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Fundamentals of the Phenomenological Theory of Nonlinear Fracture Mechanics

An introduction to the theoretical foundations of the phenomenological theory of nonlinear fracture mechanics is given. Following an outline of the full range of objectives of nonlinear fracture mechanics, the paper focuses on the phenomenological, or semiempirical, approach to the initiation of crack growth and the subsequent quasi-static crack growth and loss of stability under monotonic load histories.

1 Objectives of Nonlinear Fracture Mechanics

An individual from applied mechanics who is a newcomer to fracture mechanics is naturally predisposed to regard the subject as a means of analyzing flawed, or cracked, structural components. By contrast, his colleague from materials science is more likely to view fracture mechanics as a means of characterizing, or ordering, the fracture resistance of materials. Both aspects have provided, and continue to provide, impetus for development of the subject. The success of fracture mechanics is derived in part from the union of these two purposes. Measured fracture parameters not only serve to order the resistance of materials to fracture, but they can also be employed directly in the analysis of structural integrity in the presence of flaws. It is important for the newcomer to the subject to bear these dual purposes in mind since otherwise he or she might regard as undue, the emphasis placed on either test specimens, on the one hand, or crack solutions, on the other.

Nonlinear fracture mechanics is largely concerned with inelastic effects. Some inelasticity is almost always present in the vicinity of a stressed crack tip. Depending on material and conditions, the inelasticity can take various forms, including rate-independent plasticity, creep, and phase change. When the zone of inelasticity is small enough, solutions from linear elasticity can be used to analyze, or, more precisely, to correlate, data from test specimens. This data can, in turn, be used in conjunction with other linearly elastic crack solutions to predict failure of cracked structural components. Linear-elastic fracture mechanics has found extensive applications to high-strength, relatively brittle materials such as metals used in the aerospace industry and ceramics. For certain fracture phenomena, such as fatigue crack growth and corrosion cracking, the zone of inelasticity is often small enough to use linear-elastic fracture mechanics even in more ductile materials. However, the more ductile a material, the more likely that the inelastic zone will not be small enough at the point of fracture to justify the use of solutions based on linear elasticity. For example, components and test specimens of practical dimensions, which are made of many of the more common low-to-intermediate-strength structural metals, become fully plastic before a crack starts to advance under monotonic loadings. Under these circumstances it is essential to use solutions to crack problems based on a theory of plasticity.

Nonlinear fracture mechanics encompasses a semiempirical approach, which for the most part is an extension of linear-elastic fracture mechanics to account for large-scale inelastic effects, and a more basic approach whose aim is to predict (as opposed to correlate) fracture conditions by accounting for mechanisms of separation at the microscopic level. In the semiempirical, or phenomenological, approach crack solutions are used to correlate fracture conditions at the tip of a crack in a structural component to corresponding near-tip conditions in a previously tested specimen. It is the semiempirical approach whose foundations will be discussed in this paper. Furthermore, attention will be directed exclusively to the initiation of growth of a preexisting crack and its subsequent quasi-static advance in bodies of rate-independent materials subject to monotonic loading. While this is the most fully developed topic in nonlinear fracture mechanics, a number of other problem areas are currently under intensive investigation. These include creep crack growth, dynamic crack propagation and arrest, low-cycle fatigue cracking, and high-temperature creep-fatigue cracking. In addition, more fundamental work than that to be described in this paper is progressing on stable crack growth.

The more basic problem of relating fracture parameters to microstructural separation processes has received relatively less attention than the semiempirical approach. Nevertheless, important qualitative understanding has been gained from models for crack initiation due to both cleavage in the brittle range of behavior and hole growth in the ductile range. In addition, some progress has been made in relating high-temperature crack growth to rates of grain boundary

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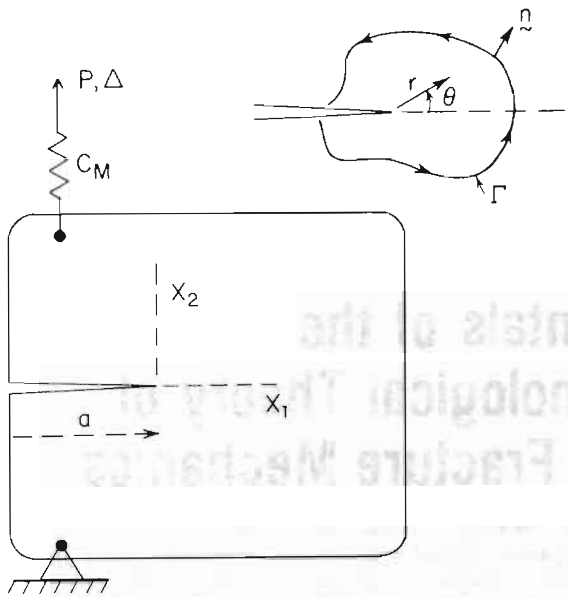

Fig. 1 Conventions for a cracked body in mode *I*

cavitation in polycrystalline materials. These and other challenging problems in the same category seem ripe for new advances.

2 Background From Linear Fracture Mechanics

The aspects of nonlinear fracture mechanics to be discussed in this paper are largely direct generalizations of concepts and results from linear-elastic fracture mechanics. For the purpose of making contact with the linear theory, we start with a brief synopsis of results from that theory which can be found in any text on the subject.

For an isotropic, linearly elastic solid under conditions of plane strain or plane stress the stress and strain fields at the crack tip always have a singularity of order $r^{-1/2}$. With the conventions shown in Fig. 1, the general form of the stress and strain singularity fields is

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}}\tilde{\sigma}_{ij}(\theta)$$

$$\epsilon_{ij} = \frac{K}{\sqrt{2\pi r}}\tilde{\epsilon}_{ij}(\theta) \tag{1}$$

where the amplitude of the singularity, K, is called the stress intensity factor. Fields that are symmetric with respect to the plane of the crack ($x_2=0$) are termed mode *I*, while antisymmetrical fields are termed mode *II*. For a given mode, the θ-variations, $\tilde{\sigma}$ and $\tilde{\epsilon}$, are fixed. It is standard practice to normalize the singular fields of the linear theory by requiring $\tilde{\sigma}_{22}(\theta=0)=1$ in mode *I* and $\tilde{\sigma}_{12}(\theta=0)=1$ in mode *II*. The θ-variations of the in-plane stress components are the same for plane stress and plane strain, but the strains depend on which condition is in effect. The stress intensity factor K depends linearly on the applied load and is a function of the crack length and other geometric parameters characterizing the body. Catalogues of results for stress intensity factors are available for a wide variety of geometries.

The stress intensity factor provides a measure of the level of deformation in the vicinity of the crack tip. Its use to characterize the onset or continuation of crack growth in a real material, which may undergo plasticity, creep, or other inelastic deformation in the most highly stressed region at the tip, relies on satisfaction of a condition known as *small-scale yielding*. In other words, this condition requires that the zone of inelasticity, whatever its source, be contained well within the region over which the singularity fields (1) provide a good approximation to the full linear elasticity solution. Small-scale yielding assures that all the pertinent information related to the geometry and loads applied to the cracked body are communicated to the crack tip only through K for a given mode of loading.

The critical stress intensity factor, K_c, associated with the onset of crack growth under monotonic loading depends on the material, its temperature, and possibly on its chemical environment. It also depends on the mode of loading and on whether plane stress or plane strain, or a set of conditions in between, pertain. The critical value of K for any given set of conditions must be obtained by experiment using a calibrated, precracked test specimen which meets the restrictions of small-scale yielding at the onset of fracture. The critical stress intensity factor in mode *I*, plane-strain conditions is designated as K_{Ic} and is called the *fracture toughness* of the material at the particular temperature.

A by-product of nonlinear fracture mechanics has been a quantitative clarification of small-scale yielding, but insight into this condition was already available through a combination of test and theory prior to the development of the nonlinear theory [1]. Rough sketches of the boundary of the plastic zone in small-scale yielding are shown in Fig. 2 for mode *I*. The material is assumed to be elastic-perfectly plastic with a tensile yeild stress σ_0 and a Mises yield surface. The zone size is proportional to K^2. With r_p as the (approximate) distance to the boundary ahead of the crack tip,

$$r_p = \frac{1}{3\pi}\left(\frac{K}{\sigma_0}\right)^2 \quad \text{for plane strain}$$

$$r_p = \frac{1}{\pi}\left(\frac{K}{\sigma_0}\right)^2 \quad \text{for plane stress} \tag{2}$$

The standards of the American Society for Testing and Materials (ASTM) for a valid K_{Ic} test [1] were arrived at by extensive testing using several types of specimens. For a compact tensile specimen, whose plane view is similar to the cracked body shown in Fig. 1, the standard requires that the uncracked ligament and the crack length itself be not less than $25r_p$ at the point of fracture. To ensure plane-strain conditions along most of the crack edge except near the faces, it is also required that the specimen thickness be at least $25r_p$. As a rule of thumb, small-scale yielding will be in force at the onset of crack growth if the applied load is less than half the limit load, modeling the cracked body as elastic-perfectly plastic with yield stress σ_0. For high-strength, relatively brittle metal alloys, values of r_p at fracture initiation in the range 0.1 to 1 mm are typical in plane strain. Thus a specimen with crack lengths on the order of 2.5 to 25 mm will suffice to ensure small-scale yielding. Similarly, many applications involving crack-like flaws in structural components of high-strength alloys will fall within the small-scale yielding range.

The utility of linear-elastic fracture mechanics is substantially diminished for applications and testing involving low and intermediate-strength metals with good ductility. For an intermediate-strength steel used in pressure containment vessels, r_p at fracture initiation under monotonic loads can be as large as 10 mm. A valid K_{Ic} test would therefore require a test specimen the size of a filing cabinet. While such specimens have indeed been produced and tested for some special applications, they are generally prohibitively expensive and too difficult to test. Fracture toughness testing on its own had provided ample motivation for the development of nonlinear fracture mechanics.

Depending on material properties, specimen geometry, and loading compliance, the crack may or may not run dynamically once the critical intensity factor is attained. Certain specimens permit the experimental determination of K versus crack advance Δa under stable quasi-static growth

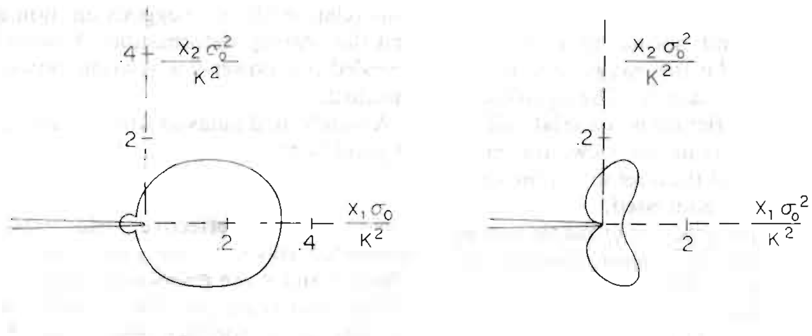

Fig. 2 Boundaries of the plastic zone in small-scale yielding for an elastic-perfectly plastic solid yielding according to the Mises criterion

conditions. A highly brittle material will undergo crack advance under essentially constant K, as depicted in Fig. 3, while a more ductile material with tearing resistance requires an incremental increase in K for each increment of crack advance. The measured curve of K_R versus Δa is called the *resistance curve* of the material. Like K_c itself, the resistance curve is a strong function of whether plane stress or plane strain is in effect, and it only has meaning under small-scale yielding. Given the crack has extended quasi-statically an amount Δa to the current crack length a, the condition for continued crack advance is $K = K_R(\Delta a)$, where K is the applied stress intensity factor. The crack poised for growth is stable (i.e., will not run dynamically) if a small increment of crack advance with the prescribed loading held fixed results in $dK < dK_R$, i.e., if

$$\left(\frac{\partial K}{\partial a}\right)_L < \left(\frac{dK_R}{d\Delta a}\right) \tag{3}$$

where L denotes the appropriate loading parameter. The body of Fig. 1 represents a cracked component or specimen subject to a generalized load P acting through a generalized spring with compliance C_M. The spring can represent the compliance of a testing machine or the compliance of the portion of the structure communicating the load to the cracked component. While K at a given P does not depend on the spring compliance, $(\partial K/\Delta a)_L$ is a strong function of C_M if Δ is prescribed, thereby influencing the stability of crack advance.

Within the context of the linear theory of elasticity, there is an important connection between the stress intensity factor and the rate of change of the potential energy of the system (cracked body, spring, and loading system) with respect to crack advance. In mode *I* (or mode *II*) the rate of decrease of the potential energy of a system such as that in Fig. 1 per unit crack advance per unit thickness is

$$\begin{aligned} \mathcal{G} &= \frac{1-\nu^2}{\mathrm{E}} K^2 \quad \text{(plane strain)} \\ &= \frac{1}{\mathrm{E}} K^2 \quad \text{(plane stress)} \end{aligned} \tag{4}$$

where E is Young's modulus and ν is Poisson's ratio. At a given current load level, the energy release rate is independent of C_M and is the same whether the load or the displacement is prescribed to be fixed during the increment of crack advance.

Griffith's original treatment of the onset of crack advance was couched in terms of an energy balance by requiring $\mathcal{G}$ to be equal to $\mathcal{G}_c$, the energy absorbed in the process of creating a unit area of separated surfaces. In small-scale yielding, the energy-based approach and stress intensity approach are mathematically equivalent for crack initiation because of the connection (4). The energy-based interpretation is still preferred in some quarters, particularly in the materials science community. In spite of the appeal of energy arguments, Irwin and others involved in the early development of fracture mechanics saw that for many purposes the intensity-based approach was more flexible and permitted application of the theory beyond what could be rationalized by energy arguments. Examples within linear-elastic fracture mechanics include fatigue crack propagation and creep cracking in brittle materials. The theory of nonlinear fracture mechancis which will be discussed in the following is strictly based on intensity arguments.

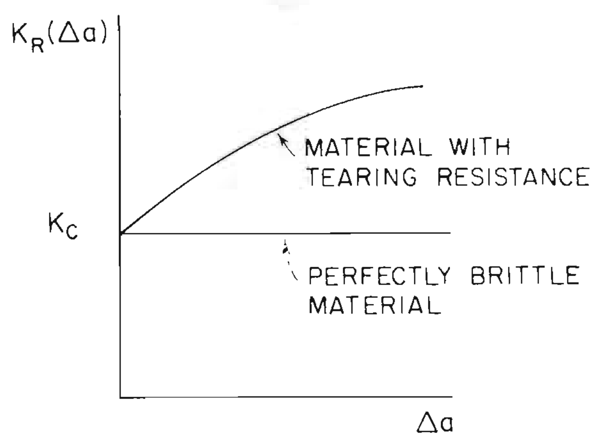

Fig. 3 Small-scale yielding resistance curves

In the introduction to the foundations of nonlinear fracture mechanics, which follows, we have not attempted to provide a comprehensive guide to the literature. Instead, we have tried for the most part to give relatively recent articles which will enable the reader to trace backward to earlier contributions.

3 The *J*-Integral and Crack-Tip Fields for Plastically Deforming Solids

The unifying theoretical idea behind nonlinear fracture mechanics for rate-independent materials under monotonic loading is the *J*-integral [2]. A small-strain deformation theory of plasticity (i.e., a small-strain, nonlinearly elastic material) is assumed as the material model. The strain energy density of the material is $W(\epsilon)$ with stress given by

$$\sigma_{ij} = \partial W/\partial \epsilon_{ij} \tag{5}$$

A cracked body such as that in Fig. 1 is considered to be in plane stress or plane strain with the crack lying along the x_1-axis. With Γ as any contour encircling the tip of the crack in a counterclockwise direction, the path-independent line integral expression for J is

$$J = \int_\Gamma (W n_1 - \sigma_{ij} n_j u_{i,1}) ds \tag{6}$$

where **u** is the displacement vector, s is the arc length along the contour, and **n** is the outward unit normal to Γ. For the fictitious nonlinearly elastic solid, J is the energy release rate. It reduces to $\mathcal{G}$ for a linearly elastic material. The significance of the energy release rate for the fictitious material will be discussed later. From a physical point of view, the more important role of J is as a measure of the intensity of the near-tip deformation and this will now be addressed.

By taking Γ as a circular contour centered at the tip and by using path independence to shrink the contour down to the tip, one concludes that if J is nonzero then the integrand in (6) should have a r^{-1} singularity. That is,

$$r(Wn_1 - \sigma_{ij}n_ju_{i,1}) = f(\theta) \quad \text{as} \quad r \to 0 \tag{7}$$

and, thus,

$$J = \int_{-\pi}^{\pi} f(\theta)d\theta \tag{8}$$

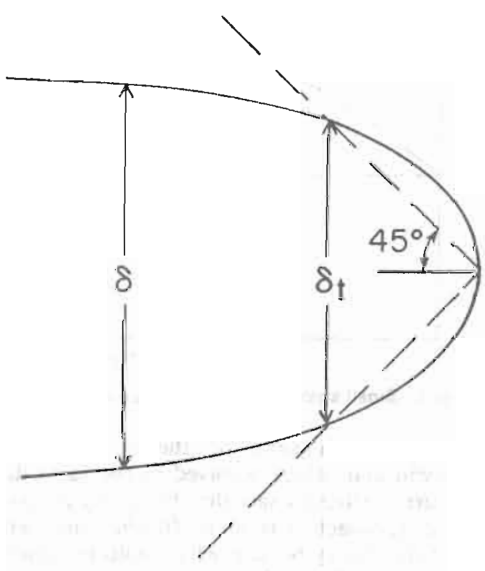

Fig. 4 Crack opening for a hardening material

This relation already suggests an intimate relation between J and the near-tip deformation. A more explicit connection is revealed if a power-law relation between stress and strain is assumed.

A widely used uniaxial stress-strain relation is the Ramberg-Osgood form

$$\epsilon/\epsilon_0 = \sigma/\sigma_0 + \alpha(\sigma/\sigma_0)^n \tag{9}$$

where σ_0 is an effective yield stress, $\epsilon_0 = \sigma_0/E$ is the associated elastic strain with E as Young's modulus, and where α and n are parameters chosen to fit data. Typical n-values range from 3 to 5 for materials with high hardening to as large as 20 for light hardening. Asymptotically, as the crack tip is approached, the contributions to the strains that depend linearly on stress are negligible compared to the power-law terms. The pure-law relation from (9) is

$$\epsilon/\epsilon_0 = \alpha(\sigma/\sigma_0)^n \tag{10}$$

If J_2 deformation theory is used to generalize (10) to multiaxial states, then

$$\frac{\epsilon_{ij}}{\epsilon_0} = \frac{3}{2}\alpha\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\frac{s_{ij}}{\sigma_0}, \quad \sigma_e = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2} \tag{11}$$

where **s** is the stress deviator. For the power-law material the r^{-1} singularity in W implies a $r^{-1/(n+1)}$ singularity in the stresses, a $r^{-n/(n+1)}$ singularity in the strains, and a $r^{1/(n+1)}$ variation in the displacements. The near-tip singularity fields can be written as

$$\sigma_{ij} = \sigma_0\left(\frac{J}{\alpha\sigma_0\epsilon_0 I_n r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(\theta,n) \tag{12a}$$

$$\epsilon_{ij} = \alpha\epsilon_0\left(\frac{J}{\alpha\sigma_0\epsilon_0 I_n r}\right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(\theta,n) \tag{12b}$$

$$u_i - \hat{u}_i = \frac{J}{I_n\sigma_0}\left(\frac{\alpha\sigma_0\epsilon_0 I_n r}{J}\right)^{\frac{1}{n+1}} \tilde{u}_i(\theta,n) \tag{12c}$$

Details of these fields are given in [3–5]. The dimensionless θ-variations $\tilde{\sigma}$, $\tilde{\epsilon}$, and $\tilde{\mathbf{u}}$ depend on the mode, on n, and on whether plane strain or plane stress is assumed, as does the normalizing constant I_n. These variations must be normalized

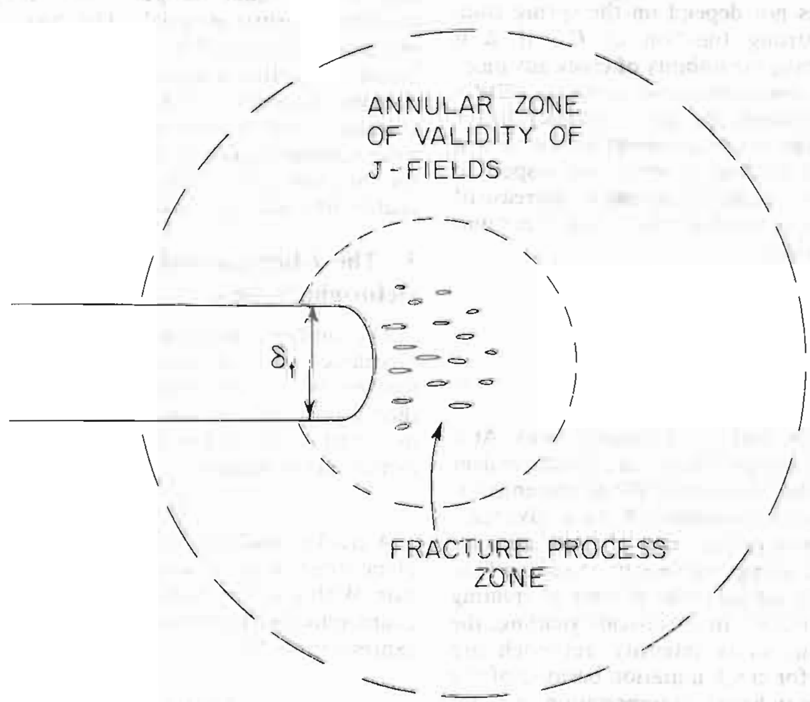

Fig. 5 Schematic of near-tip behavior under J-dominance conditions

in some manner, and in [3] and [5] the maximum value of $\bar{\sigma}_e \equiv (3\bar{s}_{ij}\bar{s}_{ij}/2)^{1/2}$ is set at unity. The contribution $\hat{\mathbf{u}}$ allows for a possible translation of the crack tip itself.

Later, ways will be discussed for calculating or estimating how J depends on the geometry and loading of the cracked body. It is clear from (12) that J can be regarded as a measure of the intensity of the crack-tip singularity fields. However, before it can be assumed that J can be used to correlate the initiation of crack growth in true elastic-plastic solids, one must be sure that the following two conditions are met. First, the deformation theory of plasticity must be an adequate model of the small-strain behavior of real elastic-plastic materials under the monotonic loads being considered. Second, the regions in which finite strain effects are important and in which the microscopic processes occur must each be contained well within the region of the small-strain solution dominated by the singularity fields. This second condition is sometimes called J-dominance and is analogous to the small-scale yielding requirement for linear fracture mechanics.

The first condition is at issue in any application of a deformation theory of plasticity. Any solution based on the J_2 deformation theory of plasticity coincides exactly with a solution to the J_2 flow (incremental) theory of plasticity if proportional loading (i.e., stress components changing in fixed proportion to one another) occurs everywhere. The singular fields (12) by themselves satisfy this condition exactly. In other words, the crack-tip singularity fields are also solutions to the corresponding J_2 flow theory equations, and J as it appears in (12) could serve as the amplitude of the singular fields, independent of its line integral definition in (6). While the full solution for a body of material characterized by the Ramberg-Osgood tensile relation (9) will not generally satisfy proportional loading exactly, most problems with a single, monotonically applied load do come sufficiently close to meeting proportionality to justify use of deformation theory. Many investigators have demonstrated that the line integral defined in (6) is approximately independent of path when finite element solutions of crack problems are obtained using an incremental theory of plasticity, even though its strict path independence is tied to its deformation theory definition. Moreover, the J-values themselves are accurately approximated by the corresponding deformation theory solutions. It is fair to say that the use of deformation theory solutions to stationary crack problems for monotonic loading is generally accepted for the purpose of evaluating J.

4 J-Dominance in Plane Strain

The requirements connected with the second condition mentioned in the foregoing are more complicated. Most efforts to assess J-dominance have focused on plane-strain configurations under mode I conditions since this combination is generally the most critical in that it leads to the lowest values of J_c (or K_c). Plane strain, mode I will be considered next.

In the discussion that follows the crack-opening displacement plays an important role. The separation of the two crack faces, $\delta = u_2(x_1, 0^+) - u_2(x_1, 0^-)$, varies like $(-x_1)^{1/(n+1)}$ as the tip is approached according to the singularity fields (12) of the small-strain theory. An effective crack-tip opening displacement δ_t can be defined as the separation where the 45 deg lines intercept the crack faces as in Fig. 4. The result is [6]

$$\delta_t = d(\alpha\epsilon_0, n)\frac{J}{\sigma_0} \tag{13}$$

with values of d ranging from about 0.8 for large n to about 0.3 for $n=3$, with a very weak dependence on $\alpha\epsilon_0$. The crack-tip opening displacement provides a measure of the size of the zone in which finite strains are important. Finite element studies based on finite strain incremental plasticity formulations [7, 8] have been performed for the plane-strain small-scale yielding problem. For distances from the tip that are greater than 2 or 3 times δ_t, the deviations from small-strain theory becomes unimportant.

Let R denote the "radius" of the zone of dominance of the singularity fields (12), i.e., the characteristic size of the region in which the singularity fields provide a good approximation to the complete small-strain deformation theory solution. As depicted in Fig. 5, one condition for J-dominance is

$$R > 3\delta_t \tag{14}$$

A second condition is that R be greater than the *fracture process zone* in which the microscopic separation processes occur. The predominant ductile fracture mechanism is void nucleation, growth, and coalescence. Since hole growth is itself a finite strain process, the fracture process zone for this mechanism is roughly comparable to the zone of finite strains, and therefore (14) again serves to ensure J-dominance. If failure due to shear localization intervenes, or if it actually precipitates hole nucleation and growth, then the fracture process zone may be somewhat larger than $3\delta_t$. When the separation process is cleavage, the size of the fracture process zone appears to be set by the grain size in many materials with the zone extending from 2 to 10 grain diameters from the tip. This sets a constraint on R for J-dominance under cleavage conditions which usually renders (14) superfluous.

The characteristic size R of the zone of dominance of the singularity fields (12) depends strongly on geometry and hardening. The geometry dependence is especially strong for low-hardening materials (high n) for reasons that will now be discussed.

The mathematical character of the equations governing behavior near the crack tip changes in the limit $n \to \infty$ corresponding to perfect plasticity. For finite n the governing equations are elliptic, and sufficiently near the tip the singularity fields necessarily do provide an asymptotic representation to the complete small-strain solution. In the perfect plasticity limit the equations are hyperbolic and there no longer exists a unique near-tip solution independent of specimen geometry [9]. The limit of the singularity fields (12) as $n \to \infty$ is just one possible near-tip solution. A unique near-tip field tied to J, or any other single parameter, necessarily involves the assumption of hardening. Certain problems do appear to have approximately the same perfectly plastic near-tip fields as those associated with the limit of (12) as $n \to \infty$. Important examples are the small-scale yielding problem itself and the fully plastic bend or compact tension configurations (see Fig. 6). In each of these problems, as well as in the limit of the singularity field (12) as $n \to \infty$, there is a significant elevation of the normal stress ahead of the tip to a level approximately 2.5 times the tensile yield stress σ_0. This high triaxiality largely accounts for the fact that the critical intensity factor (K_{Ic}) associated with mode I, plane-strain conditions is lower than that associated with other possible combinations. The small-scale yielding problem is generally taken as the basic reference configuration since K_{Ic} is determined under this condition. On the other hand, it has long been recognized [9] that a very different near-tip field occurs in a perfectly plastic plane-strain strip with a centered crack pulled to full yield in tension (Fig. 6). In this case the normal stress ahead of the tip is only slightly above σ_0 and the deformation is confined to shear bands extending from the crack tip to the edges of the strip.

Precise estimates of R are difficult to make. Numerical solutions in small-scale yielding indicate that the singularity fields (12) provide a fairly good approximation out to a distance ahead of the tip of roughly

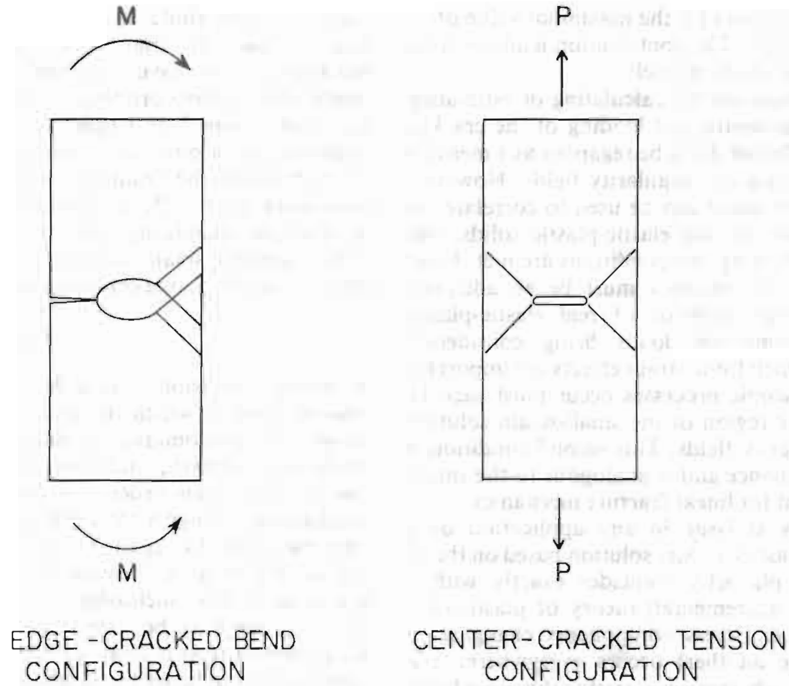

Fig. 6 Plane-strain fully plastic deformation modes

$$R = \left(\frac{1}{5} \quad \text{to} \quad \frac{1}{4}\right) r_p \tag{15}$$

with relatively little dependence on hardening. Under fully plastic conditions, when the plastic zone has reached across the entire uncracked ligament, R for the center-cracked configuration and for the bend configuration, is some fraction of the uncracked ligament b. For the bend configuration (and the compact tension specimen) numerical studies [10] suggest

$$R \cong 0.07b \tag{16}$$

and, again, for essentially all hardening levels including perfect plasticity. The center-cracked tensile configuration has a much smaller zone of dominance which becomes vanishingly small as $n \to \infty$. For light to medium hardening, i.e., $n \cong 10$) it appears that [11], very roughly,

$$R \cong 0.01b \tag{17}$$

Now we return to the issue of J-dominance as specified by (14) for ductile fracture. For low to moderate strain hardening, $\delta_t \cong 0.6J/\sigma_0$. For small-scale yielding, it is readily verified using (2), (15), and $J = (1-\nu^2)\, K^2/\text{E}$ that (14) is always satisfied. For bend-type configurations the condition for J-dominance under fully plastic conditions obtained by combining (16) and (14) is

$$b > 25J/\sigma_0 \tag{18}$$

For the center-cracked tensile configuration the condition for light to moderate hardening from (17) and (14) is

$$b > 175J/\sigma_0 \tag{19}$$

Further work defining J-dominance conditions remains to be done. Moreover, analogous specifications for plane-stress conditions have not been obtained.

5 J_{Ic} Testing

Interaction among application, experiment, and theory has long been a hallmark of fracture mechanics. No sooner was the J-integral put forward than it was exploited in fracture toughness testing [12], and this in turn provided more impetus

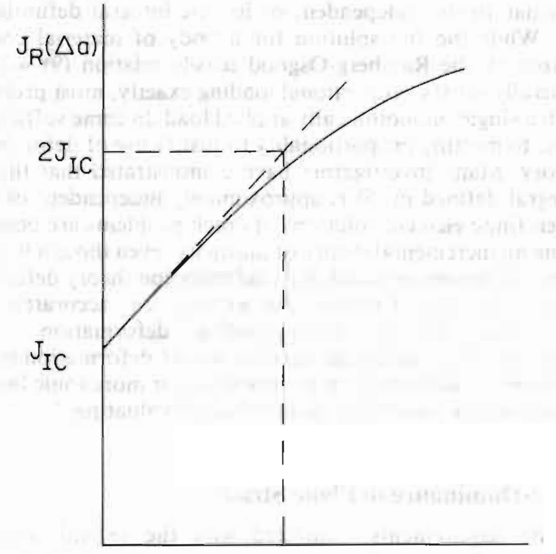

Fig. 7 Resistance curve under J-controlled growth conditions

for theoretical developments. As already mentioned, there was strong motivation to carry out fracture toughness tests on small specimens of low and intermediate-strength materials with no restriction to small-scale yielding.

Testing for J_{Ic} is an art in itself which has advanced significantly from its beginnings [13], and we will not attempt to go into the subtleties of the method here. In principle, the procedure is analogous to K_{Ic} testing. One chooses a plane-strain, mode I specimen that will meet the J-dominance condition, and one experimentally determines the critical value of J, J_{Ic}, at which the initiation of crack growth occurs. When the method was first being developed, it was essential to establish the equivalence of fracture toughness determination by any such J-test with that of valid K_{Ic} test. In small-scale yielding $J = \mathcal{G} = (1-\nu^2)K^2/\text{E}$, and thus it was essential to verify that

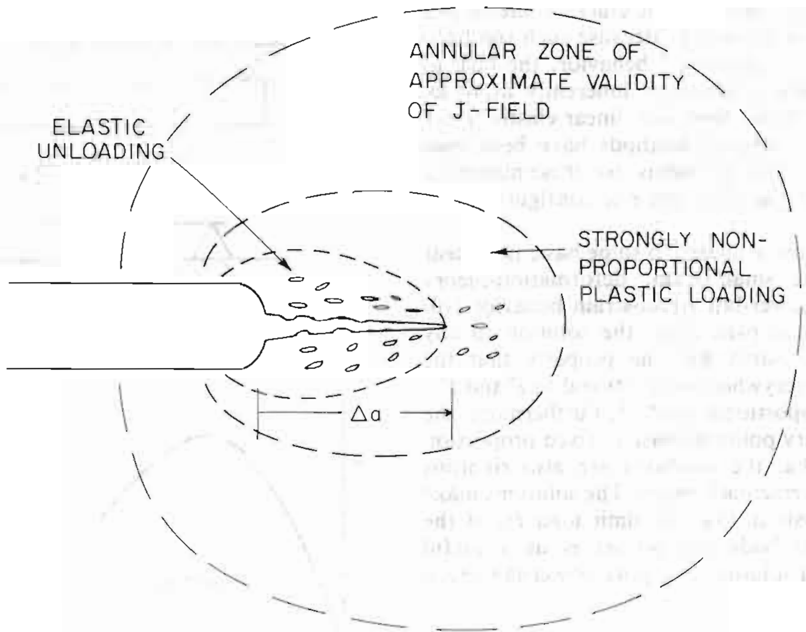

Fig. 8 Schematic of near-tip behavior under *J*-controlled growth conditions

$$J_{Ic} = (1-\nu^2)K_{Ic}^2/E \qquad (20)$$

This basic relation has been verified for a wide variety of metals using different specimen geometries and sizes with J_{Ic} being determined under conditions ranging from intermediate-scale yielding to full yielding. The determination of J_{Ic}, and of K_{Ic}, involves a somewhat arbitrary decision as to what constitutes the onset of crack growth since this is not necessarily a dramatic event especially in tough materials, as will be discussed further in the following.

The condition (18) for *J*-dominance in bend-type configurations under fully plastic yielding was first verified experimentally by noting that departures from (20) begin to appear when (18) is violated at initiation. Note that (18) is approximately equivalent to $b > 40\delta_t$. Condition (19) for the center-cracked configuration is clearly much more restrictive and should only be regarded as a rough estimate since relatively few fully plastic J_{Ic}-tests have been conducted using this configuration. Tests conducted under conditions in which (19) is violated all seem to give J_c values that are larger than J_{Ic}, sometimes by as much as a factor of two. All indications are that the mode *I* plane-strain, *J*-dominance condition is the most critical leading to the smallest J_c. Obviously, this conclusion must he used with care.

6 *J* as Energy Release Rate for the Fictitious Nonlinearly Elastic Body

Curiously, the fact that *J* is the deformation-theory energy release rate is of more mathematical consequence than physical. For this reason, we deliberately delayed discussion of *J* as the energy release rate until after we had spelled out its role as an intensity measure.

Let $PE(a)$ denote potential energy of the sytem in Fig. 1, i.e., the sum of the strain energy in the nonlinearly elastic body and in the generalized spring, assuming the total displacement Δ is prescribed. An alternative expression to the line integral (6) definition of *J* is [2]

$$J = -\frac{dPE}{da} \qquad (21)$$

generalizing the definition of $\mathcal{G}$ for linear-elastic materials. It is clear from its line integral definition that *J* depends only on the current deformation state in the cracked body. In particular, it is independent of the compliance C_M of the loading system. Counterintuitive though it may seem, the energy release rate of the specimen itself at a given current load level is the same whether the load ($C_M \to \infty$) or the displacement ($C_M = 0$) is held constant during the increment of crack advance.

Tempting though it may be, to think of the criterion for initiation of crack growth based on *J* as an extension of Griffith's energy-balance criterion, it is nevertheless incorrect to do so. This is not to say that an energy balance does not exist, just that it cannot be based on the deformation theory *J*. Crack advance in an elastic-plastic material invariably involves elastic unloading and distinctly nonproportional loading in the vicinity of the crack tip, and neither is adequately modeled by deformation theory. Efforts to develop an energy-based criterion for crack initiation and advance have been fraught with difficulty, and so far, at least, no general theory along these lines has achieved acceptance.

The energy-based definition of *J* (21) has been used to derive some very useful formulas for *J* which permit its determination directly from the experimental load-deflection record for a cracked specimen. An example is the deeply cracked bend configuration subject to a moment per unit thickness *M*. With Ω as the rotation through which *M* works minus the corresponding rotation in the absence of the crack, the expression for *J* derived in [14] is

$$J = \frac{2}{b}\int_0^{\Omega} M d\Omega \qquad (22)$$

where *b* is the length of the ligament ahead of the crack. This formula, and modifications of it, are only accurate for cracks that extend at least halfway through the specimen. Nevertheless, the existence of such a formula means that a detailed theoretical analysis of the specimen is not required in order to run a J_{Ic}-test.

7 Basic Solutions to Nonlinear Crack Problems and *J*-Estimation Formulas

Applications of nonlinear fracture mechanics requires the availability of solutions to elastic-plastic crack problems that relate J to load and crack geometry. Because such solutions depend on details of the stress-strain behavior, the task of compiling nonlinear crack solutions is inherently more involved than what has been done for linear-elastic crack problems [15–17]. Finite element methods have been used successfully for solving certain problems, but these numerical solutions are difficult to transfer from one configuration or material to another.

Simple solutions are now available to some basic nonlinear crack problems for the small-strain, deformation-theory material with the pure power-law stress-strain behavior (10) and (11). If P is the load parameter, the solution to any power-law problem necessarily has the property that the stresses and strains are everywhere proportional to P and P^n, respectively, and J is proportional to P^{n+1}. Furthermore, the stress components at every point increase in fixed proportion with increasing P so that the solutions are also rigorous solutions to J_2 flow (incremental) theory. The solutions make contact with limit analysis in that the limit load P_0 of the perfectly plastic cracked body ($n \to \infty$) serves as a useful reference load. A typical solution to a pure power-law crack problem gives

$$\frac{J}{\alpha\sigma_0\epsilon_0 a} = \left(\frac{P}{P_0}\right)^{n+1} h(n, \text{geometry}) \tag{23}$$

where h is a dimensionless function of n and of dimensionless groups of geometric parameters. For most cases, h must be computed numerically; but once computed it can be catalogued, as has been done for a number of basic configurations [18].

Approximate, but highly accurate, solutions have been obtained for two of the most basic nonlinear crack problems, a crack in plane strain and a penny-shaped crack in an infinite body of the power-law material. For a crack of length $2a$ parallel to the x_1 axis and subject to the remote in-plane stress $\sigma_{22} = \sigma$, the plane-strain solution is [19]

$$\frac{J}{\alpha\sigma_0\epsilon_0 a} = \pi\sqrt{n}\left(\frac{\sqrt{3}\sigma}{2\sigma_0}\right)^{n+1} \tag{24}$$

An estimation method has been developed for generating approximate, but reasonably accurate, solutions to elastic-plastic crack problems by combining the linear-elastic solution with the power-law solution [18]. Since the power-law solution provides an accurate estimate under fully plastic yielding, it can be combined with the linear solution to interpolate over the entire range of behavior from small-scale to fully plastic yielding. To illustrate the method consider the crack of length $2a$ in plane strain in an infinite block of material characterized by the Ramberg-Osgood tensile relation (9). The block is subject to an in-plane stress $\sigma_{22} = \sigma$ remote from the crack. In its crudest form, the estimation formula is just the linear combination of the linear and power-law formulas for J comprised to be consistent with (9), i.e.,

$$J = (1-\nu^2)\frac{\pi a}{\mathrm{E}}\sigma^2 + \alpha\sigma_0\epsilon_0 a\sqrt{n}\,\pi\left(\frac{\sqrt{3}\sigma}{2\sigma_0}\right)^{n+1} \tag{25}$$

This formula is clearly asymptotically correct for large and small σ. Improvements in accuracy in the range of intermediate-scale yielding are obtained by the introduction of a crack-length adjustment in the elastic contribution as discussed in [18].

8 J-Controlled Crack Growth

Under certain restrictions, small amounts of crack growth can be correlated in large-scale yielding using J in a way that generalizes the resistance curve analysis of small-scale yielding based on K. It is now common practice to use the formula

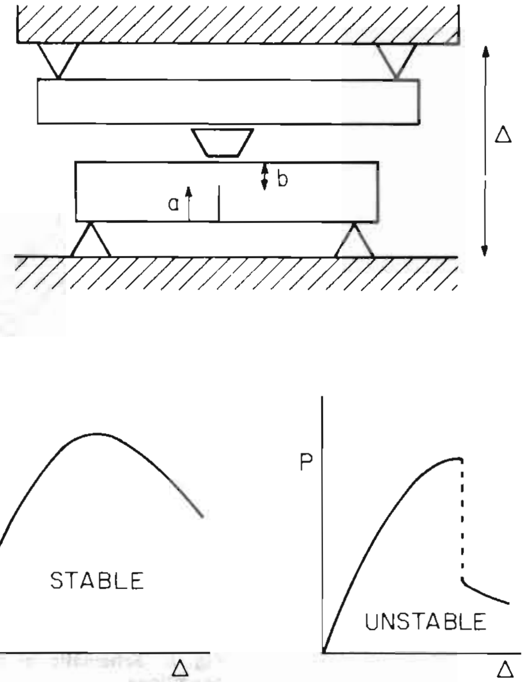

Fig. 9 Cracked specimen in a compliant loading system. Overall load-deflection behavior in presence of stable or unstable crack advance

(22), or an appropriate modification of it, to generate test data for J versus small amounts of crack advance Δa in the form of a J-resistance curve such as that depicted in Fig. 7. Several techniques are available for accurately measuring crack advance, and the observed apparent crack advance associated with crack-tip blunting has been subtracted in Fig. 7. In a typical tough, intermediate-strength steel, the amount of crack advance associated with a doubling of J above J_{Ic} (i.e., $\Delta a \cong D$ in Fig. 7) may be as little as 1 or 2 mm. For many tough materials, resistance curves have been recorded with J-values attaining 5 to 10 times J_{Ic} with J still increasing.

Initially, the J_R-curve was used only as a means of better determining J_{Ic} by using the curve to extrapolate back to $\Delta a = 0$. But it soon became clear that the J_R-curve could be regarded as a material-based curve, in the same sense that the K_R-curve is in small-scale yielding, at least under certain restrictions. Since large increases of J above J_{Ic} are possible in materials with a high tearing resistance, there are considerable practical consequences to design and structural flaw analysis of allowing for small amounts of stable crack growth. Consequently, this subject is being pursued actively in Europe, Japan, the United Kingdom, and the United States [20, 21].

As has already been emphasized, deformation theory on which the J-integral is based does not model elastic unloading or distinctly nonproportional plastic loading, and both effects are present near the crack tip when crack advance occurs. The argument for J-controlled growth requires that the region of elastic unloading and distinct nonproportional loading be contained within the J-dominance zone of the deformation-theory solution. In other words, J will still provide a unique measure for correlating near-tip fracture events if the region within which deformation theory breaks down is well within the zone of dominance. The two conditions for J-controlled growth (in addition to condition (14)) are

$$\Delta a < R \tag{26}$$

and

$$D \equiv J_R \Big/ \left(\frac{dJ_R}{d\Delta a}\right) < R \tag{27}$$

The first of these ensures that the crack advance and associated unloading all take place within the zone characterized uniquely by J, as depicted in Fig. 8. The second is less obvious. As discussed in [22], D is a rough estimate of the size of the region at the tip within which distinctly nonproportional plastic loading occurs. Both conditions should only be regarded as crude statements of the actual conditions in the sense that the numerical coefficients multiplying Δa and D must be of order unity but not necessarily unity itself.

Under fully yielded conditions R is some fraction of the uncracked ligament b. For the bend configuration with $R=0.07b$, equation (27) can be written as

$$\omega \equiv \frac{b}{J_R}\frac{dJ_R}{d\Delta a} > 14 \tag{28}$$

while (26) becomes $\Delta a < 0.07b$. Tests on bend and compact tension specimens of varying sizes and uncracked ligament lengths suggest that (28) is too restrictive for this configuration, and that perhaps $\omega > 5$ to 7 is more realistic. Furthermore, with this condition met, the J_R-curves appear to be geometry-independent for crack advance to as much as about $0.1b$. A modified J-like measure has been proposed [23] which appears promising in its ability to extend the range of the correlation to larger amounts of crack advance. This modified J makes use of the deformation theory J and reduces to it for a stationary crack. Whether it will prove to have general utility for growing cracks still remains to be seen.

9 Stability of J-Controlled Crack Growth

Substantial extra load-carrying capacity is gained if small amounts of crack advance are permitted in materials with high tearing resistance. If crack-growth initiation is permitted, the important issue then centers on the stability of crack growth. A large-scale yielding stability analysis based on the experimental resistance curve, $J_R(\Delta a)$, has been proposed and developed in [24] and exploited for design in [25]. The approach assumes J-controlled growth is in force, uses J as the intensity measure, and is otherwise similar in spirit to the K-based stability analysis in small-scale yielding.

To illustrate this approach consider the plane-strain, cracked bend bar of Fig. 9 loaded in series with a linear spring whose compliance is C_M. The total load point displacement Δ is imagined to be imposed on the system. If the crack has advanced stably by an amount Δa and is poised for further advance, then

$$J = J_R(\Delta a) \tag{29}$$

where J is regarded as the "applied" J associated with the current crack length and load carried by the bend bar. Stability of the crack under prescribed Δ requires that

$$\left(\frac{dJ}{da}\right)_\Delta < \frac{dJ_R}{d\Delta a} \tag{30}$$

in direct analog with the corresponding small-scale yielding condition (3). This condition ensures that any sufficiently small accidental advance of the crack due to some disturbance will result in a J-value that falls below that required for continued advance. Conversely, the crack is expected to start running dynamically when (30) is first violated.

A complete analysis of the bend bar of Fig. 9 is possible. In the case where the material is elastic-perfectly plastic and the bar is deeply cracked ($a/b \geq 1$) and fully yielded, the deformation theory analysis [22, 24] gives

$$\left(\frac{dJ}{da}\right)_\Delta = \frac{4P^2}{b^2}(C_{nc}+C_M) - J \tag{31}$$

Here P is the limit load per unit thickness of the cracked bar and C_{nc} is the elastic compliance of the uncracked bar by itself. The compliance of the loading system C_M strongly enters into the stability condition. Under dead load ($C_M \rightarrow \infty$) the fully yielded, perfectly plastic bend bar is necessarily unstable.

Nondimensional measures of the driving force for instability, dJ/da, and the tearing resistance, $dJ_R/d\Delta a$, were introduced in [24] according to

$$T = \frac{E}{\sigma_0^2}\left(\frac{dJ}{da}\right)_\Delta \quad \text{and} \quad T_R = \frac{E}{\sigma_0^2}\left(\frac{dJ_R}{d\Delta a}\right) \tag{32}$$

where σ_0 is some appropriate tensile yield stress when the material has a hardening capacity. In terms of the nondimensional quantities, the range of stability is

$$T < T_R \tag{33}$$

In [26] a test series was conducted using a system like that in Fig. 9. A sequence of identical specimens in series with springs of differing compliance were tested, and the validity of the stability condition was checked. The material was a

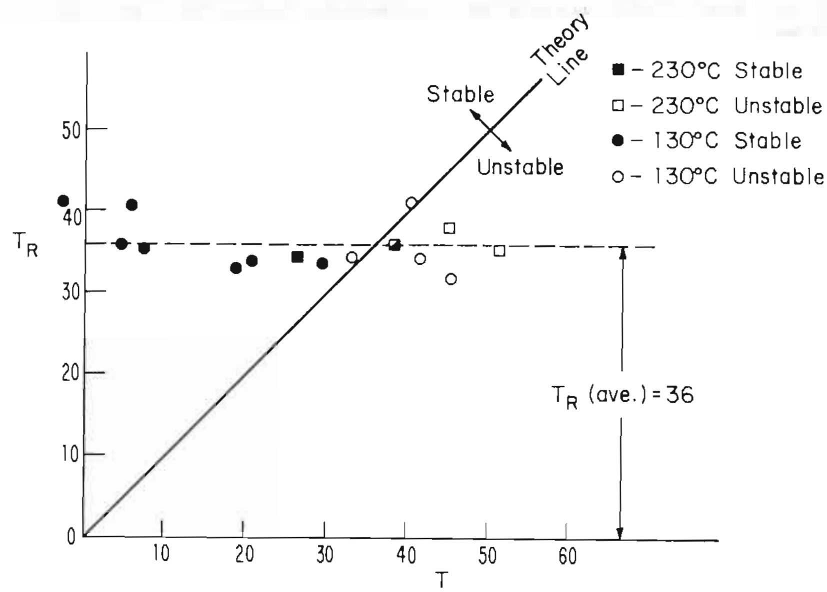

Fig. 10 Experimental data of Paris et al. (1979) showing stable or unstable crack advance

moderately high-strength steel characterized by $D \cong 1.2$ mm and $T_R \cong 36$ just following initiation. In the tests, attention was focused on the stability of crack advance at, or just following, initiation so that the first condition (26) required for J-controlled growth was certainly met. Furthermore, crack advance did occur under fully yielded, or nearly fully yielded, conditions, and the second condition (28) was also met with $\omega = 15$. The test point for each specimen is plotted in Fig. 9 as the values (T, T_R) at initiation. The solid points indicate specimens for which the crack advanced stably, while the open points designate specimens that experienced dynamic crack advance. The transition to instability occurs near $T = T_R = 36$.

Resistance curves are now generated almost routinely and data for a fairly wide variety of materials at temperatures of application are beginning to accumulate. Tough steels with high tearing resistance can have T_R-values in the range from 100 to 200 for crack advances of several millimeters or more. To put this into some perspective, the T-value for a plane-strain crack in an infinite block of fully yielded power-law material from (24) with $\alpha = 1$ is

$$T = \pi\sqrt{n}(\epsilon^{\infty}/\epsilon_0)^{\frac{n+1}{n}} \tag{34}$$

For a strain-hardening exponent associated with a typical intermediate strength steel ($n = 5$ to 10), T will not exceed 100 until the applied strain reaches about 10 times the initial yield strain. Of course, in general, T is a strong function of geometry and of loading compliance and therein lies the potential for sound design against unstable crack advance.

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