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Sandwich Test Specimens for Measuring Interface Crack Toughness*

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Abstract

A crack lying along one interface on an elastic sandwich structure is analyzed. When the thickness of the middle layer is small compared with the other length scales of the structure, a universal relation is found between the actual interface stress intensity factors at the crack tip and the apparent mode I and mode II stress intensity factors associated with the corresponding problem for the crack in the homogeneous material. Therefore, if the apparent stress intensity factors are known, for example calculated from the applied loads as if the structure was homogeneous, this information can be immediately converted into the interface stress intensity factors with the universal relation. This observation provides the theoretical basis for developing sandwich specimens for measuring interface crack toughness. The universal relation reveals the extent to which the asymmetry inherent to a bimaterial interface induces asymmetry in the near tip crack field. In particular, the result of the study can be used to infer whether stress intensity factors for a homogeneous body can be used with good approximation in place of the actual interface stress intensity factors. A proposal for simplifying the approach to interfacial fracture is made which plays down the role of the so-called oscillatory interface singularity stresses.

1. Introduction

Cracks in homogeneous, isotropic materials tend to propagate under mode I conditions in which only normal stress acts on the plane of separation ahead of the tip. For this reason, the development of fracture mechanics for such materials has tended to place heavy emphasis on mode I conditions. By contrast, the fracture mode on an interface of dissimilar materials is often mixed. Differences between elastic properties across an interface will generally disrupt the symmetry even when the geometry and loading are otherwise symmetric with respect to the crack. Moreover, an interface between dissimilar materials is frequently the weakest fracture path in a composite body, and an interface crack will tend to stay in the interface even when subject to loading combinations which give rise to shear stress as well as normal stress on the interface ahead of the tip. Some potential applications of interface fracture mechanics, such as fiber debonding from a matrix due to pull-out, involve substantial shear contributions. Thus, in general, the interfacial fracture mode is inherently mixed, and a complete characterization of an interface requires toughness data over the full range of mode combinations. Recent efforts in this direction are found in [1, 2].

A special class of sandwich specimens have been devised recently for experimental determination of interfacial toughness [3, 4], or for other related purposes such as evaluation of the toughness of adhesive joints [5, 6]. The common feature of these specimens is that each of them is homogeneous except for a very thin layer of second material which is sandwiched between the two halves comprising the bulk of the specimen. The thickness of the layer is typically a hundredth or even a thousandth of the length scale of the overall geometry (Fig. 1). A pre-existing crack lies along one of the interfaces. With such specimens, it has generally been the practice to use the stress intensity factor (or factors if mixed mode conditions pertain) determined for the homogeneous specimen with no layer to characterize the interface crack in the presence of the layer. In this paper we determine a universal relation between the stress intensity factors for the homogeneous specimen or body and the actual interfacial stress intensity factors for the crack between the layer

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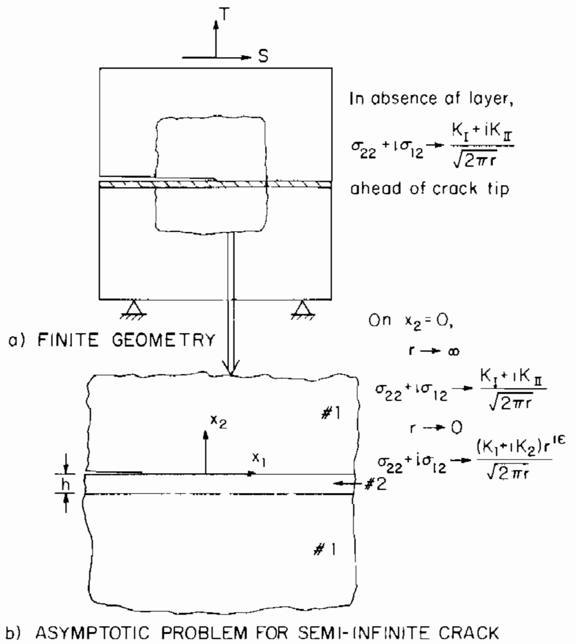

Fig. 1. Schematic of asymptotic problem.

and the material bonded to it. The universal relation is asymptotic in that it requires the thickness of the layer, h, to be very small compared with the crack length and to all other in-plane length scales of the specimen.

The mathematical problem analyzed is introduced in Fig. 1. A thin layer of material 2 is sandwiched in a homogeneous body of material 1. Each material is taken to be isotropic and linearly elastic. Attention is restricted to the plane problem, either plane strain or plane stress. The crack lies along one of the interfaces (the upper interface in Fig. 1) coincident with the x_1-axis with the tip at the origin. As indicated in Fig. 1, the asymptotic problem for the semi-infinite interface crack will be considered, as is appropriate when the layer thickness, h, is very small compared with all other in-plane length scales. The crack tip field of the homogeneous problem (with no layer present) is prescribed as the far field in the asymptotic problem. Thus the far field is characterized by the mode I and mode II stress intensity factors, K_I and K_{II}, induced by the loads on the reference homogeneous specimen. The interface crack tip field is characterized by a different set of interfacial stress intensity factors, K_1 and K_2, which will be defined precisely in Section 2 below. The universal relation developed in the next section connects these two sets of the stress intensity factors. An analogous problem and similar arguments can be found in [7] for a crack parallel to, but slightly displaced from, an interface.

With the universal relation in hand, we outline in Section 3 the procedure to convert the experimental data (*e.g.* the critical external loads) to interfacial toughness using two particular specimen configurations as illustrative examples. A proposal for simplifying the interpretation and presentation of interfacial toughness will be discussed in Section 4.

2. The universal relation

As observed in [8] the non-dimensional elastic moduli dependence of bimaterial systems, for traction prescribed plane elasticity boundary value problems, may be expressed in terms of two (rather than three) special combinations. The Dundurs' parameters adopted in this work are defined as

$$\alpha = \frac{\Gamma(\kappa_2+1)-(\kappa_1+1)}{\Gamma(\kappa_2+1)+(\kappa_1+1)}$$
$$\beta = \frac{\Gamma(\kappa_2-1)-(\kappa_1-1)}{\Gamma(\kappa_2+1)+(\kappa_1+1)} \tag{1}$$

Subscripts 1 and 2 refer to the two materials in Fig. 1, $\kappa = 3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for plane stress, $\Gamma = \mu_1/\mu_2$, ν is the Poisson ratio, and μ is the shear modulus. The physically interesting values of α and β are restricted to a parallelogram enclosed by $\alpha = \pm 1$ and $\alpha - 4\beta = \pm 1$ in the α, β plane. This will be of advantage later on when discussions of any functions depending on material moduli are made. Both α and β vanish when the dissimilarity between the materials does.

Two other bimaterial constants, Σ and ε, may help understand the roles that α and β play, respectively, *i.e.*

$$\Sigma = \frac{c_2}{c_1} = \frac{1+\alpha}{1-\alpha} \qquad \varepsilon = \frac{1}{2\pi} \ln \frac{1-\beta}{1+\beta} \tag{2}$$

where the compliance parameter c is related to Young's modulus E by

$$c = \frac{\kappa+1}{\mu} = \begin{cases} 8/E & \text{for plane stress} \\ 8(1-\nu^2)/E & \text{for plane strain} \end{cases} \tag{3}$$

From (2) α can be readily interpreted as a measure of the dissimilarity in stiffness of the two materials. Material 1 is stiffer than 2 as $\alpha > 0$ and material 1 is relatively compliant as $\alpha < 0$. The parameter ε, and thus β, as will be clear soon, is responsible for the oscillatory behavior at the interface crack tip, and it will be proposed in Section 4 that taking $\beta = 0$ may be a sensible simplifying approximation in many cases.

With K_1 and K_2 as the two interface stress intensity factors and with $K = K_1 + iK_2$ as the complex interface intensity factor [$i = (-1)^{1/2}$], the traction in the interface a distance r ahead of the crack tip is given by

$$\sigma_{22} + i\sigma_{12} = \frac{K}{(2\pi r)^{1/2}} r^{i\varepsilon} \tag{4}$$

The associated crack face displacements a distance r behind the crack tip are

$$\delta_2 + i\delta_1 = \frac{c_1 + c_2}{2(2\pi)^{1/2}(1 + 2i\varepsilon)\cosh(\pi\varepsilon)} K r^{1/2} r^{i\varepsilon} \tag{5}$$

The energy release per unit of new interfacial crack area is related to the complex stress intensity factor by

$$\mathcal{G} = \frac{c_1 + c_2}{16\cosh^2 \pi\varepsilon} |K|^2 \tag{6}$$

These results for the interface singularity field were contained in a number of papers in 1965 [9–11]. The present normalization of the interfacial stress intensity factors follows [1] and [7]. The interfacial stress intensity factors for various crack configurations have not been well documented, yet some important problems have been analyzed. Two examples are depicted in Fig. 2. For a semi-infinite crack along the interface between two elastic half-spaces loaded by equal but opposite tractions $\sigma_{22} + i\sigma_{12} = -T(x_1)$ on the crack faces,

$$K = \left(\frac{2}{\pi}\right)^{1/2} \cosh \pi\varepsilon \int_{-\infty}^{0} \frac{T(t)}{(-t)^{1/2 + i\varepsilon}} \mathrm{d}t \tag{7}$$

In the case of a finite crack of length $2a$ on the interface between two half-spaces which are subjected to equal but opposite tractions $\sigma_{22} + i\sigma_{12} = -T(x_1)$ on the crack faces, the stress intensity factor at the right-hand side tip is

$$K = \left(\frac{2}{\pi}\right)^{1/2} \cosh \pi\varepsilon (2a)^{-1/2 - i\varepsilon} \int_{-a}^{+a} \left(\frac{a+t}{a-t}\right)^{1/2 + i\varepsilon} T(t)\mathrm{d}t \tag{8}$$

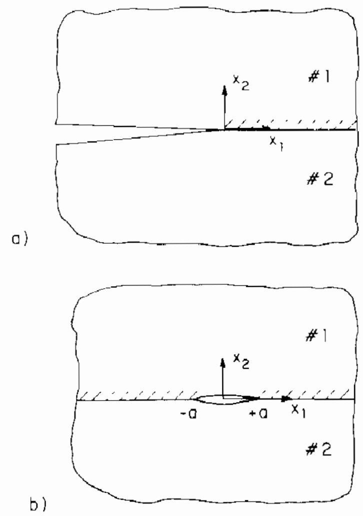

Fig. 2. Two basic interface crack configurations.

When $\beta \neq 0$, and thus by (2) $\varepsilon \neq 0$, K_1 and K_2 do not strictly measure the normal and shear traction singularities, respectively, on the interface since the two traction components do not decouple independent of r due to the term $r^{i\varepsilon} = \exp[i\varepsilon \ln r]$ in (4). Moreover, crack face interpenetration is implied by (5) at sufficiently small r (usually exceedingly small r) when $\varepsilon \neq 0$, as has been discussed in [9]. However, when $\beta = 0$, K_1 and K_2 do measure the normal and shear traction singularities on the interface ahead of the crack tip with the standard definition for the intensity factors. The utility of taking $\beta = 0$ as a pragmatic approximation will be discussed in Section 4. At this point we simply note that interpretation of the interface intensity factors is much clearer when $\beta = 0$, interpenetration is no longer an issue, and it will be argued that little of physical consequence is lost by taking β to be zero in most instances.

The far field for the asymptotic problem in Fig. 1 is characterized by mode I and mode II stress intensity factors, K_I and K_{II}, for the homogeneous specimen. With $K^\infty = K_I + iK_{II}$ as the apparent, or applied, (complex) stress intensity factor, the traction a distance r far ahead of crack tip is

$$\sigma_{22} + i\sigma_{12} = \frac{K^\infty}{(2\pi r)^{1/2}} \tag{9}$$

The energy release rate computed using the far field is

$$\mathcal{G} = \frac{c_1}{8} |K^\infty|^2 \tag{10}$$

By elementary energy arguments, or by application of the J-integral, one notes that the energy release rate in (6) and (10) must be equal and thus

$$|K| = p|K^{\infty}| \tag{11}$$

where

$$p = \left(\frac{1-\alpha}{1-\beta^2}\right)^{1/2} \tag{12}$$

Following an argument similar to that used in ref. 7, dimensional considerations (*cf.* (4) and (9)) and linearity require that

$$Kh^{i\varepsilon} = aK^{\infty} + b\overline{K^{\infty}} \tag{13}$$

where $(\bar{\ })$ denotes complex conjugation and a and b are dimensionless complex constants depending only on α and β. Equations (11) and (13) imply

$$|a|^2 + |b|^2 = p^2 \qquad \bar{a}b = a\bar{b} = 0 \tag{14}$$

The solution to (14) which gives $K = K^{\infty}$ in the limit $\alpha = \beta = 0$ is

$$a = pe^{i\omega} \qquad b = 0 \tag{15}$$

where ω is a real function of only α and β. Now it is possible to rewrite (13) as

$$Kh^{i\varepsilon} = pK^{\infty}e^{i\omega} \tag{16a}$$

or

$$K_1 + iK_2 = p(K_{I} + iK_{II})h^{-i\varepsilon}e^{i\omega} \tag{16b}$$

The universal relation (16) thus has been fully determined apart from $\omega(\alpha, \beta)$. An integral equation formulation of the interface crack problem is given in Appendix A. The function $\omega(\alpha, \beta)$ extracted from the numerical solution to the integral equation is presented in Table 1. As discussed in Appendix A, the error is believed to be within a few tenths of a degree.

The basic relation (16) can be expressed another way. Noting $|Kh^{i\varepsilon}| = |K|$, one can write

$$Kh^{i\varepsilon} = |K|e^{i\psi} \tag{17}$$

as in [1], where ψ is a real phase angle. With $\phi = \tan^{-1}(K_{II}/K_{I})$ so that

$$K^{\infty} = |K^{\infty}|e^{i\phi} \tag{18}$$

Equation (16) can be written equivalently as

$$|K| = p|K^{\infty}| \qquad \psi = \phi + \omega(\alpha, \beta) \tag{19}$$

In words, the complex interface stress intensity factor combination $Kh^{i\varepsilon}$ has magnitude scaled by a factor p and phase angle shifted by ω with respect to the far field stress intensity factor.

For material combinations with $\beta = 0$ and thus $\varepsilon = 0$, K_1 and K_2 have conventional interpretations, as already emphasized, and (16) becomes

$$K = (1-\alpha)^{1/2}K^{\infty}\exp[i\omega(\alpha, 0)] \tag{20}$$

independent of h. Now, $\psi = \tan^{-1}(K_2/K_1)$ measures the relative proportion of "mode II" to "mode I" on the interface. By (19), the shift in the relative proportion in the applied (remote) field to that in the interface field is given by $\omega(\alpha, 0)$. From Table 1, it is noted that this shift is not large, varying from 4.4° for $\alpha = -0.8$ (when the layer material is stiff compared with the bulk material) to $-14.3°$ for $\alpha = 0.8$ (when the layer material is relatively compliant). Thus, for example, a mode I specimen with a sandwich layer will not be "mode I" at the interface crack tip, but the shift will not be large. Much to the expectation of the designers of these kind of specimens, the fracture mode at the crack tip is essentially the same as that induced for the crack in the corresponding homogeneous specimen under the same external loading, but with a scaling factor p.

TABLE 1 **$\omega(\alpha, \beta)$ (in degrees)**

β \ α	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
−0.4	2.2	3.5							
−0.3	3.0	4.0	3.3	1.4					
−0.2	3.6	4.1	3.4	2.0	−0.3	−3.3			
−0.1	4.0	4.1	3.3	2.0	0.1	−2.3	−5.5	−10.8	
0.0	4.4	3.8	2.9	1.6	0.0	−2.1	−4.7	−8.4	−14.3
0.1			2.3	1.1	−0.5	−2.3	−4.5	−7.4	−11.6
0.2					−1.3	−3.0	−4.9	−7.3	−10.5
0.3							−5.8	−7.8	−10.4
0.4									−11.1

3. Applications

The application of the universal relation to a particular sandwich specimen is straightforward. One may start with any specimen which has been successfully used for homogeneous crack fracture test. Proper techniques are required to sandwich a second material layer into the bulk of the specimen and ensure that the crack stays along one of the interfaces, as discussed in [4]. Critical external loads are recorded as the crack starts to propagate. The apparent stress intensity factor, K^{∞}, is then calculated from the critical external loads as if the specimen were homogeneous. The actual interfacial stress intensity factors are readily evaluated using the universal relation (16), or its simplified version (20). Two particular specimens are discussed below for purpose of illustration.

First consider the specimen shown in Fig. 3. A layer of material 2 with thickness h is sandwiched in a large plate of material 1, with overall length scale L. A crack of length $2a$ is introduced at the center of the specimen along the interface. To apply the universal relation, the specimen should be devised such that h is very small compared with the crack half-length a. We will also assume that $L \gg a$ so that the formula for the infinite plane applies, but any formula which accounts for the influence of L on the intensity factors for the homogeneous problem could be used. A uniaxial tensile stress σ is applied at an angle θ to the direction of the layer and crack. The apparent stress intensity factor is simply that of an internal crack in an infinite homogeneous plate due to remote stress [12], *i.e.*

$$K^{\infty} = K_{\mathrm{I}} + \mathrm{i}K_{\mathrm{II}} = \sigma(\pi a)^{1/2} \sin\theta\, \mathrm{e}^{\mathrm{i}\theta} \tag{21}$$

The interface stress intensity factor is then obtained by substituting (21) into (16). That is

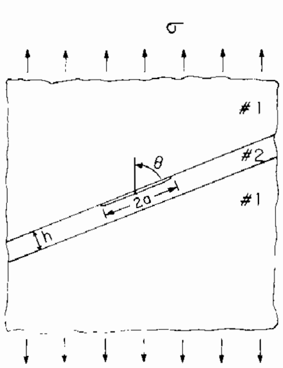

Fig. 3. Sandwich layer and interface crack in finite plane at angle θ to applied tension.

$$Kh^{\mathrm{i}\varepsilon} = p\sigma(\pi a)^{1/2} \sin\theta \exp[\mathrm{i}(\theta+\omega)] \tag{22}$$

With $\beta = 0$, the interface stress intensity factors are

$$K_1 = \sigma[\pi(1-\alpha)a]^{1/2} \sin\theta \cos(\theta+\omega)$$

$$K_2 = \sigma[\pi(1-\alpha)a]^{1/2} \sin\theta \sin(\theta+\omega) \tag{23}$$

Observe that (23) implies the toughness $|K_c|$ can be measured using this specimen over a wide range of phase angles $\psi = \tan^{-1}(K_2/K_1)$ by continuously varying the direction of the load θ.

Next consider the double cantilever beam (DCB) specimen proposed in [4], consisting of a thin film of medium 2 bonded between substrates of medium 1 (Fig. 4). The apparent stress intensity factor associated with the corresponding homogeneous specimen, determined from the previous numerical solution [13] in terms of applied load per unit thickness P, crack length a, and half-height l, is

$$K^{\infty} = K_1 = Pl^{-3/2} a[3.467 + 2.315(l/a)] \tag{24}$$

Substituting (24) into (16) gives the interface stress intensity factor of the crack tip as

$$Kh^{\mathrm{i}\varepsilon} = pPl^{-3/2} a[3.467 + 2.315(l/a)]\mathrm{e}^{\mathrm{i}\omega} \tag{25}$$

If the simplifying assumption is made, *i.e.* $\beta = 0$, (25) becomes

$$K = (1-\alpha)^{1/2} Pl^{-3/2} a[3.467 + 2.315(l/a)]\mathrm{e}^{\mathrm{i}\omega} \tag{26}$$

Since ω is typically very small according to Table 1, this is essentially a mode I specimen, as anticipated [4].

4. On the virtues of taking $\beta = 0$

By conducting fracture tests over a full range of external loading (*i.e.* a full range of $\phi = \tan^{-1}(K_{\mathrm{II}}/K_{\mathrm{I}})$, one generates a locus of the critical combinations of the interface intensity factors, K_1 and K_2. A thorough discussion of several approaches to recording and using interfacial fracture data is given in [1]. In particular, if

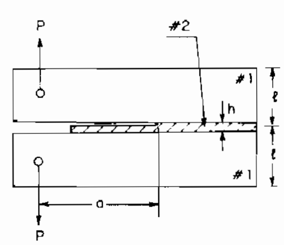

Fig. 4. Double-cantilever beam specimen with sandwich layers.

full accounting for the ε-effects is made, then a critical value of $K = K_1 + iK_2$ must be reported in units of stress σ and length L as

$$K = \Lambda \sigma L^{1/2 - i\varepsilon} \tag{27}$$

where Λ is a dimensionless complex number. A peculiar consequence of (27) is that the phase, *i.e.* the relative proportion of K_1 to K_2, changes when length units are changed. Moreover, from (4) it can be seen that the relative proportion of shear to normal stress acting on the interface directly ahead of the tip is not constant but varies (weakly) according to $r^{i\varepsilon} = \exp(i\varepsilon \ln r)$ when $\varepsilon \neq 0$.

Plane strain values of α, β and ε were listed in [7] for six representative material pairs. In most of these cases, ε is very small, often less than 0.01 in magnitude, even when α is substantial corresponding to ratios of 4 or 5 of the plane strain modulus $E/(1-\nu^2)$ of the two materials. At the present stage of the development of the mechanics of interfacial fracture it is likely that other problems and issues, such as the difficulty in preparing specimens and in measuring interfacial toughness, are much more pertinent than ε-effects. Certainly, there is no compelling experimental evidence to date which suggests an important role for ε, and various proposals for ignoring ε-effects have been considered [1]. A consistent approach proposed in [14] is to systematically take $\beta = 0$, both in the determination of critical toughness data from test specimens and in subsequent application of such data to predict fracture. Of the two non-dimensional parameters, α and β, measuring dissimilarity in material elastic properties, α appears to be the more important. For example, in the present solution (16), β enters in the factor p only as β^2 and for a typical plane strain β value makes a very small numerical contribution to p. Similarly, its lowest-order influence on the relation between K and the energy release rate in (6) is only of order β^2 through cosh $\pi\varepsilon$. It is also noted that ω in Table 1 is a stronger function of α than of β and for typical β values is hardly influenced. A similarly weak dependence on β of the solution variables for intensity factors for a crack kinking out of an interface was noted in [14]. Curiously, the several solutions for intensity factors for cracks on the interface between two semi-infinite blocks of materials produced in 1965 [9–11], two of which are listed here as (7) and (8), have no dependence on α but do depend weakly on ε and, therefore, on β. This may partly explain why ε-effects may have been overemphasized. In any case, at this stage in the development of the subject it seems sensible to take $\beta = 0$, especially when β is small, in view of the clarification in interpretation and simplification in approach which thereby follows. A safe procedure would be to report data in a manner which would permit conversion to an ε-based scheme at a later date if that turns out to be necessary. Guidelines can be found in ref. 1.

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Appendix A. Integral equation formulation and solution

In this Appendix we set up and solve the integral equation for the plane elasticity problem specified in Fig. 1. Similar solution procedure can be found in refs. 7, 15, 16. A layer of material 2 is sandwiched in an infinite medium of material 1. Each material is taken to be isotropic and linearly elastic with the x_1-axis coincident with the upper interface. The thickness of the layer is set to be unity since the h dependence is known. A semi-infinite crack lies along the x_1-axis with the tip at the origin. The external loading is prescribed in the far field as the standard crack tip field of a homogeneous crack characterized by the classical stress intensity factor

$$K^\infty = K_{\mathrm{I}} + \mathrm{i}K_{\mathrm{II}} \tag{A1}$$

Let $b_i(\xi)$ be the x_i component of an edge dislocation located on the interface at $x_1 = \xi$. The stresses at point $x_1 = x$ on the interface induced by the dislocation are given by

$$\sigma_{22}(x) + \mathrm{i}\sigma_{12}(x) = \frac{2\bar{B}(\xi)}{x-\xi} + 2\pi\beta\mathrm{i}\delta(x-\xi)\bar{B}(\xi) + B(\xi)F_1(x-\xi) + \bar{B}(\xi)F_2(x-\xi) \tag{A2}$$

where $\delta(x)$ is the Dirac delta function and

$$B(\xi) = \frac{1+\alpha}{c_2(1-\beta^2)}\frac{1}{\pi\mathrm{i}}[b_1(\xi) + \mathrm{i}b_2(\xi)] \tag{A3}$$

and the complex-valued functions $F_i(\zeta)$ are constructed in Appendix B. The functions F_1 and F_2 are well behaved in the whole range $-\infty < \zeta < +\infty$, with asymptotes

$$F_1(\zeta) = O\left(\frac{1}{\zeta}\right), F_2(\zeta) = O\left(\frac{1}{\zeta}\right) \qquad \text{as } \zeta \to \infty \tag{A4}$$

The semi-infinite crack is represented by a distribution of dislocations lying along the negative x_1-axis such that the traction vanishes along the negative x_1-axis. That is, the distribution $B(\xi)$ for $\xi < 0$ must be governed by

$$\int_{-\infty}^{0}\left(\frac{2}{x-\xi} + F_2(x-\xi)\right)\bar{B}(\xi)\mathrm{d}\xi + \int_{-\infty}^{0} F_1(x-\xi)B(\xi)\mathrm{d}\xi + 2\pi\beta\mathrm{i}\bar{B}(x) = 0 \qquad \text{for } x<0 \tag{A5}$$

where the first integral is the Cauchy principal value integral.

The crack face displacements are related to the dislocation distribution by

$$\delta_1(x) + \mathrm{i}\delta_2(x) = \int_x^0 [b_1(\xi) + \mathrm{i}b_2(\xi)]\mathrm{d}\xi = \pi\mathrm{i}c_2\frac{1-\beta^2}{1+\alpha}\int_x^0 B(\xi)\mathrm{d}\xi \qquad \text{for } x<0 \tag{A6}$$

The relation between the complex stress intensity factor K and the dislocation distribution B can be derived by combining (5) and (A6). That is

$$\bar{K} = (2\pi)^{3/2}(1-\beta^2)^{1/2}\lim_{x\to 0^-}\frac{B(x)}{(-x)^{-1/2-\mathrm{i}\epsilon}} \tag{A7}$$

The behavior of $B(\xi)$ as $\xi \to -\infty$ can be specified to give the correct far field loading (A1), *i.e.*

$$B(\xi) = (2\pi)^{-3/2}\frac{1-\alpha}{1-\beta^2}\bar{K}^\infty(-\xi)^{-1/2} \qquad \text{as } \xi \to -\infty \tag{A8}$$

Notice that with asymptotic behaviors (A4) and (A8), the integrands in (A5) are integrable.

Make the change of variables

$$x = \frac{u-1}{u+1} \qquad -1<u<1$$

$$\xi = \frac{t-1}{t+1} \qquad -1<t<1 \tag{A9}$$

and let

$$\zeta \equiv x - \xi = \frac{2(u-t)}{(u+1)(t+1)} \tag{A10}$$

Then with $A(t) \equiv B(\xi)$, the integral equation (A15) can be reduced to

$$\int_{-1}^{1}\frac{\bar{A}(t)}{u-t}\mathrm{d}t + \pi\beta\mathrm{i}\bar{A}(u) + \int_{-1}^{1}\frac{F_1(\zeta)A(t) + [1+t+F_2(\zeta)]\bar{A}(t)}{(1+t)^2}\mathrm{d}t = 0 \qquad \text{for } -1<u<1 \tag{A11}$$

where the first integral is the Cauchy principal value integral. With the asymptotic behaviors

(A7) and (A8) in mind, one can take the approximation for $A(t)$ as

$$A(t)=\left(\frac{1-t}{2}\right)^{-1/2-\mathrm{i}\varepsilon}\left[a_0\left(\frac{1+t}{2}\right)^{1/2}+(1+t)\sum_{k=1}^{N}a_kT_{k-1}(t)\right] \tag{A12}$$

where

$$a_0=(2\pi)^{-2/3}\frac{1-\alpha}{1-\beta^2}\bar{K}^{\infty} \tag{A13}$$

and $T_j(t)$ is the Chebyshev polynomial of the first kind of degree j, and the a values are complex coefficients which must be determined in the solution process. When substituted into (A11), the representation for A leads to an equation of the form

$$\sum_{k=1}^{N}[a_kI_1(u,k)+\bar{a}_kI_2(u,k)]+a_0I_3(u)=0 \tag{A14}$$

where the terms I_j for $j=1$, 3 involve integrals such as

$$I_1(u,k)=\int_{-1}^{1}F_1(\zeta)T_{k-1}(t)(1+t)^{-1}\left(\frac{1-t}{2}\right)^{-1/2-\mathrm{i}\varepsilon}\mathrm{d}t \tag{A15}$$

These integrals must be evaluated numerically for given values of u and k.

The solution procedure is as follows. Let a set of $2N$ real unknowns be the real and imaginary parts of a_k for $k=1$, N. This set of $2N$ unknowns is used to satisfy the real and imaginary parts of (A11) at N Gauss–Legendre points $\{u_i\}$ on the interval $-1<u<1$. Once the a values have been determined, the complex stress intensity factor can be computed, using (A7) and (A12), from

$$\bar{K}=(2\pi)^{3/2}(1-\beta^2)^{1/2}\left(a_0+2\sum_{k=1}^{N}a_k\right) \tag{A16}$$

The general expression for K in (14) applies to the present case with $K_{\mathrm{I}}=1$ and $K_{\mathrm{II}}=0$, or equivalently, with $K^{\infty}=1$, and $h=1$, so that

$$K=p\mathrm{e}^{\mathrm{i}\omega} \tag{A17}$$

which yields $\sin\omega$ and $\cos\omega$ independently. The relation $\sin^2\omega+\cos^2\omega=1$ provides a consistency check on the accuracy of the solution. The results reported in Table 1 were computed with N between 10 and 20. The consistency check was satisfied to better than 0.3%. It is believed that the error of ω is within a few tenths of a degree.

Appendix B. A dislocation in the sandwich structure

The dislocation solution used as the kernel in the integral equation (A5) is summarized here.

The plane elasticity problem is specified in Fig. 5. An edge dislocation with components b_1 and b_2 at the origin lies on the upper interface of the bonded sandwich structure. The solution to the problem in Fig. 5 is obtained by the similar superposition technique used in [16]. Only the final results are reported below. The stresses at $(\zeta, 0)$ induced by the dislocation at the origin are given by

$$\sigma_{22}(\zeta,0)+\mathrm{i}\sigma_{12}(\zeta,0)=\bar{B}\left(\frac{2}{\zeta}+2\pi\beta\mathrm{i}\delta(\zeta)+F_2(\zeta)\right)+BF_1(\zeta) \tag{B1}$$

where $\delta(x)$ is the Dirac delta function and

$$B=\frac{1+\alpha}{c_2(1-\beta^2)}\frac{1}{\pi\mathrm{i}}(b_1+\mathrm{i}b_2) \tag{B2}$$

The complex-valued functions $F_i(\zeta)$ are determined by

$$\begin{aligned}F_1(\zeta)&=[Q_2(\zeta)-R_1(\zeta)]+\mathrm{i}[Q_1(\zeta)+R_2(\zeta)]\\F_2(\zeta)&=[Q_2(\zeta)+R_1(\zeta)]+\mathrm{i}[R_2(\zeta)-Q_1(\zeta)]\end{aligned} \tag{B3}$$

where the Q and R are defined by Fourier integrals

$$\begin{aligned}Q_1(\zeta)&=\int_0^{\infty}(-A_1)\cos\zeta\lambda\,\mathrm{d}\lambda\\R_1(\zeta)&=\int_0^{\infty}(-A_1+A_2)\sin\zeta\lambda\,\mathrm{d}\lambda\\Q_2(\zeta)&=\int_0^{\infty}(-B_1)\sin\zeta\lambda\,\mathrm{d}\lambda\\R_2(\zeta)&=\int_0^{\infty}(B_1-B_2)\cos\zeta\lambda\,\mathrm{d}\lambda\end{aligned} \tag{B4}$$

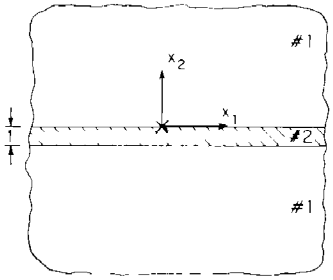

Fig. 5. Geometry for dislocation solution used in constructing the integral equation.

The A and B are solved from the linear algebraic equations

$$[P_1 P_2]\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \\ A_4 & B_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix} \tag{B5}$$

where $P_1 = CD$ and

$$C = -\begin{bmatrix} e^{\lambda} & -\lambda e^{\lambda} & e^{-\lambda} & -\lambda e^{-\lambda} \\ -e^{\lambda} & (1+\lambda)e^{\lambda} & e^{-\lambda} & (1-\lambda)e^{-\lambda} \\ 0 & \frac{\Sigma}{2}e^{\lambda} & 0 & -\frac{\Sigma}{2}e^{-\lambda} \\ 0 & \frac{\Sigma}{2}e^{\lambda} & 0 & \frac{\Sigma}{2}e^{-\lambda} \end{bmatrix}$$

$$D = \frac{1}{1+\alpha}\begin{bmatrix} 1+\beta & -\beta \\ 0 & 1-\beta \\ \alpha-\beta & \beta \\ -2(\alpha-\beta) & \alpha-\beta \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & -\lambda \\ 1 & 1-\lambda \\ \frac{\alpha-\beta}{1-\alpha} & -\frac{\alpha-\beta}{1-\alpha}\lambda - \frac{1}{2} \\ \frac{\alpha-\beta}{1-\alpha} & \frac{\alpha-\beta}{1-\alpha}(1-\lambda)+\frac{1}{2} \end{bmatrix} e^{-\lambda} \tag{B6}$$

where $\Sigma = (1+\alpha)/(1-\alpha)$, and

$$\begin{aligned} X_1 &= \left[-\frac{\alpha-\beta^2}{1-\alpha} + \frac{(\alpha-\beta)(1-\beta)}{1-\alpha}\lambda\right]e^{-\lambda} \\ X_2 &= \left[\beta + \frac{(\alpha-\beta)(1-\alpha)}{1-\alpha}\lambda\right]e^{-\lambda} \\ Y_1 &= \left[-\beta + \frac{(\alpha-\beta)(1-\alpha)}{1-\alpha}\lambda\right]e^{-\lambda} \\ Y_2 &= \left[\frac{\alpha-\beta^2}{1-\alpha} + \frac{(\alpha-\beta)(1-\alpha)}{1-\alpha}\lambda\right]e^{-\lambda} \end{aligned} \tag{B7}$$