Hugo Ernst,[1] *P. C. Paris,*[1] *Mark Rossow,*[1] *and J. W. Hutchinson*[2]

Analysis of Load-Displacement Relationship to Determine J-R Curve and Tearing Instability Material Properties

REFERENCE: Ernst, Hugo, Paris, P. C., Rossow, Mark, and Hutchinson, J. W., **"Analysis of Load-Displacement Relationships to Determine J-R Curve and Tearing Instability Material Properties,"** *Fracture Mechanics, ASTM STP 677*, C. W. Smith, Ed., American Society for Testing and Materials, 1979, pp. 581–599.

ABSTRACT: Exact methods, based on dimensional analysis of the dependence of the load on crack length and displacement using deformation theory of plasticity, are used to obtain J from a single load-displacement record for different configurations. The methods also permit the evaluation of the crack length increment Δa, and hence, complete J-R curves can be constructed. Formulae for T_{mat} and T_{app} are presented as well.

KEY WORDS: crack propagation, fatigue (materials), mechanical properties, load-displacement records, methods of analysis, J-integral, J-R curves, tearing modulus (T)

This analysis is based on recent work of Hutchinson and Paris et al [*1,2*],[3] which suggested that load-displacement records for pure bending could be analyzed to determine J-R curve and instability related material properties. However, here the analysis will be generalized and shown to be applicable to all configurations (2-D) and especially useful for typical test configurations such as compact, three-point bend, center-cracked, etc., configurations.

Indeed, the analysis to be presented is "exact," from an analytical viewpoint, and is based on dimensional considerations in the spirit of Rice-Paris-Merkle analysis [*3*] based on Rice's J-integral concepts [*4*].

[1] Research assistant, professor of mechanics, and associate professor of civil engineering, respectively, Washington University, St. Louis, Mo. 63130.
[2] Professor of engineering, Harvard University, Cambridge, Mass. 02138.
[3] The italic numbers in brackets refer to the list of references appended to this paper.

The strict deformation theory of plasticity interpretation of J which will be used here has been shown to be "exact" under the size restrictions described by Hutchinson and Paris [1,5] for determining J-R curves and related material properties.

Finally, the analysis will lead to the following useful results:

1. Correct (exact) methods for computing J from load-displacement records with crack growth present will be developed.
2. Methods for determining crack length change, Δa, from load-displacement records (without further instrumentation) will be developed.
3. From the results of (1) and (2), it is possible to construct J-R curves from load-displacement records alone.
4. Moreover, a material's tearing instability properties, that is, the tearing modulus, T_{mat}, may be determined from load-displacement records (without further instrumentation).
5. Similarly, a system's tendency for tearing instability, T_{appl}, may be found from further analysis and load-displacement information from the system.

Therefore, the methods to be developed here are anticipated to have extensive applicability to J-R curve methods of determining material properties controlling crack extension and stability. Moreover, since only load-displacement records are required to determine desired properties, the methods are ideally suited to certain special testing problems, such as hot cell testing of irradiated materials or dynamic testing, or both (both of which present especially difficult problems if additional instrumentation is required).

Development of a Convenient Form of the Rice J-Integral

The original familiar form of the J-integral is [4]

$$J = \oint_{\Gamma} W dy - T_i \frac{\partial u_i}{\partial x} ds \tag{1}$$

which, based on deformation theory, is path independent, when integrating around a crack tip, and can be represented equally well by the alternate forms [4]

$$J = \int_0^P \left(\frac{\partial \Delta}{\partial a}\right)_P dP = - \int_0^\Delta \left(\frac{\partial P}{\partial a}\right)_\Delta d\Delta \tag{2}$$

where, Δ is the work producing component of load point displacement for the load, P. It is noted that Eq 2 appropriately implies either $\Delta = \Delta(a, P)$ or $P = P(a, \Delta)$ as functional relationships between the variables including the crack length, a, as the third variable.

In the analysis to follow here, it is convenient to subdivide J into elastic and plastic parts, J_{eL} and J_{pL}. This is done by noting that for actual elastic-plastic situations the displacement, Δ, may always be divided into its linear-elastic and plastic parts. That is

$$\Delta = \Delta_{eL} + \Delta_{pL} \tag{3}$$

Then the first form in Eq 2 may be written

$$J = \int_0^P \left(\frac{\partial \Delta_{eL}}{\partial a}\right)_P dP + \int_0^P \left(\frac{\partial \Delta_{pL}}{\partial a}\right)_P dP \tag{4}$$

Now the first term in Eq 4 is the linear-elastic component, J_{eL}, of J, which is also the Griffith, G. That is

$$J_{eL} = \int_0^P \left(\frac{\partial \Delta_{eL}}{\partial a}\right)_P dP = G(P, a) \tag{5}$$

Moreover, the second term in Eq 4, J_{pL}, may be reinterpreted by referring to Fig. 1. This figure shows schematically load, P, versus plastic displacement, Δ_{pL}, curves for crack sizes a and $a + da$. The area between curves is noted to be $J_{pL}da$ by integrating over elements of the area by

$$J_{pL}(da) = \int_0^P \left(\frac{\partial \Delta_{pL}}{\partial a}\right)_P dP(da) \tag{6}$$

but using the alternate element it is observed that two forms are equally appropriate, that is,

$$J_{pL} = \int_0^P \left(\frac{\partial \Delta_{pL}}{\partial a}\right)_P dP = - \int_0^{\Delta_{pL}} \left(\frac{\partial P}{\partial a}\right)_{\Delta_{pL}} d\Delta_{pL} \tag{7}$$

Now substituting Eqs 5 and 7 into Eq 4, the result is

$$J = J_{eL} + J_{pL}$$

or

$$J = G(P, a) - \int_0^{\Delta_{pL}} \left(\frac{\partial P}{\partial a}\right)_{\Delta_{pL}} d\Delta_{pL} \tag{8}$$

This form, Eq 8, gives a convenient method for computing J without any

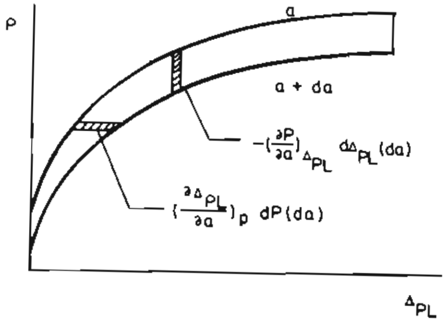

FIG. 1—*Load-plastic displacement records for two neighboring crack lengths.*

loss in analytical precision compared to the original forms, Eqs 1 or 2. Without ambiguity G is always to be computed using linear-elastic fracture mechanics formulas using the actual load, P, and crack length, a (without plastic zone correction). It remains to be shown here that the second term of Eq 8 may be evaluated appropriately.

Dimensional Analysis of Relationships Between Load, Crack Length, and Plastic Displacement

Normally the plasticity in a cracked member or specimen is confined to the remaining ligament at the cracked section. In order to avoid certain ambiguities requiring some modifications in the analysis, plasticity confined to the ligament region is assumed here. Under this condition, dimensional analysis leads to the following form for the relationship between load, P, crack length, a, and plastic displacement

$$\frac{\Delta_{pL}}{W} = f\left(\frac{P}{W}, \frac{a}{W}, \frac{L}{W}, \frac{B}{W}, \text{ etc}\right) \tag{9}$$

where P is taken as load per unit thickness, W is a typical characteristic (nonvarying) dimension such as width, and L, B, etc., are other characteristic dimensions. In this form a basic argument proposed by Rice [3] is that load can only appear as P/W with units, force per length squared, since f depends only on stress-strain properties having like units or nondimensional units. Furthermore, this form is correct for arbitrary monotonic stress-strain properties so that the analysis to follow will be "exact" for all stress-strain curves.

Inverting this function then

$$\frac{P}{W} = F\left(\frac{\Delta_{pL}}{W}, \frac{a}{W}, \frac{L}{W}, \frac{B}{W}, \text{ etc.}\right) \tag{10}$$

Now more appropriate forms for analysis can be modified by defining the width of the remaining uncracked ligament, b (from other dimensions, such as W-a). Then either b/W or $(b/W)^2$ may be factored out of F ( ) to give the special forms of Eq 10

$$P = \frac{b^2}{W} F_1\left(\frac{\Delta_{pL}}{W}, \frac{a}{W}, \frac{L}{W}, \frac{B}{W}, \text{ etc.}\right) \tag{11a}$$

or

$$P = b\ F_2\ \left(\frac{\Delta_{pL}}{W}, \frac{a}{W}, \frac{L}{W}, \frac{B}{W}, \text{ etc.}\right) \tag{11b}$$

Indeed, an additional form which is equally appropriate dimensionally and similar to Eq (11b) is

$$P = bF_3\ \left(\frac{\Delta_{pL}}{b}, \frac{a}{W}, \frac{L}{W}, \frac{B}{W}, \text{ etc.}\right) \tag{11c}$$

Now it will be shown that these forms, in combination with Eq 8, will lead to significant results. Indeed, all of the results in Ref *3* as well as important new results will follow.

Analysis to Determine J Using First Special Functional Form, F_1

The first functional form, Eq 11a may be substituted into Eq 8, noting that $db = -da$, to give

$$J = G + \frac{2b}{W}\int_0^{\Delta_{pL}} F_1(\ )d\Delta_{pL} - \frac{b^2}{W^2}\int_0^{\Delta_{pL}} \frac{\partial F_1}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL} \tag{12}$$

Resubstituting for $F_1(\ )$ from Eq 11a into the first integral of Eq 12 leads to

$$J = G + \underbrace{\frac{2}{b}\int_0^{\Delta_{pL}} P\, d\Delta_{pL}}_{\text{Rice's term}} - \underbrace{\frac{b^2}{W^2}\int_0^{\Delta_{pL}} \frac{\partial F_1}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL}}_{\text{Merkle-Corten term}} \tag{13}$$

It is noted that Eq 13 is an exact form for computing J from any load displacement record but prior to the initiation of crack growth (but it could be used to compute J after crack growth provided that the final value of the crack length is used, and the load displacement record for that crack length with no crack extension is known.) Indeed, it is observed that the first integral in Eq 13 is the usual Rice [3] area under the load-displacement curve (work) with the normal $2/b$ coefficient and the second integral is a correction of the Merkle-Corten [6] type, which is "exact" in Eq 13, however.

Now Eq 13 may be used to evaluate J for any configuration, as will be discussed later. It is best applied when the correction term, the final integral, is small compared to the others, which in this case will be the situation where the uncracked ligament is subject principally to bending.

In developing J-R curves it is appropriate to extend this analysis to the situation where crack growth has commenced. This is most clearly accomplished by taking the differential of Eq 12, noting that the integral terms are functions of the independent variables, the crack size, a or b, and the plastic displacement, Δ_{pL}. The differential of Eq 12 is thus

$$dJ = dG + \left[\frac{2b}{W}F_1 - \frac{b^2}{W^2}\frac{\partial F_1}{\partial\left(\frac{a}{W}\right)}\right] d\Delta_{pL} + \left[-\frac{2}{W}\int_0^{\Delta_{pL}} F_1(\ )\,d\Delta_{pL} + \frac{4b}{W^2}\int_0^{\Delta_{pL}}\frac{\partial F_1}{\partial\left(\frac{a}{W}\right)}d\Delta_{pL} - \frac{b^2}{W^3}\int_0^{\Delta_{pL}}\frac{\partial^2 F_1}{\partial\left(\frac{a}{W}\right)^2}d\Delta_{pL}\right] da \quad (14)$$

Now it is convenient for integration of Eq 14 to first define $\bar{J}_{pL}$ as the last two terms in Eq 12, that is, it is J_{pL} computed as if no crack growth occurred, or

$$\bar{J}_{pL} = \frac{2b}{W}\int_0^{\Delta_{pL}} F_1 d\Delta_{pL} - \frac{b^2}{W^2}\int_0^{\Delta_{pL}}\frac{\partial F_1}{\partial\left(\frac{a}{W}\right)}d\Delta_{pL} \quad (15)$$

The reintegration of Eq 14 with crack growth from the initial crack size, a_0, to a size, a, and plastic displacement 0 to Δ_{pL} is appropriate over any path, which gives

$$J = G + \bar{J}_{pL}\Big|_{b=b_0} + \int_{a_0}^{a}\frac{1}{b}\left[-\bar{J}_{pL} + \frac{3b^2}{W^2}\int_0^{\Delta_{pL}}\frac{\partial F_1}{\partial\left(\frac{a}{W}\right)}d\Delta_{pL} - \frac{b^3}{W^3}\int_0^{\Delta_{pL}}\frac{\partial^2 F_1}{\partial\left(\frac{a}{W}\right)^2}d\Delta_{pL}\right] da \quad (16)$$

This result, Eq 16 in combination with Eq 15, defines an exact method of computing J in any configuration. Applied in a general way, it could be very complicated to compute in terms involving F_1 in Eqs 15 and 16. However, in applications to be cited here, the cases will be restricted to situations where the second term in Eq 15 is small compared to the first and in such cases it will be possible to ignore the terms in Eq 16 explicitly containing F_1.

Determination of Crack Length Change and T Using the First Special Functional Form

Equation 11a, the first form, expresses the load, P, as a function of crack length a or b and plastic displacement, Δ_{pL}, and other fixed quantities. Forming the differential of P, it is

$$dP = \frac{b^2}{W^2}\frac{\partial F_1}{\partial\left(\frac{\Delta_{pL}}{W}\right)}d\Delta_{pL} + \left[\frac{b^2}{W^2}\frac{\partial F_1}{\partial\left(\frac{a}{W}\right)} - \frac{2b}{W}F_1\right] da \quad (17)$$

Solving this expression for the crack length change, da, it is

$$da = \frac{\dfrac{b^2}{W^2}\dfrac{\partial F_1}{\partial\left(\frac{\Delta_{pL}}{W}\right)}d\Delta_{pL} - dP}{\dfrac{2b}{W}F_1 - \dfrac{b^2}{W^2}\dfrac{\partial F_1}{\partial\left(\frac{a}{W}\right)}} \quad (18)$$

Assuming that F_1 can be found and the initial crack size, b, is known, then the crack length change can be found during an increment along a load-displacement record, dP and $d\Delta_{pL}$.

During such an increment, dJ can also be found using Eq 14. The results from Eqs 14 and 18 are substituted into the definition of the tearing modulus, that is

$$T_{mat} = \frac{dJ}{da}\frac{E}{\sigma_0^2} \tag{19}$$

It is seen that the tearing modulus of the material, T_{mat}, can also be found for an increment of the load displacement record.

An Example—the Case of Pure Bending of a Small Remaining Ligament

For the case of a half plane with a semi-infinite crack approaching perpendicular to the free edge, leaving a remaining ligament of size, b, the load, P, may be thought of as being removed from the ligament a large distance, W, to form a pure moment, M, where

$$M = PW \tag{20}$$

Now P is regarded as tending to zero while W tends to infinity, so that M is a finite moment. Moreover, the moment will then do work through the plastic angle,

$$\theta_{pL} = \frac{\Delta_{pL}}{W} \tag{21}$$

These results, Eqs 20 and 21, may be substituted into Eq 11a leading to the form

$$M = b^2F_1(\theta_{pL}) \tag{22}$$

where other variables disappear compared to W. Equation 13 then becomes

$$J = G + \frac{2}{b}\int_0^{\theta_{pL}} Md\theta_{pL} = \frac{2}{b}\int_0^{\theta} Md\theta \tag{23}$$

since $G = \frac{2}{b}\int_0^{\theta_{eL}} Md\theta_{eL}$ (for the semi-infinite case where $\theta = \theta_{eL} + \theta_{pL} = \theta_{crack}$) which is the familiar Rice [3] pure bending result. Moreover, Eq 14 becomes

$$dJ = dG + [2b\, F_1 + 0]\, d\theta_{pL} + \left[-\frac{2}{b^2}\int_0^{\theta_{pL}} F_1 d\theta_{pL} + 0\right] da$$

or

$$dJ = 2bF_1d\theta - \frac{J}{b}da \tag{24}$$

Reintegrating to give a result analogous to Eq 16 leads to

$$J = 2b_0\int_0^{\theta}\frac{M}{b^2}\, d\theta - \int_{a_0}^{a}\frac{J}{b}da \tag{25}$$

The analogue of Eq 18 becomes

$$da = \frac{b\,\dfrac{\partial F_1}{\partial\theta}d\theta - \dfrac{dM}{b}}{2F_1} \tag{26}$$

and hence Eqs 24 and 26 can be substituted into Eq 19 to give

$$T_{mat} = \frac{E}{\sigma_0^2}\frac{dJ}{da} = \frac{E}{\sigma_0^2}\left[\frac{4b^2F_1^2}{b^2\dfrac{\partial F_1}{\partial\theta} - \dfrac{dM}{d\theta}} - \frac{J}{b}\right] \tag{27}$$

Therefore, given F_1, it is noted that Eq 26 can be used to compute crack length changes increment by increment along a load-displacement (M versus θ) record. Then Eq 25 may be used to compute J and Eq 27 may be used to obtain T_{mat} at any point on the record. These results for pure bending were obtained in Refs *1* and *2* and in both, a simple experimental method of determining F_1 was discussed. Of course F_1 also could be determined by analytical methods, such as finite element method, using the stress-strain curve for the material to which the analysis here is applied. It is emphasized that this analysis, Eq 24 through 27, is exact, independent of material property assumptions for this example case of pure bending.

This example, pure bending, has been given because it produces the simplest results and is most likely to be familiar to the reader. However, more important is the application of these procedures to configurations which will be used for testing or the analysis of structural cracking problems. In such applications for any configuration, any one of the special forms, Eqs 11a, 11b, or 11c may be used. However, careful choice of the form, for most configurations, will lead to much more practical results. Moreover, the proper procedures for applying the results to load-displacement record analysis bears further comment here.

Additional Example—the Compact Specimen

For a compact specimen which is deeply cracked, say $a/W \geq 0.8$, the remaining ligament, $b(= W - a)$, is subjected to substantially pure bending with other planar dimensions, a, W, H, etc., being very large compared to b. In this case the analysis of pure bending in the preceding section may be applied with reasonable accuracy. However, in such applications J must be computed using analysis such as Eq 25, where the second integral is required to correct for the effects of crack growth on J. This correction has not been used throughout the previous literature except for Refs *1* and *2*. Indeed, the reader is cautioned that J-R curves in the literature are only approximately correct and that their slopes, dJ/da, for the crack growth portion may be in error by 20 to 30 percent due to neglect of this correction term in previous work. Thus, it is clear that more careful analysis is warranted in even the simplest cases, especially those used in testing where material properties so evaluated will reside permanently in the published literature. It is relevant then to proceed with a general analysis of the compact specimen for all a/W values and neglect terms only where they are justifiably small compared to others.

In general the remaining ligament of a compact specimen is subjected mostly to bending but also to a moderate axial force which cannot be neglected entirely. For such cases where bending is a dominant factor, the use of the first special functional form, F_1, as defined by Eq 11*a* is most appropriate. This is because in applying the analysis using that form, Eqs 12 through 19, the terms involving derivatives with respect to a/W of F_1 will be weak, that is, considerably smaller than the main terms. It will be noted here that some of these terms may then be neglected justifiably to simplify the analysis.

In reviewing the analysis, Eqs 12 through 19 will be repeated with appropriate modifications for the compact specimen. First for the determination of J prior to crack growth, Eqs 13 and 15 may be used, hence

$$J = G + \bar{J}_{pL}$$

where

$$\bar{J}_{pL} = \frac{2}{b_0}\int_0^{\Delta_{pL}} P d\Delta_{pL} - \frac{b_0^2}{W^2}\int_0^{\Delta_{pL}} \frac{\partial F_1}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL} \tag{28}$$

(for $\Delta a = 0$)

Now the second term in Eq 28 involving $\partial F_1/\partial(a/W)$ is as noted previously a Merkle-Corten type correction *[6]* which is known to be smaller than other terms but which disappears only at very high a/W values. Now this term cannot be neglected in Eq 28. However, if the computation of J with growing cracks is considered using Eq 16, then note that the integral term in Eq 16 is small compared to other terms. Within this integral, the terms in the integrand explicitly involving the derivations of F_1 are small compared to $\bar{J}_{pL}$, thus their effect on the overall J can be neglected. That is, *with small amounts of crack growth present* compared to the ligament size, b, Eq 16 may be simplified for compact specimens to give

$$J = G + \bar{J}_{pL} - \int_{a_0}^{a} \frac{\bar{J}_{pL}}{b}\, da$$

where (as before)

$$\bar{J}_{pL} = \frac{2b_0}{W}\int_0^{\Delta_{pL}} F_1 d\Delta_{pL} - \frac{b_0^2}{W^2}\int_0^{\Delta_{pL}} \frac{\partial F_1}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL} \tag{29}$$

(for all small Δa, $0 \leq \Delta a << b$)

Now the limitation of Eq 29 to small Δa is no loss in generality, since J-controlled crack growth is restricted to small crack length changes *[1]*. That is, for large amounts of crack growth, J-controlled crack-tip fields disappear so that J analysis itself tends to become inappropriate.

Therefore, Eq 29 is a general form which may be used to compute J accurately in compact specimens. It is noted that two types of "correction terms" exist, that is, the last terms of both parts of Eq 29. The term involving $\partial F_1/\partial(a/W)$ is similar to the Merkle-Corten correction now in use *[7]*, but since F_1 obviously depends on material properties and is thought to be influenced strongly by hardening, etc., the terms are definitely not identical. The conclusion here is that until demonstrated otherwise, Eq 29 is the only sure way to obtain accurate J values from compact specimen results.

In order to evaluate increments of crack length change, da, in compact specimens, Eq 18 may be used without modification. It is, again

$$da = \frac{\dfrac{b^2}{W^2}\dfrac{\partial F_1}{\partial\left(\frac{\Delta_{pL}}{W}\right)} d\Delta_{pL} - dP}{\dfrac{2b}{W}F_1 - \dfrac{b^2}{W^2}\dfrac{\partial F_1}{\partial\left(\frac{a}{W}\right)}} \tag{30}$$

It is likely that the term in the denominator with $\partial F_1/\partial(a/W)$ will be negligible, but it is carried here for further evaluation. At this point it is seen that Eqs 29 and 30 are sufficient to evaluate crack length changes, da, and J, so that J-R curves may be constructed, given F_1 and a load-displacement record. Subsequently, the means of evaluating F_1 shall be discussed. However, first it is relevant to develop the procedure to directly evaluate T_{mat}.

Adopting the usual definition of T_{mat} (see Eq 19), the remaining component required is dJ. This is obtained most directly by differentiating Eq 29 and referring to earlier expressions such as Eq 14. The simplest format neglecting terms appropriately is

$$dJ = dG + \left[\frac{2b}{W} F_1 - \frac{b^2}{W^2} \cdot \frac{\partial F_1}{\partial\left(\frac{a}{W}\right)}\right] d\Delta_{pL} - \left[\frac{2}{W}\int_0^{\Delta_{pL}} F_1\, d\Delta_{pL}\right] da \quad (31)$$

Then using Eq 19, and substituting Eqs 30 and 31, T_{mat} becomes

$$T_{mat} = \frac{E}{\sigma_0^2}\left[\frac{dG}{da} + \frac{\left[\frac{2b}{W} F_1 - \frac{b^2}{W^2} \frac{\partial F_1}{\partial\left(\frac{a}{W}\right)}\right]^2}{\frac{b^2}{W^2} \frac{\partial F_1}{\partial\left(\frac{\Delta_{pL}}{W}\right)} - \frac{dP}{d\Delta_{pL}}} - \frac{2}{W}\int_0^{\Delta_{pL}} F_1 d\Delta_{pL}\right] \quad (32)$$

Therefore, for a compact specimen Eqs 29 through 32 are sufficient not only to construct a J-R curve, but also to evaluate T_{mat} directly at any point of a load-displacement record, if F_1 and its derivatives, $\partial F_1/\partial(\Delta_{pL}/W)$ and $\partial F_1/\partial(a/W)$, are known.

Determination of F_1 and Its Derivatives for Compact Specimens

As defined by Eq 11a, F_1 is the relationship between load and plastic displacement, for a given set of specimen proportions, including a/W, and for a given material's stress-strain curve. Then it is evident that one way to obtain the function, F_1, is analytically through finite element method or other such methods. For example, it would be relevant to tabulate this function and its derivatives for common test configurations, such as the standard compact specimen, for Ramberg-Osgood stress-strain curves of various hardening coefficients, n.

However, it is also possible to determine F_1 and its derivatives experimentally for the actual material to be considered in a reasonably practical way. The procedure shall be described here.

More specifically it is assumed that a single load-displacement record is to be used to determine a material's crack growth properties. But in order to calibrate the method, that is, to determine F_1 and its derivatives, then assume that a few sub-size specimens of the same material are also available, where the sub-size specimens are made in exactly the same dimensional proportions as the full-size specimen, except their initial a/W values vary slightly from that of the full-size specimen to larger values. Now let all of the load-displacement records from these specimens be plotted on the basis of PW/b_0^2 versus Δ/W as shown on Fig. 2a. The elastic parts of the displacement on Fig. 2a may be noted, using initial slopes for each of the specimens, to develop Fig. 2b a plot of PW/b_0^2 versus Δ_{pL}/W in which case PW/b_0^2 is F_1. Now the sub-sized specimens will go to larger deformations, Δ_{pL}/W, prior to having crack growth occur. Therefore, each plot of data from a sub-sized specimen is F_1 associated with its a/W value up to its own initiation of crack extension. Furthermore, the slopes of the sub-sized specimen curves give $\partial F_1/\partial(\Delta_{pL}/W)$ and the spacing of curves will give $\partial F_1/\partial(a/W)$. Therefore, all of the information on F_1 required for analysis using Eqs 29 through 32 is available. Then the full-size specimen's load displacement curve may

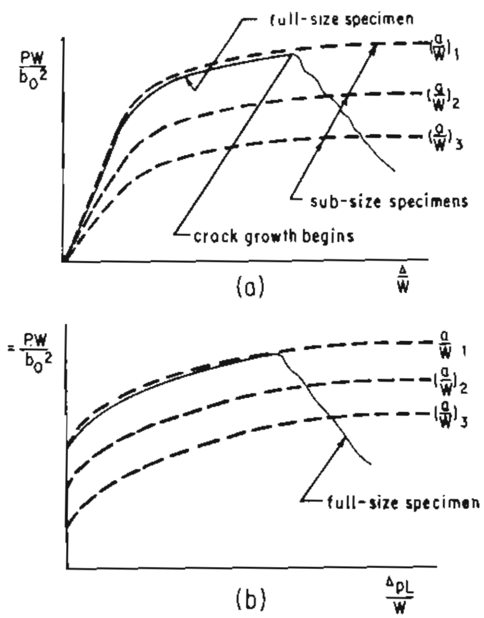

FIG. 2—*Normalized load-displacement records for full and subsize specimens.*

be analyzed fully using Eqs 29 through 32, integrating along this curve increment by increment as indicated by the equations to determine J, Δa, and T_{mat} which can then be used to plot a J-R curve if desired.

All of the theory behind this procedure is exact. It is calibrated for the actual material tested employing exactly the same degree of plane stress versus plane strain in the sub-size specimens of the identical proportion and material. (Indeed, after testing a full-size compact specimen, the broken halves contain enough material to make four sub-size specimens just slightly smaller than half size.) (Moreover, many reactor surveillance programs contain scaled specimen sizes of identical proportions which fortuitously turns out to be ideal for this purpose.) Finally, using this method, the terms which were neglected in developing Eqs 29 through 32 from Eqs 12 through 19 can be evaluated from the information from the sub-size specimens technique if any doubt exists.

It is noted here that other approaches, rather than direct application of Eqs 29 through 32 to the full-size specimen using sub-size specimens as passive calibrations of F_1 and its derivatives can be devised to determine J, Δa, and T_{mat} from this test information. However, each makes less than maximum use of information from the full-size specimen and therefore, is regarded as less precise. Finally, it is noted that a set of full-size specimens when tested can be used to calibrate each other, (that is, determine F_1 and derivatives without sub-size specimens), but this would be less than precise if points of beginning of crack extension, etc. were not clearly distinguishable. The practical aspects of such short cuts are to be left to experimental programs.

In summary, for evaluation of a material's cracking properties from compact specimens, the calibration method described here along with Eqs 29 through 32 forms a precise and practical basis for determining J, Δa, T_{mat}, etc. from nothing more than load-displacement records. Since other methods require costly additional instrumentation, etc., this method shows great promise.

Bending Test Specimens of the Three-Point and Four-Point Bending Type

For typical bend test specimens with a/W large enough so that plasticity is confined to the remaining ligament region, the analysis in the preceding sections on compact specimens also applies. However, all terms involving $\partial F_1/\partial(a/W)$ become very weak and can be neglected within normal accuracy requirements. Thus simply repeating Eqs 29 through 32, deleting these terms, the results are

$$J = G + \hat{J}_{\mathrm{pL}} - \int_{a_0}^{a} \frac{\hat{J}_{\mathrm{pL}}}{b}\, da$$

where

$$\hat{J}_{\mathrm{pL}} = \frac{2}{b_0}\int_0^{\Delta_{\mathrm{pL}}} P\, d\Delta_{\mathrm{pL}} \tag{33}$$

and

$$da = \frac{\dfrac{b^2}{W^2}\dfrac{\partial F_1}{\partial\left(\dfrac{\Delta_{\mathrm{pL}}}{W}\right)} d\Delta_{\mathrm{pL}} - dP}{\dfrac{2b}{W}F_1} \tag{34}$$

and

$$T_{\mathrm{mat}} = \frac{E}{\sigma_0^2}\left[\frac{dG}{da} + \frac{\left[\dfrac{2b}{W}F_1\right]^2}{\dfrac{b^2}{W^2}\dfrac{\partial F_1}{\partial\left(\dfrac{\Delta_{\mathrm{pL}}}{W}\right)} - \dfrac{dP}{d\Delta_{\mathrm{pL}}}} - \frac{2}{W}\int_0^{\Delta_{\mathrm{pL}}} F_1\, d\Delta_{\mathrm{pL}}\right] \tag{35}$$

using Eqs 33 through 35 and a single sub-size specimen to determine F_1, the evaluation procedures described in the previous section hold to determine T_{mat}, etc., from a load-displacement record of a full-size specimen. On the other hand, if doubt exists on the precision of deleting terms containing $\partial F_1/\partial(a/W)$ here, then the previous analysis, Eqs 29 through 32 can be employed in full (with further sub-size specimens required) to evaluate the effect of neglecting these terms.

It seems relevant to point out here that the final form of Eq 23 for three-point bending has been used erroneously in the past. In this connection it should be noted that the first form of Eq 23 contains G and that it must be modified for three-point bending of a finite specimen whereas the plastic term remains reasonably accurate (based on θ_{crack}). The errors were the result of the inappropriate use of linear-elastic components of the analyses. Moreover, it is also noted here that in developing J-R curves from bending tests, the final term in Eq 33, correcting for the effect of crack growth, has been neglected throughout the literature.

Applications to Predominately Tension Loading

Any of the three special functional forms, Eqs 11a, 11b, and 11c may be applied to any two-dimensional configuration. Therefore, for predominately tension, the form F_1 and its associated analysis, Eqs 12 through 19, could be employed correctly. However, in that analysis the terms involving $\partial F_1/\partial(a/W)$ and its second derivative as well, would become dominant and difficult to evaluate accurately either by experimental methods or further numerical analysis. Thus for cases where the remaining ligament is subjected principally to tension, it is equally correct but significantly more practical to apply the alternate forms F_2 or F_3, or both. The latter form, F_3, is only of substantial advantage whereas the remaining ligament, b, is very small compared to other dimensions, W, L, a, etc., so that F_3 is a function only of Δ_{PL}/b, and other variables disappear, as discussed originally in Ref *3*. Consequently, the continuing discussion here will center on the functional form, F_2 from Eq 11*b*.

Substituting Eq 11*b* into Eq 8 gives

$$J = G + \int_0^{\Delta_{pL}} F_2 d\Delta_{pL} - \frac{b}{W}\int_0^{\Delta_{pL}} \frac{\partial F_2}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL} \tag{36}$$

Resubstituting from Eq 11*b* for F_2 in the first integral term leads to

$$J = G + \frac{1}{b}\int_0^{\Delta_{pL}} \mathrm{P} d\Delta_{pL} - \frac{b}{W}\int_0^{\Delta_{pL}} \frac{\partial F_2}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL} \tag{37}$$

(for $\Delta a = 0$)

These results, Eqs 36 and 37 are analogous to Eqs 12 and 13 and so upon comparison we note that Rice's term in Eq 13 appears in Eq 37 but with a coefficient of one instead of two. Upon noting that the last two terms in Eqs 13 and 37 are weak (small) in bending and tension respectively, then the principal term in J_{PL}, the plastic part of J is

$$J_{PL} \cong \frac{\eta}{b}\int_0^{\Delta_{pL}} P d\Delta_{pL}' \tag{38}$$

where

$\eta \rightarrow 2$ (pure bending)
$\eta \rightarrow 1$ (pure tension)

as noted by Sumpter [*8*], quite some time ago. Indeed, this analysis agrees with Turner's results but in addition offers Eqs 13 and 37 as exact analysis, both being applicable to all cases if correction terms (the final terms in each) are evaluated properly.

The analysis is continued as with Eqs 14 through 19, hence

$$dJ = dG + \left[F_2 - \frac{b}{W}\frac{\partial F_2}{\partial\left(\frac{a}{W}\right)}\right] d\Delta_{pL} + \left[\frac{2}{W}\int_0^{\Delta_{pL}} \frac{\partial F_2}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL} - \frac{b}{W^2}\int_0^{\Delta_{pL}} \frac{\partial^2 F_2}{\partial\left(\frac{a}{W}\right)^2} d\Delta_{pL}\right] da \tag{39}$$

and

$$J = G + \int_0^{\Delta_{pL}} \left[F_2 - \frac{b}{W}\frac{\partial F_2}{\partial\left(\frac{a}{W}\right)}\right] d\Delta_{pL} + \int_{a_0}^{a}\left[\frac{2}{W}\int_0^{\Delta_{pL}} \frac{\partial F_2}{\partial\left(\frac{a}{W}\right)} d\Delta_{pL} - \frac{b}{W^2}\int_0^{\Delta_{pL}} \frac{\partial^2 F_2}{\partial\left(\frac{a}{W}\right)^2} d\Delta_{pL}\right] da \tag{40}$$

($0 < \Delta a << b$)

and

$$da = \frac{\frac{b}{W}\frac{\partial F_2}{\partial\left(\frac{\Delta_{pL}}{W}\right)} d\Delta_{pL} - dP}{F_2 - \frac{b}{W}\frac{\partial F_2}{\partial\left(\frac{a}{W}\right)}} \tag{41}$$

where

$$T_{mat} = \frac{E}{\sigma_0^2}\frac{dJ}{da} \tag{42}$$

As before, if Eqs 39 through 42 are applied to center crack or double edge notch specimens where little or no bending is present, terms containing $\partial^2 F_2/\partial(a/W)^2$ in Eqs 39 and 40 can be neglected and the influence of the $\partial F_2/\partial(a/W)$ term in the denominator will be small in some circumstances for certain computations. Thus a simplification of these equations for some practical cases is possible. Moreover, in such cases, experimental procedures very much like those suggested in the preceding section on determination of F_1 and its derivatives for compact specimens are appropriate. However, that analysis and subsequent experimental procedures are so similar to the preceding discussions that it is not repeated here.

A Note on Determination of $T_{appl.}$ for Analyzing the Tearing Instability

Following the analysis provided by Hutchinson [1], T_{appl} can be found from

$$T_{appl} = \frac{E}{\sigma_0^2}\left(\frac{\partial J}{\partial a}\right)_{\Delta_T} \tag{43}$$

Now $(\partial J/\partial a)_{\Delta_T}$ is the increase in J applied by a loading system per increment of crack extension with the overall system displacement, Δ_T held constant. Therefore, following Hutchinson's anaysis procedures [1], the $(\partial J/\partial a)_{\Delta_T}$ may be computed making use of the forms for analyzing J herein, for example, Eqs 16, 25, 29, 33, 40, etc. Once suggested, carrying out such computations is straightforward and consequently omitted from further discussion here.

Conclusions

1. Methods of properly (exactly) computing J for various test configurations have been developed herein.
2. The methods developed correctly account for the effects of crack growth on J, which has often been in error in previous works.
3. Procedures for determining J-R curves, T_{mat}, etc. are discussed which require no more than load-displacement records from tests.

Acknowledgments

This work was sponsored by a contract from the U.S. Nuclear Regulatory Commission, Operating Reactors Division, at Washington University, St. Louis. The continued encouragement of R. Gamble and W. Hazelton of Nuclear Regulatory Commission is most gratefully acknowledged. The assistance of this work through consultation with

J. R. Rice (Brown University), and upon whose works this analysis was based, is acknowledged with thanks.

References

[1] Hutchinson, J. W. and Paris, P. C. in *Elastic Plastic Fracture, ASTM STP 668*, American Society for Testing and Materials, 1979, pp. 37–64.
[2] Paris, P. C., Tada, H., Zahoor, A., and Ernst, H., "A Treatment of the Subject of Tearing Instability," Report NUREG-0311, U. S. Nuclear Regulatory Commission, Aug., 1977.
[3] Rice, J. R., Paris, P. C., and Merkle, J. G. in *Progress in Flaw Growth and Fracture Toughness Testing, ASTM STP 536*, American Society for Testing and Materials, 1973, pp. 231–235.
[4] Rice, J. R., *Journal of Applied Mechanics*, American Society of Mechanical Engineers, 1968, pp. 379–386.
[5] Paris, P. C., Discussion of work by J. A. Begley and J. D. Landes in *Fracture Toughness, ASTM STP 514*, American Society for Testing and Materials, 1972, pp. 1–39.
[6] Merkle, J. and Corten, H., *Journal of Pressure Vessel Technology; Transactions*, American Society of Mechanical Engineers, 1974, pp. 286–292.
[7] Clarke, G. A., "Tentative Recommendation Procedure for J_{Ic} Testing," American Society for Testing and Materials, Committee E-24 meeting, Norfolk, Virginia, March, 1977.
[8] Sumpter, J. D. and Turner, C. E. in *Cracks and Fracture, ASTM STP 601*, American Society for Testing and Materials, 1976, pp. 3–18.