilon_o I_n r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(\theta, n) \tag{9}$$

$$\varepsilon_{ij} \sim \alpha \varepsilon_o \left(\frac{J}{\alpha \sigma_o \varepsilon_o I_n r}\right)^{\frac{n}{n+1}} \tilde{\varepsilon}_{ij}(\theta, n) \tag{10}$$

where r and θ are polar coordinates centered at the tip. The dimensionless θ variations, $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$, depend on the symmetry of the fields with respect to the crack and on whether plane strain or plane stress prevails, as does the normalizing constant I_n.

The separation of the two crack faces varies like $r^{1/(n+1)}$ as $r \to 0$. Defining an effective crack-tip opening displacement δ_t as the separation where the 45° lines intercept the crack faces, as in Fig. 2, gives

$$\delta_t = d(\varepsilon_o, n)\frac{J}{\sigma_o} \tag{11}$$

Values of d have been given by Shih [9] for plane strain and plane stress. In plane strain d ranges from about .8 for $n \to \infty$ to .3 for $n = 3$ with a relatively weak dependence on ε_o ; in plane stress the same variation is from 1.0 to about .4. In the range of low strain hardening d is a fairly strong function of n . Reported results [9, 10] for d for plane strain obtained from finite element calculations for low and zero strain-hardening materials range from .8 to about .5. Although the connection between δ_t and J is not as well established as it should be, the implication of (11) is that δ_t also measures the intensity of the crack-tip fields.

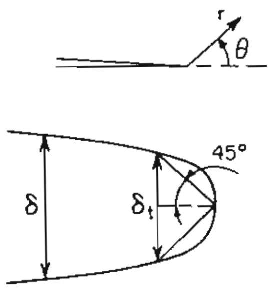

Fig. 2 Crack-tip opening displacement

ZONE OF DOMINANCE OF CRACK-TIP FIELDS AND LIMITATIONS OF SINGLE PARAMETER CRACK-TIP CHARACTERIZATIONS

For a stationary crack subject to a monotonically increased single loading variable, it is expected that plastic loading will not depart radically from proportionality. Thus, the deformation theory solution should be a good approximation to the corresponding solution based on incremental plasticity theory. A number of numerical studies have shown this to be the case. In particular, the line integral representation of J (6) is found to be essentially independent of the path in the plastic zone when calculated using the standard incremental theories. The argument for using a critical value of J or of δ_t to identify the onset of crack propagation, independent of other geometric and loading parameters, assumes that the crack-tip fields (9) and (10) dominate (i.e., are a good approximation to) the behavior over a zone at the tip which surrounds the region of finite strains and fracture processes where (9) and (10) break down. Since the region of finite strains (and also usually the fracture process zone) is on the order of δ_t , the zone of dominance of (9) and (10) must therefore be sufficiently large compared to δ_t.

McMeeking [10] employed a finite element method, based on a finite strain version of J_2 flow theory of plasticity, to study the near-tip behavior in small scale yielding under mode I plane strain conditions. (Small scale yielding is the asymptotic situation where the plastic zone is small compared to the crack length and other relevaut in-plane length quantities. In mode I the fields are symmetric with respect to the line of the crack.) McMeeking found that finite strain effects are important over distances of about 2 or 3 times δ_t for values of the initial yield strain less than .01. For distances from the tip greater than $3\delta_t$ the small strain theory predictions were accurate and the J-integral was essentially independent of the path. We will use R to characterize the size (radius) of the zone of dominance of the crack-tip fields (9) and (10) in the small strain problem. From McMeeking's work one concludes that a necessary condition for using J or δ_t as a single, configuration-independent parameter to characterize the near-tip behavior in plane strain is approximately

$$R > 3\delta_t \tag{12}$$

Very recently there have been several efforts [11, 12] to ascertain the size of the zone of dominance R under large scale yielding conditions where the cracked body has become fully yielded. As background to these studies we recall that when J was first discussed a possible intensity measure for fracture analysis under large scale yielding conditions, McClintock [13] pointed out the following limitation on J (or on any other single parameter such as δ_t). He noted that neither the stress nor the strain fields near the tip of a crack can be configuration-independent in elastic-perfectly plastic bodies under fully yielding conditions. Examples of two plane strain slip line fields with fundamentally different near-tip stress and strain fields are sketched in Fig. 3. The edge-cracked strip in bending develops a high triaxial and normal stress ahead of the crack, similar to that associated with the well-known Prandtl slip-line field. The stress ahead of the crack in the center-cracked tension strip is the plane strain tensile yield stress which is significantly below that attained in the other case. The strain fields are different, as well, with straining concentrated on planes emanating from the tips at 45° to the crack in the center-cracked strip.

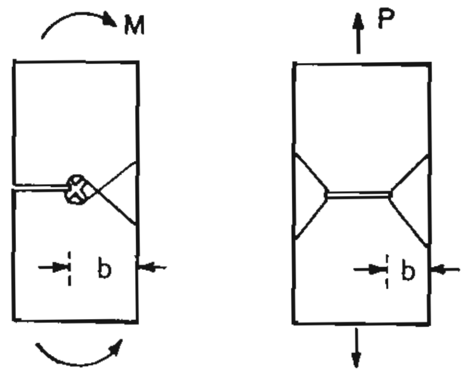

Fig. 3 Fully yielded edge-cracked strip in bending and center-cracked strip in tension

These observations would appear to be at odds with the assertion that the stress and strain fields, (9) and (10), uniquely determine asymptotic conditions at the tip once J is given. That assertion relies on the existence of some strain hardening (i.e., finite n). In the limit of elastic-perfectly plastic behavior $(n \to \infty)$, singular terms not considered become potentially as important as (9) and (10). Put differently, the θ-variations $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are only unique for finite n. This is reflected in the two cases of Fig. 3. In general, some strain hardening is required to justify the use of a single parameter such as J or δ_t to correlate fracture of different cracked configurations under large scale yielding conditions.

A quantitative assessment of the limitations of a single parameter approach as related to strain-hardening and configuration dependence is just beginning to emerge. The edge-cracked strip in bending seems to be reasonably well in hand. The standard compact tension specimen can be regarded as a bend-type configuration for the purposes of this discussion.

McMeeking and Parks [11] employed the same finite strain, finite element procedure referred to earlier. They showed that the near-tip fields of the small scale yielding problem were essentially identical to the near-tip fields in the fully plastic edge-cracked strip in bending at corresponding values of J as long as

$$b > 25J/\sigma_o \tag{13}$$

This condition does not seem to be strongly dependent on the initial yield strain. Of greater importance is the fact that the correspondence held up even without strain hardening. Condition (13) had been suggested earlier and has been indirectly verified experimentally [2, 14]. Since $\delta_t \approx .6J/\sigma_o$ in plane strain for moderate to low strain hardening, (13) states that the uncracked ligament b must satisfy (approximately)

$$b > 40\delta_t \tag{14}$$

Under fully plastic conditions the zone of dominance R discussed earlier is necessarily some fraction of the uncracked ligament b, assuming yielding is confined to the ligament. The functional connection is of the form

$$R = g(n,\varepsilon_o)b \tag{15}$$

The above discussion suggests that for the edge-cracked strip in bending g is not strongly dependent on either n or ε_o. A comparison of (12) and (14), noting (15), gives the estimate $g \cong .07$. The more fundamental expression of the condition for J-dominance, i.e. $R = .07b$ with $R > 3\delta_t$, translates into the better known expressions (14) or (13) when R is eliminated. Work of Shih and German [12] lends additional support to the value $g \cong .07$ in (15). Using a small strain finite element procedure, they compared calculated stress and strain fields in the fully plastic edge-cracked strip in bending with the dominant singnlarity fields (9) and (10) at corresponding values of J. The agreement between the two predictions was reasonably good within a distance R of the tip less than abont $R \cong .07b$ for the two levels of hardening exponent considered, $n = 3$ and $n = 10$.

At the other extreme is the center-cracked plane strain strip in tension. As already discussed, the radius R of the zone of dominance must vanish as $n \to \infty$ since the near-tip fields in the elastic-perfectly plastic limit are inherently different from the corresponding limit of (9) and (10). Studies along the lines of those described above [11, 12] suggest that the counterpart to (13) for the center-cracked strip in tension is (tentatively)

$$b > 200J/\sigma_o \tag{16}$$

for fully yielded conditions with moderately low strain hardening $(n = 10)$. At this level of strain hardening, Eqs. (12), (14) and (15) imply $g \cong .01$. That is, the singnlarity fields dominate a region of only about one percent of the uncracked ligament when $n = 10$. Condition (16) places a severe limitation on the applicability of a single parameter characterization for fully plastic center-cracked tensile configurations, as will be discussed further below.

INITIATION OF CRACK GROWTH

The potential of J for extending engineering fracture mechanics into the large scale yielding range was appreciated immediately after it was first introduced [6, 15, 16]. But it was the innovative experimental work of Begley and Landes [15, 14] that established the feasibility of using J and that provided the initial impetus for much of the work of the last five years, including some of that just described in the previous section.

Begley and Landes showed that it was possible to determine the fracture toughness under large scale yielding conditions using various types of test specimens. With K_{IC} denoting the fracture toughness (i.e., the stress intensity factor at initiation as determined by a plane strain small scale yielding test), the corresponding value of J at initiation should be [6]

$$J_{IC} = (1-\nu^2)K_{IC}^2/E \qquad (17)$$

where ν is Poisson's ratio. The test series of Begley and Landes verified this connection. Subsequent work in a number of laboratories has refined and improved upon these first studies (see the discussion in [2] and various references in [17]). There now appears to be a consensus that bend-type test specimens can be employed under large scale yielding conditions to determine fracture toughness. For testing purposes alone this is a major accomplishment since it eliminates the necessity of employing the huge test specimens required in small scale yielding testing of relatively high toughness metals with intermediate yield strength.

There has been some success in relating measured values of the crack tip opening displacement at initiation, δ_t^c, to J_{IC} through (11) -- see [18] and the discussion in [10]. A difficulty involved in making this comparison is the apparent relatively strong dependence of d in (11) on strain hardening. Typical values of δ_t^c range from less than .01 mm for high strength low toughness metals to several tenths of a millimeter for intermediate strength high toughness metals. For a bend-type specimen where $\delta_t^c = .2$ mm, say, (14) implies that the uncracked ligament must be at least 8 mm. For a center-cracked tension specimen of the same material, (16) requires a ligament about eight times as large. If either specimen were sized such that initiation occurred under small scale yielding conditions (i.e., under valid K_{IC} testing conditions) a ligament of at least about 250 mm would be required. The advantage of the fully plastic bend-type specimen is obvious!

Shortly after Begley and Landes's preliminary work was finished, a very useful formula for J for deeply edge-cracked bend-type specimens, such as that in Fig. 4, was obtained by Rice, Paris and Merkle [19]. For a deeply-cracked specimen they found a rigorous formula for J in terms of load and displacement quantities measurable in a test. With Δ_{nc} denoting the load-point deflection of the specimen in Fig. 4 without a crack $(a = 0)$ at load P, let

$$\Delta_c = \Delta - \Delta_{nc} \tag{18}$$

where Δ is the total deflection in the presence of the crack. (A spring has been inserted in series with the specimen in anticipation of the discussion of stability given later. The spring does not alter the relation between J and P or Δ as previously discussed.) The result of [19] is

$$J = \frac{2}{b} \int_0^{\Delta_c} P d\Delta_c \tag{19}$$

The existence of simple formulas such as (19) tend to favor the use of J as a crack-tip parameter over other potential candidates for which analogous simple formulas are not available.

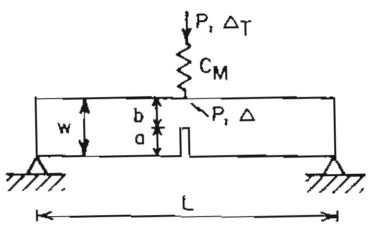

Fig. 4 Three-point bend specimen loaded in series with a linear spring

J-CONTROLLED CRACK GROWTH

A relatively simple means of analyzing limited amounts of stable, quasi-static crack growth has been proposed by Paris, et al. [20] and Garwood, et al. [21] as a result of experimental findings which will now be described. In conducting tests to determine the critical value of J associated with initiation (J_{IC} in plane strain), experimentalists [17] used (19), or a formula like it, to measure the relation between J and crack advance Δa for small amounts of growth. A representative J-resistance curve, $J_R(\Delta a)$, for a typical intermediate strength, high toughness steel is depicted in Fig. 5. A small apparent growth due to crack-tip blunting prior to initiation has been subtracted off in Fig. 5. For such steels the advance D needed to double J above J_{IC} is typically less than a few millimeters. These curves were used to extrapolate back to the initiation value J_{IC}. But it became evident that under certain restrictive conditions, called J-controlled growth, the J-resistance curve could be regarded as a material characterizing curve which was independent of geometry -- see, for example, the discussion in Rice's review [3].

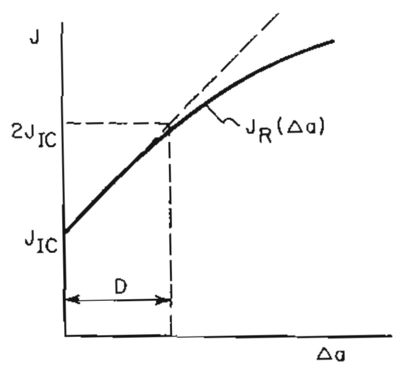

Fig. 5 J-resistance curve

The J-integral is based on the deformation theory of plasticity which cannot model effects of elastic unloading or highly nonproportional plastic loading. Thus the argument for J-controlled growth relies on the conditions that the region of elastic unloading and non-proportional loading, which is a region on the order Δa in radius, be embedded within, and controlled by, the singularity fields (9) and (10), as depicted in Fig. 6. The two conditions for J-controlled growth [22] are

$$\Delta a \ll R \tag{20}$$

and

$$D \ll R \tag{21}$$

where D shown in Fig. 5 is

$$D = J_{IC}/(dJ_R/da)_c \tag{22}$$

The first condition is apparent. The second, (21), ensures that J increases sufficiently rapidly as the crack advances such that deformation theory is a good approximation within an annular region inside R, as shown in Fig. 6.

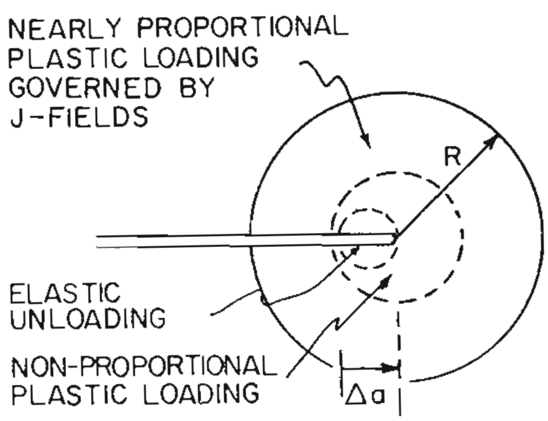

Fig. 6 Schematic of near-tip conditions for J-controlled growth

For fully yielded configurations, such as those in Fig. 3, in which yielding is confined to an uncracked ligament, R is related to b by (15). In such cases the condition (21) can be stated nondimensionally as

$$\omega \equiv \frac{b}{J_{IC}}\left(\frac{dJ_R}{da}\right)_c \gg 1 \tag{23}$$

Judging from (20) and the discussion on the size of R, the amount of crack growth possible under J-controlled conditions is small. But for many of the intermediate strength alloys relatively large increases of J above J_{IC} are nevertheless possible under J-controlled conditions since D is very small. Efforts to refine the conditions (20) and (23) have only recently been made. For edge-cracked bend-type configurations Shih and Dean [23] have performed numerical calculations which have led to the tentative proposal that (20) and (23) should be (approximately)

$$\Delta a < 0.06b \tag{24}$$

$$\omega > 10 \tag{25}$$

For center-cracked tension configurations it is expected that these conditions will be much more restrictive, as has already been indicated by a few tests [24].

STABILITY OF J-CONTROLLED CRACK GROWTH

Paris and coworkers [20] have proposed a stability analysis based on the J-resistance curve which is similar in spirit to the resistance curve analysis of linear elastic fracture mechanics. If small amounts of crack growth are to be tolerated, with the attendant relatively large increase in J, it becomes essential to be certain that such growth is stable.

To illustrate the approach of [20] and [22] consider the system in Fig. 4. We will assume that the total load-point displacement, Δ_T, is imposed. The compliance of the linear spring C_M can be regarded as the compliance of a test machine or as the compliance of the surrounding structure transmitting load to the cracked element. Assume the crack had advanced an amount Δa and is currently loaded for further possible advance, i.e.,

$$J = J_R(\Delta a) \tag{26}$$

Stability at this state, with Δ_T prescribed, requires

$$\left(\frac{\partial J}{\partial a}\right)_{\Delta_T} < \frac{dJ_R}{da} \tag{27}$$

which simply ensures that any small "accidental" advance of the crack can be sustained by the tearing resistance of the material. Paris, et al. [20] introduced a nondimensional tearing force and tearing resistance as

$$T = \frac{E}{\sigma_o^2}\left(\frac{\partial J}{\partial a}\right)_{\Delta_T} \quad \text{and} \quad T_R = \frac{E}{\sigma_o^2}\frac{dJ_R}{da} \tag{28}$$

so that the stability condition becomes

$$T < T_R \tag{29}$$

A relatively simple formula for T can be obtained for the system in Fig. 4 which is exact in the deeply-cracked limit [22]. For the case of a fully yielded, elastic-perfectly plastic cracked beam that result is

$$T = \frac{4EP^2}{\sigma_o^2 b^2}(C_{nc} + C_M) - \frac{EJ}{\sigma_o^2 b} \tag{30}$$

Here P is the limit load of the cracked beam and C_{nc} is the elastic compliance of the uncracked beam. As expected, the system compliance has a significant influence on the stability through T, whereas it does not affect J. Of course, under dead load ($C_M \to \infty$) the fully yielded, perfectly plastic beam is unstable.

Paris et al. [25] conducted a test series in which a spring of adjustable compliance was inserted in series with just such a deeply-cracked bend specimen. By testing a sequence of identical specimens in series with springs of differing compliance, they were able to check the validity of the stability condition (29). Their material had a tearing resistance at initiation of $T_R \cong 36$, $D \cong 1.2$ mm, and their specimens met (25) with $\omega = 15$. Their tests did reveal a transition from stability to instability at T-values very close to $T_R = 36$.

A table of values of T_R for a wide variety of steels at various temperatures has been compiled in [20]. For high strength, low toughness alloys T_R is often as small as or below unity. On the other hand, many of the intermediate strength steels have T_R-values which exceed 30, some being as large as 200. In many circumstances the T-values will be far smaller so that small amounts of crack growth can be safely sustained. As an illustration, consider a finite crack in an infinite body whose material behaves in simple tension as $\varepsilon/\varepsilon_o = (\sigma/\sigma_o)^n$. If ε^∞ is the remote strain due to a remote uniaxial stress normal to the crack face, then

$$T = h(n)(\varepsilon^\infty/\varepsilon_o)^{\frac{n+1}{n}} \tag{31}$$

where $h(n)$ is roughly 3 for n less than 10 [26]. Only when the overall strain ε^∞ exceeds approximately 10 times the effective yield strain will T exceed 30.

An approach with common features to that described above is also being developed by Garwood, Robinson and Turner [25 and unpublished work].

PROGRESS TOWARDS A UNIFIED NEAR-TIP FRACTURE CRITERION FOR INITIATION AND GROWTH

The approach described above is inherently empirical in that J_{IC} and the resistance curve must be obtained experimentally for each material for every set of conditions. In addition the range of potential application, although important, is quite restricted, particularly in that it is limited to relatively small amounts of crack growth. Thus the basic problem of identifying a near-tip fracture criterion based on the fracture processes very close to the tip is of considerable practical importance as well as fundamental scientific interest. The problem is far from being "solved" but some significant first attempts have been made. Probably the most ambitious attempt to understand the mechanics of ductile crack initiation is that of Rice and Johnson [27] who carried out an approximate analysis of the linking-up process of a void with the crack-tip. Other approaches, one level removed from dealing with the micro-mechanical fracture processes, have been proposed for combined initiation and growth [28, 29, 30].

We will make use of McClintock's [28, 29] early results in anti-plane shear (mode III) to indicate the source of stable crack growth and to predict initiation and growth in terms of near-tip fracture criteria of the type used in [28, 29]. With γ denoting the total shear strain ahead of the crack, the condition for growth is a critical strain criterion

$$\gamma = \gamma_c \quad \text{at} \quad r = r_c \qquad (32)$$

where r_c is a material length characterizing the fracture process zone.

Small scale yielding is assumed. The material is elastic-perfectly plastic with initial yield stress in shear as τ_o and yield strain as $\gamma_o = \tau_o/G$ where G is the elastic shear modulus. A Mises yield condition is used. Prior to initiation the strain ahead of the crack in the plastic zone ($r \le r_p$, where r_p is the plastic zone extent ahead of the crack) is

$$\gamma = \gamma_o \frac{r_p}{r} \quad \text{where} \quad r_p = \frac{2}{\pi} \frac{J}{\tau_o \gamma_o} \qquad (33)$$

Imposition of (32) using (33) gives the value of J at initiation

$$J_c = \frac{\pi}{2} r_c \gamma_c \tau_o \quad \text{and} \quad r_p^c = \beta r_c \qquad (34)$$

where

$$\beta \equiv \gamma_c / \gamma_o \qquad (35)$$

Next consider steady-state growth where the crack has grown sufficiently far ahead such that it is able to progress at constant J . In this case the strain ahead of the crack is [28, 29, 6]

$$\gamma = \gamma_o [1 + \ln(r_p/r) + \tfrac{1}{2}\ln^2(r_p/r)] \qquad (36)$$

Chitaley and McClintock [31] have shown that r_p is still given by (33), to a very good approximation. A comparison of (36) with (33) shows that the strain near the tip in a growing crack at steady-state is much less than the corresponding strain the same distance ahead of the crack in the stationary problem at the same value of J. The significantly weaker singularity in (36) is a consequence of the highly nonproportional plastic flow which occurs ahead of the crack in the growing crack. It is the substantial resistance of plastic flow to nonproportional stressing which is the primary source of stable crack growth. Invoking the growth criterion (32) using (36), together with (33) for r_p in terms of J, gives the value of J necessary to drive the crack under steady-state conditions

$$J_{ss} = \frac{\pi}{2} r_c \gamma_o \tau_o \exp[\sqrt{2\beta - 1} - 1] \qquad (37)$$

The ratio of J_{ss} to J_c is

$$\frac{J_{ss}}{J_c} = \frac{1}{\beta} \exp[\sqrt{2\beta - 1} - 1] \qquad (38)$$

Large values of $\beta = \gamma_c/\gamma_o$ imply substantial potential stable crack growth. Approximate calculations of the full J-resistance curve in mode III based on the criterion (32) have been reported iu [29, 6]. Here we will be content to report the result for the initial slope of the J_R-curve following initiation which has been obtained using McClintock's analysis for the transient case. In non-dimensional form that result is

$$T_R = \frac{G}{\tau_o^2}\left(\frac{dJ_R}{da}\right)_c = \frac{\pi}{2}(\beta - 1 - \ln\beta) \qquad (39)$$

A "perfectly brittle" material with $\beta = 1$ corresponds to $T_R = 0$, while for large β $T_R \cong \pi\beta/2$. The material-based length quantity, D, is

$$D \equiv \frac{J_c}{(dJ_R/da)_c} = r_c \frac{\beta}{\beta - 1 - \ln\beta} \qquad (40)$$

It is interesting to note, that for β larger than about 10. D is essentially the characteristic length associated with the fracture process zone, r_c.

The model suggests certain implications relating macroscopic fracture resistance to features of the fracture process zone. In particular, note that the ratio, J_{ss}/J_c, in (38) and the nondimensional tearing modulus T_R in (39) depend only on $\beta = \gamma_c/\gamma_o$. Furthermore, for large β, J_{ss}/J_c increases exponentially while T_R increases linearly in β. Note that for $\beta = 60$, $T_R \cong 100$ and $J_{ss}/J_c \cong 1000$. For larger values of β the small strain assumptions will certainly be violated for typical values of γ_o at the point where $r = r_c$. But the model does suggest the source of the large values of T_R which are observed. The very large values of J_{ss}/J_c for large β result from the considerable resistance an elastic-plastic material offers to nonproportional straining, as has already been noted. This effect is undoubtedly overestimated by the simple smooth yield surface of Mises (and Tresca in mode III) used in the present analysis. In this sense the values of J_{ss}/J_c for large β may be considerably in excess of observable values.

Rice and Sorensen [30] have considered the more difficult mode I, plane strain problem in small scale yielding. Qualitatively the findings are similar to mode III and several features of the analysis are closely analogous. While the criterion (32) is sensible in mode III, a critical strain condition cannot be taken to be met ahead of the crack in plane strain mode I since the strains are most intense above and below the tip in the small strain solution. Instead, Rice and Sorensen used an alternative criterion which is essentially an integration of the near-tip strains. They require the crack opening displacement to reach a critical value at some fixed small distance back behind the tip. By making contact with numerical results they are able to obtain an approximate integration of the equations relating the crack opening displacement, the crack advance and J. Resistance curves are determined. Large tearing resistance is found, typical of observed values, with realistic choices for the near-tip fracture criterion.

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