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# The path-independent M integral implies the creep closure of englacial and subglacial channels

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*Drainage channels are essential components of englacial and subglacial hydrologic systems. Here we use the M integral, a path-independent integral of the equations of continuum mechanics for a class of media, to unify descriptions of creep closure under a variety of stress states surrounding drainage channels. The advantage of this approach is that the M integral around the hydrologic channels is identical to same integral evaluated in the far-field. In this way, the creep closure on the channel wall can be determined as a function of the far-field loading, e.g. involving antiplane shear as well as overburden pressure. We start by analyzing the axisymmetric case and show that the Nye solution for the creep closure of the channels is implied by the path-independence of the M integral. We then examine the effects of superimposing antiplane shear. We show that the creep closure of the channels acts as a perturbation in the far-field, which we explore analytically and numerically. In this way, the creep closure of channels can be succinctly written in terms of the path-independent M integral and understanding the variation with applied shear is useful for glacial hydrology models.*

## Nomenclature

$A$ Ice softness
$\varepsilon_{ij}$ Strain field
$I_i$ Strain rate tensor invariants
$M$ Path-independent integral
$W_s$ Strain energy density
$W$ Flow potential
$u_i$ Displacement field
$\ell$ Dimension, i.e. 2 or 3.
$m$ Homogeneity degree, $W_s = \sigma_{ij}\varepsilon_{ij}/m$
$n$ Rheological exponent, i.e. 3 for Glen's law
$p$ Isotropic ice pressure
$v_i$ Velocity field
$x_i$ Position field
$D_{ij}$ Strain rate field
$\sigma_{ij}$ Stress field
$\sigma'_{ij}$ Deviatoric stress field, $\sigma_{ij} + p\delta_{ij}$
$D_E$ Effective strain rate, $\sqrt{D_{ij}D_{ij}/2}$
$\tau_E$ Effective stress, $\sqrt{\sigma'_{ij}\sigma'_{ij}/2}$
$I_k$ Invariant of strain rate tensor
$a$ Channel radius
$\Delta p$ Effective pressure
$\sigma_0$ Ice overburden pressure, $\rho_i g H$
$\rho_i$ Ice density
$g$ Acceleration due to gravity
$H$ Ice thickness
$p_f$ Fluid pressure within channel
$\dot{\gamma}_{far}$ Far-field antiplane strain rate
$S$ Strain rate ratio, $\dot{\gamma}_{far}/(A\Delta p^n)$

## 1 Introduction

Glacial melt water from surface ablation, precipitation, and internal deformation drains via englacial conduits to the base of the glacier where it is evacuated through a subglacial hydrologic network of channels melted into the ice or cavities in the sediments [1, 2, 3]. In this paper, we focus

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on channelized drainage through Röthlisberger channels [2], where these channels are melted into the ice by the heat dissipation of the turbulently flowing melt water and close by viscous creep of the surrounding ice. We model these channels as very long straight conduits that are oriented along the direction of glacier flow and we use a conserved integral approach to derive the classical solution found by Nye [4] for the radial, or in-plane, creep closure velocity of the ice into the channel. We then show how the creep closure of the channels increases when we take the downstream shear present within the ice column, referred to as antiplane shear, into account, such as in an ice stream shear margin.

Path-independent (conserved) integrals are important mathematical tools that are often employed in mechanics as a method of solution to the equations or as a supplemental constraint. Günther [5], and independently Knowles and Sternberg [6], introduced a path-independent integral for linear elastic solid mechanics, which Budiansky and Rice [7] called the $M$ integral. Using the Noether [8] procedure, the $\ell$-dimensional, linear elastic $M$ integral is the conservation integral that results from a self-similar scale change by the infinitesimal factor $\gamma$, i.e.

$$x'_\alpha = x_\alpha + \gamma x_\alpha \quad \text{and} \quad u'_\alpha(\underline{x}') = u_\alpha(\underline{x}) + \left(1 - \frac{\ell}{2}\right)\gamma u_\alpha(\underline{x}),$$

That is, coordinates $x_\alpha$ and displacements $u_\alpha$ are self-similarly scaled from the reference configuration [9]. In the framework of linearized kinematics, the strain is equal to the symmetric part of the displacement $u_i$ gradient tensor as $\varepsilon_{ij} = \mathrm{sym}\,(\partial u_i/\partial x_j)$. We can then write a strain energy density $W_s$ as a product of stresses $\sigma_{ij}$ and strains $\varepsilon_{ij}$ by $W_s = \sigma_{ij}\varepsilon_{ij}/2$. The general conserved integral resulting from the Noether procedure, with $y_\alpha = x'_\alpha - x_\alpha$ and $f_\alpha(\underline{x}) = u'_\alpha(\underline{x}') - u_\alpha(\underline{x})$, is given by

$$\oint_\Gamma \left\{ W_s y_\alpha n_\alpha + \sigma_{\alpha k} n_k \left[ f_\alpha(\underline{x}) - y_\beta \frac{\partial u_\alpha}{\partial x_\beta} \right] \right\} ds.$$

where this integral is a line integral for 2-dimensional problems and a surface integral for 3-dimensional problems. In this way, the the $M$ integral for a linear elastic material in two dimensions with linearized kinematics is given as

$$M = \oint_\Gamma \left\{ W_s x_\alpha n_\alpha - \sigma_{\alpha k} n_k x_\beta \frac{\partial u_\alpha}{\partial x_\beta} \right\} ds.$$

as written by [6].

Budiansky and Rice [7] extended the earlier definitions of $M$ to a generalized elastic material with a strain energy density $W_s$ that is homogeneous of degree $m$ in the strains $\varepsilon_{ij}$, and, therefore, $W_s = \sigma_{ij}\varepsilon_{ij}/m$. Unfortunately, the [7] expression for the generalized $M$ integral contains an error. Whether it is typographical, conceptual, or due to the printing process is unknown. He and Hutchinson [10] give a correct expression (although without derivation) for the three-dimensional generalized $M$ integral in a different geometry than used here or in [7]: a closed 3-D void or flat crack of axially symmetric shape, such that stresses vary with $z$ and $\sqrt{x^2+y^2}$. Rice [9] gives the correct Noether invariant transformation to generate the $\ell$-dimensional $M$ integral for a power law solid, although, subsequently, only writes the expression of the $M$ integral for the linear ($m = 2$) material in two dimensions. To set the record straight, the generalized $M$ integral in two dimensions with combined in-plane and antiplane straining in the $y$-$z$ plane, and with void aligned in the $x$ direction, is

$$M = \oint_\Gamma \left\{ W_s x_k n_k + \sigma_{ik} n_k \left[ \left(\frac{m-2}{m}\right) u_i - x_j \frac{\partial u_i}{\partial x_j} \right] \right\} ds, \quad (1)$$

where $n_i$ is the unit outer normal to the closed contour $\Gamma$ and $s$ is arc length anti-clockwise around the path, such that $n_1 ds = dx_2$ and $n_2 ds = -dx_1$, where the tensor subscripts correspond as $(x,y,z) = (x_1,x_2,x_3)$ and $(u,v,w) = (u_1,u_2,u_3)$. Figure 1 shows the domain, the coordinate system, and a path of integration about a void.

The generalized definition of $M$ for a power-law nonlinear elastic solid is exactly equivalent to the definition for a power-law nonlinear viscous fluid, where the displacement $u_i$ is replaced by the velocity $v_i$, and the strain $\varepsilon_{ij}$ by the strain rate $D_{ij}$ [11]. The strain rate is the symmetric part of the velocity gradient tensor, as $D_{ij} = \mathrm{sym}\,(\partial v_i/\partial x_j)$. Under the definition for a power-law viscous fluid, the elastic strain energy density $W_s$ is a replaced by a function called the flow potential $W$. Just as $dW_s = \sigma_{ij} dD_{ij}$ is an exact differential in generalised elasticity, $dW$ is also an exact differential, satisfying $dW = \sigma_{ii} dD_{ij}$. From here on, we will use the notation related to the flow of a viscous fluid. This change of notation and extension to viscous fluids allow us to apply the $M$ integral to the flow of ice in glaciers. In the notation of viscous fluids, the 2-dimensional $M$ integral is written as

$$M = \oint_\Gamma \left\{ W x_k n_k + \sigma_{ik} n_k \left[ \left(\frac{m-2}{m}\right) v_i - x_j \frac{\partial v_i}{\partial x_j} \right] \right\} ds. \quad (2)$$

We include a proof of the path independence of Equation (2) in appendix A.

The flow potential $W$ for a viscous fluid is given as

$$\frac{\partial W}{\partial D_{ij}} = \sigma_{ij} + p\delta_{ij}, \quad W = \frac{1}{m}\sigma_{ij}D_{ij}, \quad \text{and} \quad dW = \sigma_{ij} dD_{ij},$$

where $p$ $(= -\sigma_{kk}/3)$ is the isotropic pressure (understood here as a Lagrange multiplier to enforce mass conservation, $\varepsilon_{kk} = 0$) and $dW$ is an exact differential. The strain rates are defined in terms of derivatives of the velocity $v_i$ as

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad \text{with} \quad \frac{\partial v_k}{\partial x_k} = 0,$$

where the first condition shows that $D_{ij}$ is symmetric and by the second condition (mass conservation for an incompressible substance) $D_{ij}$ is trace free.

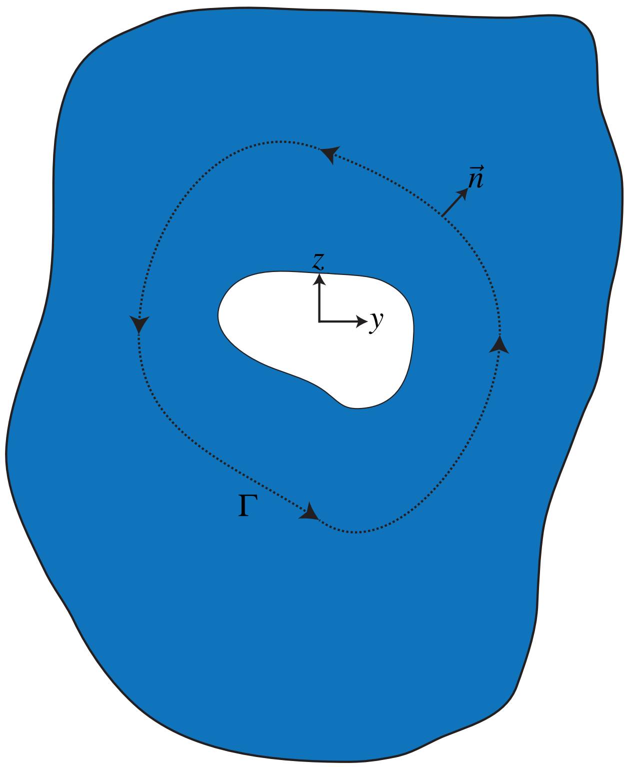


Fig. 1. Schematic of the path-independent $M$ integral around a void. The path of integration is $\Gamma$ and the vector normal to this integration path is $\vec{n}$. The down glacier component is $x$ which points out of the page, i.e. antiplane, the in-plane coordinates are $y$ (across glacier) and $z$ (depth).

### 1.1 Ice Rheology

Here we describe how the rheology sets the value for the parameter $m$. In glaciology, a commonly assumed rheology is Glen's law,

$$D_E = A\tau_E^n, \tag{3}$$

where $D_E$ is a function of the second invariant of the strain rate tensor, i.e. $D_E = \sqrt{D_{ij}D_{ij}/2}$, $\tau_E$ is defined in the same way for the deviatoric stress tensor $\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}$, and the subscript $E$ stands for effective. The two parameters are $A$, the ice softness, and $n$, the rheological exponent. The standard value used in glaciology is $n = 3$ and is called Glen's law, which is appropriate for the typical values of stress and strain rate encountered in the field [12]. Now relating deviatoric stress and strain rate tensors using Equation (3) implies that

$$D_{ij} = A\tau_E^{n-1}\left(\sigma_{ij} + p\delta_{ij}\right). \tag{4}$$

Using this rheology the flow potential $W$ can be determined as

$$W = \int \sigma_{ij} dD_{ij} = \frac{n}{n+1}\sigma_{ij}D_{ij} = \frac{1}{m}\sigma_{ij}D_{ij},$$

and, thus, $m = 1 + 1/n$. From here on we will use the rheological exponent $n$ instead of $m$; for a Newtonian viscous fluid (or in linear elasticity), $n = 1$ and, therefore, $m = 2$, so the term proportional to $v_i$ in Equation (2) will disappear.

In appendix B we show the general class of incompressible viscous fluids for which a flow potential $W$ exists, which is required for the $M$ integral. All purely viscous incompressible fluids fall into the general class of Reiner-Rivlin fluids, which have a rheology given by

$$\sigma_{ij} = -p\delta_{ij} + \phi_1(I_2, I_3)D_{ij} + \phi_2(I_2, I_3)D_{in}D_{nj},$$

where $I_k$ is the $k$th invariant of the tensor $D_{ij}$ [13]. For a three-dimensional flow, the three invariants are

$$I_1 = D_{kk}, \quad I_2 = \frac{1}{2}D_{lm}D_{ml}, \quad I_3 = \frac{1}{3}D_{lm}D_{mn}D_{nl}.$$

For a fluid that is incompressible,

$$I_1 = 0.$$

Truesdell and Noll [14] assert that there is little experimental evidence for fluids with $\phi_2 \neq 0$. This assertion is based on Markovitz and Williamson [15], who find the polymeric data collected by [16] to be incompatible with $\phi_2 \neq 0$. In glaciology, it is not fully resolved whether Glen's law should be expanded to include a dependence on $I_3$ or $\phi_2 \neq 0$. Glen [17] describes ice as a Reiner-Rivlin fluid and concludes that the experimental data show sufficient scatter to warrant further study. Baker [18] reviews the subsequent experiments in determining the effects of the third invariant on the flow of ice and compares the results with his own experimental set-up, which show that there is a significant dependence on $I_3$. Still there appears to be little evidence that $\phi_2 \neq 0$ in ice. Such a fluid, as [17] notes, would to susceptible to swelling or contraction in the direction perpendicular to the plane of shear. Schoof and Clarke [19] exploit this generation of deviatoric normal stress and use a Reiner-Rivlin fluid to model subglacial flutes by way of a secondary transverse flow. Here we show that only Reiner-Rivlin fluids that are independent of $I_3$ with $\phi_2 = 0$ have a flow potential $W$, unless

$$\frac{\partial \phi_1}{\partial I_3} = \frac{\partial \phi_2}{\partial I_2}.$$

Therefore, our analysis primarily applies for ice modeled using a power-law rheology for ice, such as Glen's law.

## 2 Analysis

To analyze the creep closure of an drainage channel, it is convenient to write $M$ in polar coordinates and adopt a circular path of radius r ($ds = rd\theta$) as

$$M = \int_0^{2\pi} \left\{ Wr - \left(\frac{n-1}{n+1}\right) [\sigma_{rr} v_r + \sigma_{r\theta} v_\theta + \sigma_{rx} v_x] - \left[ \sigma_{rr} \frac{\partial v_r}{\partial r} + \sigma_{r\theta} \frac{\partial v_\theta}{\partial r} + \sigma_{rx} \frac{\partial v_x}{\partial r} \right] r \right\} rd\theta. \quad (5)$$

In this expression there are two types of terms: in-plane and antiplane. The in-plane terms are those in the $r$ and $\theta$ direction, such as $v_r$, $v_\theta$, and $\sigma_{r\theta}$. The antiplane terms, quantities with a subscript $x$, represent motion into and out of page as a function of only the in-plane coordinates (using the standard glaciology coordinate system with $z$ vertical, $y$ across glacier, and $x$ down glacier). We consider a very long channel with constant ice thickness and, in this way, we can reduce a three dimensional problem to two dimensions where the quantities are homogeneous along $x$.

### 2.1 Nye solution

Nye [4] derived the rate of closure of a circular channel in a Glen rheology subject to a stress $\sigma_{rr}(r = a) = -p_w$ (water pressure) applied at the channel and the stress $\sigma_{rr}(r \to \infty) = -\sigma_0 = -\rho_i gH$ (overburden ice pressure for a glacier of height $H$ and density $\rho_i$) far away. By adding a uniform tensile stress $\sigma_0$ to the mass of ice, we transform our problem and apply a tensile stress $\sigma_{rr}(r = a) = \sigma_0 - p_w = \Delta p$ at the channel wall and a traction free condition at infinity. We are able to do this without changing the problem because of incompressibility and the pressure independence of Glen's law. Thus, the boundary conditions are

$$\sigma_{rr}(r = a) = \sigma_0 - p_w = \Delta p \quad \text{and} \quad \sigma_{rr}(r \to \infty) = 0.$$

The set-up for the problem and these conditions can be seen in Figure 2. What is also evident is that the problem is purely in-plane and, therefore, we disregard the antiplane terms in Equation (5). Furthermore, the problem is axisymmetric and so we can neglect the in-plane shear terms. Hence, we have the integral

$$M = \int_0^{2\pi} \left\{ Wr^2 - \left(\frac{n-1}{n+1}\right) \sigma_{rr} v_r r - \sigma_{rr} \frac{dv_r}{dr} r^2 \right\} d\theta. \quad (6)$$

Mass conservation gives that

$$\frac{dv_r}{dr} + \frac{v_r}{r} = 0,$$

and, therefore, we can simplify terms in Equation (6) as

$$M = \int_0^{2\pi} \left\{ Wr^2 + \frac{2\sigma_{rr} v_r}{n+1} r \right\} d\theta. \quad (7)$$

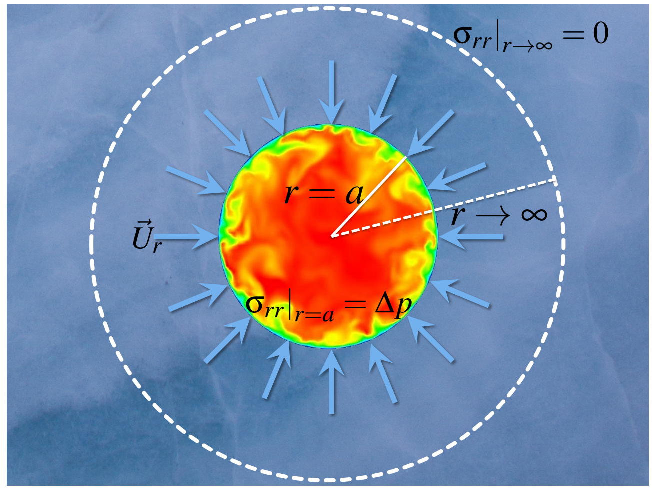


Fig. 2. Nye problem set-up and boundary conditions. Blue arrows indicate the creep closure of the ice $(\vec{U}_r)$ and the turbulence in the channel is suggested by the warm colors. (turbulent pipe flow simulation by [20]. photo credit: Noel Fitzpatrick)

For in-plane, axisymmetric motion, the flow potential $W$ can be written as

$$W = \frac{2n}{n+1} A^{-\frac{1}{n}} D_E^{1+1/n} = \frac{2n}{n+1} A^{-\frac{1}{n}} \left| \frac{dv_r}{dr} \right|^{1+1/n}.$$

This can be inserted into Equation (7). Then, using the fact that $M$ is path independent, we can evaluate two contours: first, along $r = a$ and, second, around $r \to \infty$. These two contours are chosen because these are the locations where the boundary conditions are applied. Starting with the latter, we can see from mass conservation that $dv_r/dr \to 0$ as $r \to \infty$. This means that the flow potential $W$ also decays to zero in the far-field. Although $v_r r$ is a constant as $r \to \infty$, the stress $\sigma_{rr}(r \to \infty) = 0$ (boundary condition) and, therefore, $M = 0$ as $r \to \infty$.

Thus, the $M$ integral around the channel must also be zero. Using the boundary condition $\sigma_{rr}(r = a) = \Delta p$, mass conservation, and the expression for the flow potential, we can write Equation (7) as

$$M = \frac{2a}{n+1} \int_0^{2\pi} \left\{ nA^{-\frac{1}{n}} |-v_r|^{1+1/n} a^{-1/n} + \Delta p v_r \right\} d\theta = 0.$$

Now, due to axisymmetry, the integrand must be independent of $\theta$ and is therefore equal to zero. Taking care with the absolute value term, we can rearrange the integrand to find

$$v_r(a) = -Aa \left( \frac{\Delta p}{n} \right)^n, \quad (8)$$

which is the Nye solution for the creep closure rate at the edge of the channel [4, 21].

This analysis can be easily extended to the case where the outer boundary is finite, i.e. where $r = b$ on the outer edge in Figure 2. Following the same method, and using $D_{\theta\theta} = v_r/r$ from geometry and $D_{\theta\theta} = -D_{rr}$ from mass conservation, we have that

$$M = \frac{2na^2}{n+1}\int_0^{2\pi}\left\{A^{-\frac{1}{n}}\left|-\frac{v_r(a)}{a}\right|^{1+\frac{1}{n}} + \frac{\Delta p v_r(a)}{na}\right\} d\theta,$$
$$= \frac{2nb^2}{n+1}\int_0^{2\pi} A^{-\frac{1}{n}}\left|-\frac{v_r(b)}{b}\right|^{1+\frac{1}{n}} d\theta. \tag{9}$$

Now the $M$ integral around the outer edge of the domain is no longer zero. Since there is a constant volume flux through any radius, i.e. $2\pi r v_r =$ constant, we can related the radial velocity at $v_r(r = a)$ to $v_r(r = a)$ as

$$av_r(a) = bv_r(b).$$

Solving for the creep closure rate at the edge of the channel $v_r(r = a)$ using Equation (9), we find that the finite domain Nye solution is given as

$$v_r(a) = -\frac{Aa}{[1-(a/b)^{2/n}]^n}\left(\frac{\Delta p}{n}\right)^n,$$

which corroborates [22] and, also, reduces to the standard result as $b \to \infty$.

## 2.2 Antiplane shear

A natural extension for the $M$ integral around an englacial or subglacial channel is to include antiplane terms. These are the terms in Equation (5) that include $x$ dependence. In glaciology, the antiplane terms can represent the shear flow of ice downstream, which is often ignored in the creep closure of channels [23, 24, 25]. However, the downstream flow decreases the effective viscosity of the ice, due to the fact that Glen's law is a shear-thinning rheology, and channels close more quickly than in environments free of antiplane stress. Nye [4] and Glen [26] compare the Nye solution to tunnel closure measurements in the field and find that some tunnels close much faster than predicted. Thus, the coupling between the in-plane creep closure and the antiplane motion of the glacier may be important in modeling subglacial hydrologic systems.

Ice stream shear margins are also examples of where antiplane effects can affect the size of drainage channels. Perol *et al.* [27] give theoretical arguments for the existence of subglacial channels beneath ice stream shear margins which is backed up by observations of running water at the base of the dormant Kamb ice stream [28]. Figure 3 shows a schematic for an idealized ice stream shear margin, where the velocity transitions from the fast flowing centerline to the nearly stagnant ridge. In the margin we have schematically drawn a Röthlisberger [2] subglacial channel. Meyer *et al.* [29] show

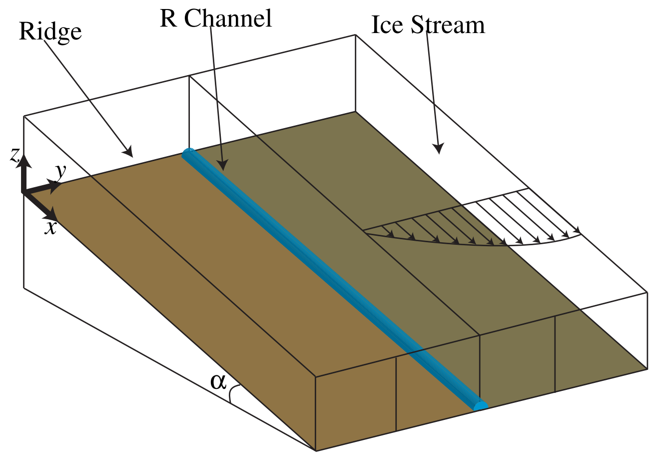


Fig. 3. Schemcatic for an idealized ice stream shear margin with a Röthlisberger [2] subglacial channel. Surface ice velocity increases away from the margin and is nearly stagnant in the ridge (adapted from [29]).

that the shear in the margin leads to faster closure velocities of drainage channels than would be predicted by the Nye solution, due to a decrease in effective viscosity from adding the antiplane shear.

The problem set-up for including antiplane shear follows Figure 2 with the additional boundary conditions

$$\sigma_{rx}|_{r=a} = 0 \quad \text{and} \quad v_x|_{r\to\infty} = \dot{\gamma}_{far} r\cos(\theta), \tag{10}$$

which define the antiplane field. Using these boundary conditions, we evaluate the $M$ integral around two contours: the channel wall $r = a$ as well as in the far-field $r \to \infty$.

### 2.2.1 Evaluation of the $M$ integral at the channel

We start by evaluating $M$ at the channel. We cancel the in-plane shear terms (e.g. $\sigma_{r\theta}$) and antiplane terms in Equation (5), due to the boundary conditions in Equation (10). Thus, we arrive at

$$M = \int_0^{2\pi}\left\{Wr^2 - \left(\frac{n-1}{n+1}\right)\sigma_{rr}v_r r - \sigma_{rr}\frac{\partial v_r}{\partial r}r^2\right\} d\theta.$$

Using mass conservation, we can write the radial velocity derivative as

$$\frac{\partial v_r}{\partial r} = -\frac{v_r}{r} - \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}.$$

The second term in this expression can be neglected because it will always be zero when integrated around a closed loop for a constant boundary stress. If we insert the effective pressure $\Delta p$ for the radial stress, we find that

$$M = \int_0^{2\pi}\left\{Wa^2 + \frac{2\Delta p}{n+1}v_r a\right\} d\theta, \tag{11}$$

which is nearly identical to Equation (7) in symbols. The difference is in the value of $W$. Along $r=a$ with superimposed antiplane shear, we have that

$$\begin{aligned} W_{r=a} &= \frac{2n}{n+1}A^{-1/n}D_E^{1+1/n}\Big|_{r=a}, \\ &= \frac{2n}{n+1}A^{-\frac{1}{n}}\left[\sqrt{\left(\frac{\partial v_r}{\partial r}\right)^2+\left(\frac{1}{2a}\frac{\partial v_x}{\partial \theta}\right)^2}\right]^{1+1/n}, \end{aligned}$$

where the shear term $\partial v_x/\partial r$ was neglected due to the boundary condition specified in Equation (10). Inserting this expression for $W_{r=a}$ into Equation (11), we find that

$$M = \frac{2a^2\Delta p}{n+1}\int_0^{2\pi}\left\{\frac{nA^{-\frac{1}{n}}}{\Delta p}\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{2a}\frac{\partial v_x}{\partial \theta}\right)^2\right]^{\frac{n+1}{2n}} + \frac{v_r}{a}\right\} d\theta. \quad (12)$$

#### 2.2.2 Evaluation of $M$ in the far-field

Following the derivation of the Nye solution, we seek to determine the value of $M$ by evaluating the contour as $r\to\infty$. Starting from Equation (5), we now cancel all the in-plane terms, as they must decay as $r\to\infty$. This leaves us with

$$M = \int_0^{2\pi}\left\{Wr^2 - \left(\frac{n-1}{n+1}\right)\sigma_{rx}v_x r - \sigma_{rx}\frac{\partial v_x}{\partial r}r^2\right\} d\theta.$$

We star by assuming that the antiplane fields are given by the boundary conditions, we have that

$$v_x = \dot{\gamma}_{far} r\cos(\theta) \quad \text{and} \quad \sigma_{rx} = A^{-1/n}(\dot{\gamma}_{far}/2)^{1/n}\cos(\theta),$$

which is a state of constant strain rate. Inserting these expressions into $M$ gives

$$M = \int_0^{2\pi}\left\{W - \frac{2n}{n+1}(2A)^{-1/n}\dot{\gamma}_{far}^{1+1/n}\cos^2(\theta)\right\}r^2\, d\theta.$$

Evaluating $W$ as $r\to\infty$ gives

$$W|_{r\to\infty} = \frac{2n}{n+1}A^{-1/n}\left(\dot{\gamma}_{far}/2\right)^{1+1/n}.$$

Thus, the integral for $M$ as $r\to\infty$ reduces to

$$M = \frac{n}{n+1}(2A)^{-1/n}\dot{\gamma}_{far}^{1+1/n}\int_0^{2\pi}\left\{1-2\cos^2(\theta)\right\}r^2\, d\theta = 0,$$

using the half-angle formula for $\cos(\theta)$. In order for $M$ to be finite, this integral must be zero because of the $r^2$ term in the integrand. However, this integral does not represent the full far-field contributions to $M$. The boundary conditions represent a constant strain rate, a situation where $M=0$ and this evaluation of $M$ in the far-field disregards the coupling between the antiplane and in-plane motion. Therefore, we must retain a small perturbation away from a constant background strain rate.

#### 2.2.3 Far-field perturbation

In the far-field, we consider a perturbation from the constant antiplane strain rate boundary condition. We write the far-field velocities in Cartesian coordinates as

$$v_x = \dot{\gamma}_{far}y + \varepsilon h(y,z), \qquad v_y = \varepsilon g(y,z), \quad \text{and} \quad v_z = \varepsilon f(y,z).$$

Here $\varepsilon$ is an unknown small parameter and $f(y,z)$, $g(y,z)$, and $h(y,z)$ are unknown functions. To first order in $\varepsilon$, the effective strain rate is given as

$$D_E = \frac{\dot{\gamma}_{far}}{2}\sqrt{1+\frac{2\varepsilon}{\dot{\gamma}_{far}}\frac{\partial h}{\partial y} + O(\varepsilon^2)}.$$

Consequently, the antiplane problem in the far-field is coupled to the in-plane motions through $\varepsilon$ and we are able to ignore the velocities $v$ and $w$. Thus, we concentrate on writing an equation for $h(y,z)$. The antiplane stresses are given by

$$\begin{aligned} \sigma_{yx} &= A^{-1/n}D_E^{(1-n)/n}\left(\frac{\dot{\gamma}_{far}}{2}+\frac{\varepsilon}{2}\frac{\partial h}{\partial y}\right) \\ \sigma_{zx} &= A^{-1/n}D_E^{(1-n)/n}\frac{\varepsilon}{2}\frac{\partial h}{\partial z}. \end{aligned}$$

Inserting the effective strain rate and linearizing the stress about $\varepsilon$ gives

$$\begin{aligned} \sigma_{yx} &= \left[\frac{A}{2\dot{\gamma}_{far}}\right]^{1/n}\left(1+\frac{\varepsilon}{n\dot{\gamma}_{far}}\frac{\partial h}{\partial y}\right), \\ \sigma_{zx} &= \left[\frac{A}{2\dot{\gamma}_{far}}\right]^{1/n}\frac{\varepsilon}{\dot{\gamma}_{far}}\frac{\partial h}{\partial z}. \end{aligned}$$

We then insert the linearized stress into the force balance equation

$$\frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0,$$

and we find that

$$\frac{1}{n}\frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0.$$

By making the transformation, $\eta = \sqrt{n}y$, we can write this equation as

$$\hat{\nabla}^2 h = 0,$$

i.e. Laplace's equation with $\hat{\nabla} = (\partial_\eta, \partial_z)$. If we return to polar coordinates, now with

$$\hat{r} = \sqrt{\eta^2 + z^2} \quad \text{and} \quad \hat{\theta} = \arctan\left(\frac{z}{\eta}\right),$$

we can write

$$h = \hat{r}^\lambda f(\hat{\theta}).$$

Inserting this ansatz into Laplace's equation for $h$, we find that

$$f = B\cos(k\hat{\theta}) \quad \text{and} \quad \lambda = \pm k,$$

where $k$ is some unknown integer. The term proportional to $\sin(k\hat{\theta})$ can be ignored due to symmetry and the positive solutions for $\lambda$ can be disregarded as they are singular as $r \to \infty$. The term that decays the slowest is $k = -1$ and, therefore, we have

$$h = \frac{B}{\hat{r}}\cos(\hat{\theta}) = \frac{B}{r}\left(\frac{\cos\left(\arctan\left(\frac{\tan(\theta)}{\sqrt{n}}\right)\right)}{\sqrt{1+(n-1)\cos^2(\theta)}}\right), \tag{13}$$

which we refer to as

$$h(r,\theta) = \frac{B}{r}\chi(\theta).$$

Now inserting the perturbation ansatz $h = \varepsilon r^\lambda \chi(\theta)$ into the $M$ integral in the far-field, we find that

$$M = \varepsilon\left[\frac{\dot{\gamma}_{far}}{2A}\right]^{1/n} r^{\lambda+1}\int_0^{2\pi}\{\lambda\cos(\theta)\chi - \sin(\theta)\chi'\} - \frac{2(n-1)}{n+1}\left[\lambda\cos^3(\theta)\chi - \frac{1}{2}\cos(\theta)\sin(2\theta)\chi'\right] - \frac{n-1}{n+1}(\lambda+1)\cos(\theta)\chi + 2\lambda\cos(\theta)\chi\, d\theta,$$

where we have incorporated the unknown factor $B$ into $\varepsilon$ as the unknown far-field amplitude. We know that $\lambda = -1$ and, therefore, we have that

$$M = \varepsilon\left[\frac{\dot{\gamma}_{far}}{2A}\right]^{1/n}\int_0^{2\pi}\frac{2(n-1)}{n+1}\left[\cos^3(\theta)\chi + \sin(\theta)\cos^2(\theta)\chi'\right] + 2\cos(\theta)\chi\, d\theta,$$

which is of the form

$$M = \varepsilon\left[\frac{\dot{\gamma}_{far}}{2A}\right]^{1/n} I_M(n),$$

where $I_M(n)$ is given by

$$\int_0^{2\pi}\frac{2(n-1)}{n+1}\left[\cos^3(\theta)\chi + \sin(\theta)\cos^2(\theta)\chi'\right] + 2\cos(\theta)\chi\, d\theta,$$

Inserting our expression for $\chi(\theta)$ from Equation (13), we can integrate over $\theta$ numerical to find $I_M(n)$. For $n = 3$, i.e. Glen's law for ice, we find that

$$I(3) = -0.113\ldots$$

This integral should be negative because in $v_x = \dot{\gamma}_{far}y + \varepsilon h(y,z)$ the antiplane velocity should be less than the boundary value as it approaches the edge.

Now, to determine the far-field amplitude factor $\varepsilon$, we need to know the behavior of $v_z$ for small $r$, which requires solving the fully coupled (in-plane and antiplane) PDE. Simultaneously, the path independence of the $M$ integral relates the perturbation integral in the far-field to the integral around the channel, i.e. Equation (12). This gives

$$\begin{aligned} M &= \frac{2a^2\Delta p}{n+1}\int_0^{2\pi}\left\{\frac{nA^{-\frac{1}{n}}}{\Delta p}\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{2a}\frac{\partial v_x}{\partial\theta}\right)^2\right]^{\frac{n+1}{2n}} + \frac{v_r}{a}\right\} d\theta. \\ &= \varepsilon\left[\frac{\dot{\gamma}_{far}}{2A}\right]^{1/n} I_M(n). \end{aligned} \tag{14}$$

Thus, the perturbation amplitude $\varepsilon$ is related to the unknown strain rates at the edge of the channel. Furthermore, we can see that that if $v_x = 0$ (or if $v_x$ is a function of $r$ only), then this integral reduces to the integral for the Nye solution and $\varepsilon = 0$.

#### 2.2.4 Nondimensional equations and numerics

We now write the expression for $M$ in far-field nondimensionally. The natural lengthscale is the channel radius and, therefore, we write $r = aR$, where $R$ is the nondimensional radial coordinate (using capital letters to denote nondimensional variables). We proceed by using the boundary conditions to scale the stress and velocity. For the in-plane components of stress, $\Delta p$ is a sensible scaling. This leads to a scaling for the in-plane velocity, by dimensional arguments alone, that is reminiscent of the Nye solution (Equation (8)), i.e. $v_r = Aa\Delta p^n V_r$. From the antiplane boundary conditions we can see that $v_x = \dot{\gamma}_{far}aV_x$. It is evident immediately that the in-plane and antiplane velocities do not scale in the same manner and, thus, a logical control parameter for the system is their ratio $S$, which is

$$S = \frac{\dot{\gamma}_{far}}{A\Delta p^n},$$

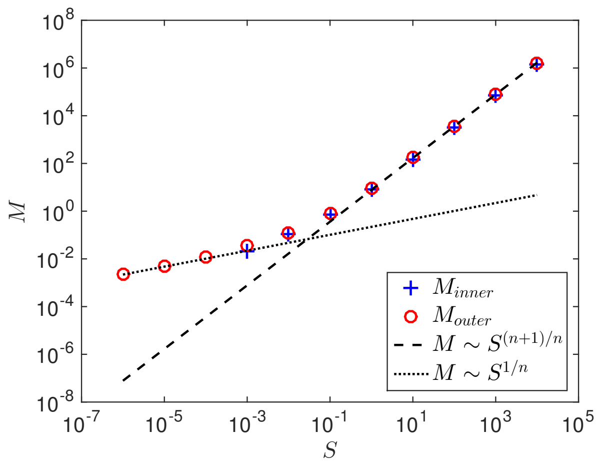


Fig. 4. Numerical evaluation of $M$ at the edge of the channel $M_{inner}$ and in the far-field $M_{outer}$ at R=500. The scalings show that for $S \ll 1$, $M \sim S^{1/n}$ and for $S \gg 1$, $M \sim S^{(n+1)/n}$. Numerical artifacts lead to negative values for $M_{inner}$ when $S$ is small, therefore these simulations are omitted but yield absolute values close to $M_{outer}$.

and represents the importance of antiplane shearing to in-plane creep closure [29]. Using $S$ we can rewrite the antiplane velocity as $v_x = Aa\Delta p^n SV_x$.

Using these scalings we can rewrite $M$ as

$$M = a^2 A\Delta p^{n+1}\hat{M},$$

where we immediately drop the variable hat. Thus, Equation 14 gives

$$\begin{aligned} M &= \frac{2}{n+1}\int_0^{2\pi}\left\{n\left[\left(\frac{\partial V_r}{\partial R}\right)^2 + \frac{S^2}{4}\left(\frac{\partial V_x}{\partial \theta}\right)^2\right]^{\frac{n+1}{2n}} + V_r\right\}d\theta, \\ &= \frac{\varepsilon}{a^2 A\Delta p^n}\left(\frac{S}{2}\right)^{1/n} I_M(n). \end{aligned} \quad (15)$$

When the far-field of the domain is dominated by anitplane shear, we have that $S \gg 1$ and the appropriate scaling for $\varepsilon$ is

$$\varepsilon = \kappa\dot{\gamma}_{far}a^2,$$

where $\kappa$ is now a constant related to the unknown strain rates at the edge of the channel. When $S \ll 1$, we scale $\varepsilon$ using the Nye strain rate as $\varepsilon = \kappa A\Delta p^n a^2$. From this definition of $\varepsilon$, we can see that $M$ in the far-field is either

$$M = \frac{\kappa I_M(n)}{2^{1/n}}S^{1/n} \quad (S \ll 1) \quad \text{or} \quad \frac{\kappa I_M(n)}{2^{1/n}}S^{(n+1)/n} \quad (S \gg 1).$$

Although we cannot solve for the constant $\kappa$ analytically, we can its value by numerical simulations. We implement the numerical method described in Meyer et al. [29] in the existing finite element software COMSOL [30]. These simulations allows us to compute the value of $M$ as a function of the strainrate ratio $S$, which is shown in figure 4. The two regimes, where $M$ scales as $M \sim S^{1/n}$ for $S \ll 1$ and $M \sim S^{1/n}$ for $S \gg 1$ are clearly visible. The best-fit value of $\kappa$ determined from the simulations is given by

$$\kappa = -2.45 \quad (S \ll 1) \quad \text{or} \quad \kappa = -86.97 \quad (S \gg 1).$$

These results show that as amount of antiplane shear with respect to in-plane shear is increased, as measured by an increase in the strain rate ratio $S$, the $M$ integral also increases. This is due to a simultaneous increase in both the creep closure velocity $V_r$ as well as the antiplane straining along the edge of the channel. The increase in channel closure velocity is due to a decrease in effective viscosity, as described in Meyer et al. [29].

We now describe the evolution of the creep closure velocity as a function of $S$. When there is very little antiplane motion as compared to in-plane straining, the creep closure velocity is given by the Nye solution, Equation (8), which is written nondimensionally as

$$V_r = -\frac{n^{-n}}{R}.$$

When the deformation is dominated by antiplane motion, i.e. $S \gg 1$, the effective strain rate scales as $\varepsilon_E \sim S$. The radial force balance gives that the averaged creep closure velocity around the channel is given as

$$V_r \sim -S^{(n-1)/n}, \quad (16)$$

where more details are provided in [29]. In figure 5, we show the two limits of the creep closure velocity: for $S \ll 1$, the simulations approach the Nye solution and for $S \gg 1$, we verify the scaling given in Equation (16).

This increase in creep closure velocity due to the addition of antiplane shear is consistent with the increase in tunnel closure velocities observed by Nye [4] and Glen [26] due to changes in the glacier stress state. Furthermore, the increase in creep closure velocity due to antiplane straining is analogous to the Rice and Tracey [31] effect where the growth of voids is strongly enhanced by triaxiality. Intuitively, the in-plane creep closure velocity for large $S$ grows less than linearly with $S$ as the antiplane field only influences the in-plane motion through the viscosity. The consequence is that the dominant balance for large $S$ in Equation (15) is still between the perturbation in the far field and the antiplane shear at the channel wall.

## 3 Concluding remarks

In this paper, we apply the path-independent $M$ integral to the creep of ice around subglacial and englacial channels. We correct a longstanding error in the implementation of the

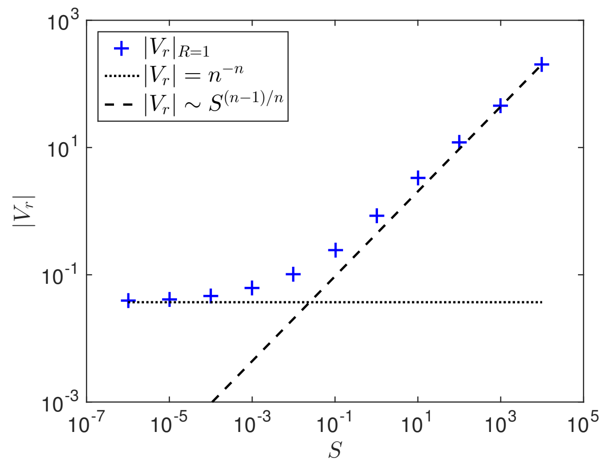


Fig. 5. Numerical simulations showing the nondimensional channel creep closure velocity $|V_r|$ at the edge of the channel $R = 1$. For $S \ll 1$, the antiplane straining is negligible and the creep closure velocity approaches the nondimensional Nye solution. When the antiplane motions are dominant, i.e. $S \gg 1$, the creep closure velocity increases as $S^{(n-1)/n}$.

$M$ integral to problems in generalized elasticity and non-Newtonian power-law fluids. We then use this integral to derive the Nye solution for the creep closure of an englacial or subglacial channel. Building on this solution, we consider applications where the flow of ice is not entirely in-plane and axisymmetric but includes components of flow down glacier (i.e. parallel to the channel axis). Using a simple far-field shear as a representation for ice stream shear margins, we find that an in-plane perturbation exists in the far-field. We solve for the perturbation explicitly, up to a constant factor ε. Then, using the $M$ integral, we are able to relate this perturbation back to the in-plane strain rates at the channel. We show that the factor ε exhibits two scaling regimes based on the size of the strain rate ratio $S \equiv \dot{\gamma}_{far}/(A\Delta p^n)$: for small antiplane velocity relative to in-plane creep closure, ε is a constant and the $M$ integral approaches zero as $M \sim S^{1/n}$ for vanishingly small $S$. In the other limit, where antiplane straining dominates, $M$ grows as $M \sim S^{(n+1)/n}$. These two scaling regimes are also present in creep closure velocity where for small $S$ we retrieve the Nye solution and for $S \gg 1$, the closure velocity increases as $V_r \sim -S^{(n-1)/n}$ due to a decrease in ice viscosity. Thus, $M$ provides a succinct description of the processes affecting channel closure with superimposed antiplane shear.


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## Appendix A: Proof of the path independence of $M$

Here we start with the two dimensional generalized $M$ integral as written in Equation (2) with $m = 1 + 1/n$, i.e.

$$M = \oint_\Gamma \left\{ W x_k n_k - \sigma_{ik} n_k \left[ \left( \frac{n-1}{n+1} \right) v_i + x_j \frac{\partial v_i}{\partial x_j} \right] \right\} ds,$$

where $\Gamma$ is a closed path, as shown in Figure 1. We use the divergence theorem to write this integral as

$$M = \oint_\Omega \frac{\partial}{\partial x_k} \left( W x_k - \sigma_{ik} \left[ \left( \frac{n-1}{n+1} \right) v_i + x_j \frac{\partial v_i}{\partial x_j} \right] \right) dA.$$

The $M$ integral is path independent if the terms inside the area integral are zero. That is, if

$$\frac{\partial}{\partial x_k} \left( W x_k - \sigma_{ik} \left[ \left( \frac{n-1}{n+1} \right) v_i + x_j \frac{\partial v_i}{\partial x_j} \right] \right) = 0.$$

Taking the derivatives we find that

$$\frac{\partial W}{\partial x_k} x_k + W \frac{\partial x_k}{\partial x_k} - \frac{\partial \sigma_{ik}}{\partial x_k} \left[ \left( \frac{n-1}{n+1} \right) v_i + x_j \frac{\partial v_i}{\partial x_j} \right] - \sigma_{ik} \left[ \left( \frac{n-1}{n+1} \right) \frac{\partial v_i}{\partial x_k} + \frac{\partial x_j}{\partial x_k} \frac{\partial v_i}{\partial x_j} + x_j \frac{\partial^2 v_i}{\partial x_j \partial x_k} \right] = 0. \quad (17)$$

The equilibrium equations are

$$\frac{\partial \sigma_{ik}}{\partial x_k} = 0,$$

which allow us to cancel the third group of terms in Equation (17) and, therefore, write

$$\frac{\partial W}{\partial x_k} x_k + W \frac{\partial x_k}{\partial x_k} - \sigma_{ik} \left[ \left( \frac{n-1}{n+1} \right) \frac{\partial v_i}{\partial x_k} + \frac{\partial x_j}{\partial x_k} \frac{\partial v_i}{\partial x_j} + x_j \frac{\partial^2 v_i}{\partial x_j \partial x_k} \right] = 0.$$

The derivatives of coordinates lead to Kronecker delta functions as

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij},$$

where in two dimensions, $\delta_{kk} = 2$ and thus,

$$\frac{\partial W}{\partial x_k} x_k + 2W - \sigma_{ik} \left[ \left( \frac{n-1}{n+1} \right) \frac{\partial v_i}{\partial x_k} + \delta_{jk} \frac{\partial v_i}{\partial x_j} + x_j \frac{\partial^2 v_i}{\partial x_j \partial x_k} \right] = 0.$$

We can simplify this expression slightly by multiplying through by the stress and combining like terms as

$$\frac{\partial W}{\partial x_k} x_k + 2W - \frac{2n}{n+1} \sigma_{ik} \frac{\partial v_i}{\partial x_k} - \sigma_{ik} x_j \frac{\partial^2 v_i}{\partial x_j \partial x_k} = 0.$$

The strain rate energy density function $W$ can be related to the stress and strain rates as

$$W = \frac{n}{n+1} \sigma_{ij} D_{ij} = \frac{n}{n+1} \sigma_{ij} \frac{\partial v_i}{\partial x_j}.$$

This allows us to write

$$\frac{\partial W}{\partial x_k} x_k + 2W - 2W - \sigma_{ij} \frac{\partial^2 v_i}{\partial x_k \partial x_j} x_k = 0.$$

Canceling the $2W$ terms, we can use the chain rule to relate spatial derivatives on $W$ to derivatives of strain as

$$\frac{\partial W}{\partial D_{ij}} \frac{\partial D_{ij}}{\partial x_k} x_k - \sigma_{ij} \frac{\partial^2 v_i}{\partial x_k \partial x_j} x_k = 0.$$

If $W$ is written to depend symmetrically on $D_{ij}$, with constraint $D_{kk} = 0$, then

$$\sigma_{ij} + p\delta_{ij} = \frac{\partial W}{\partial D_{ij}},$$

and thus, the integrand is

$$\sigma_{ij} \frac{\partial D_{ij}}{\partial x_k} x_k - \sigma_{ij} \frac{\partial^2 v_i}{\partial x_k \partial x_j} x_k = 0,$$

where the isotropic components of the stress tensor cancel when multiplied by $\partial D_{ij}/\partial x_k$ because $\partial D_{ll}/\partial x_k = 0$. Now, using the symmetry of the stress tensor, we have that

$$\sigma_{ij} \frac{\partial^2 v_i}{\partial x_k \partial x_j} x_k - \sigma_{ij} \frac{\partial^2 v_i}{\partial x_k \partial x_j} x_k = 0,$$

which is zero. Thus, we have shown that the $M$ integral, written as

$$M = \oint_\Gamma \left\{ W x_k n_k - \sigma_{ik} n_k \left[ \left( \frac{n-1}{n+1} \right) v_i + x_j \frac{\partial v_i}{\partial x_j} \right] \right\} ds,$$

is path independent.

**Appendix B: Flow potential for a Reiner-Rivlin fluid**

A Reiner-Rivlin fluid is an incompressible fluid ($I_1 = D_{kk} = 0$) for which

$$\sigma_{ij} = -p\delta_{ij} + \phi_1(I_2, I_3) D_{ij} + \phi_2(I_2, I_3) D_{in} D_{nj}, \qquad (18)$$

where $\sigma_{ij}$ is the stress tensor, $D_{ij}$ is the symmetric part of the velocity gradient tensor, $p$ is the isotropic pressure, and $I_k$ is the $k$th invariant of the tensor $D_{ij}$ [13]. By writing the stress as an isotropic matrix function of $D_{ij}$, assuming a symmetric dependence on $D_{ij}$ and $D_{ji}$, and expanding this function of as a power series, we can use the Cayley-Hamilton theorem to show that the stress is a quadratic polynomial in $D_{ij}$ with coefficients that are functions of the invariants of $D_{ij}$. For an incompressible fluid with isotropic pressure, this reduces to Equation (18).

Here we ask what is the most general fluid rheology that still posses a flow potential, i.e. where $dW$ is a perfect differential of $\sigma_{ij} dD_{ij}$. This requires that

$$\frac{\partial \sigma_{ij}}{\partial D_{kl}} = \frac{\partial \sigma_{kl}}{\partial D_{ij}}, \qquad (19)$$

which is called Maxwell reciprocity. For an incompressible fluid, we include pressure as a Lagrange multiplier to enforce mass conservation, and write that

$$\frac{\partial \left( \sigma'_{ij} + p\delta_{ij} \right)}{\partial D_{kl}} = \frac{\partial \left( \sigma'_{kl} + p\delta_{ij} \right)}{\partial D_{ij}},$$

where the isotropic pressure $p$ is independent of the strain rate. This shows that the deviatoric stress also satisfies Maxwell reciprocity.

If we insert the Reiner-Rivlin fluid rheology into Equation (19), we find that

$$\left( \frac{\partial \phi_1}{\partial I_3} - \frac{\partial \phi_2}{\partial I_2} \right) (D_{ij} D_{kn} D_{nl} - D_{kl} D_{in} D_{nj}) = 0.$$

Now since $D_{ij} D_{kn} D_{nl} \neq D_{kl} D_{in} D_{nj}$ for all $i$, $j$, $k$, and $l$, the only way for this condition to be satisfied for all flows is for

$$\frac{\partial \phi_1}{\partial I_3} = \frac{\partial \phi_2}{\partial I_2}. \qquad (20)$$

This condition also arises from an equality requirement of the mixed partial derivatives of the flow potential $W$. If we start with the invariants of $D_{ij}$, given as

$$I_1 = D_{kk}, \quad I_2 = \frac{1}{2} D_{lm} D_{ml}, \quad I_3 = \frac{1}{3} D_{lm} D_{mn} D_{nl},$$

we can write the relationship between the flow potential and the stress as

$$\sigma_{ij} + p\delta_{ij} = \frac{\partial W}{\partial D_{ij}} = \frac{\partial W}{\partial I_2} \frac{\partial I_2}{\partial D_{ij}} + \frac{\partial W}{\partial I_3} \frac{\partial I_3}{\partial D_{ij}}. \qquad (21)$$

Now using the facts that

$$\frac{\partial D_{ij}}{\partial D_{kl}} = \delta_{ik}\delta_{jl}, \quad \frac{\partial I_2}{\partial D_{kl}} = D_{kl} \quad \text{and} \quad \frac{\partial I_3}{\partial D_{kl}} = D_{kn} D_{nl},$$

we have that

$$\sigma_{ij} + p\delta_{ij} = \frac{\partial W}{\partial D_{ij}} = \frac{\partial W}{\partial I_2} D_{ij} + \frac{\partial W}{\partial I_3} D_{in} D_{nj}. \qquad (22)$$

from which we can see that Equation (22) is Reiner-Rivlin fluid with

$$\phi_1 = \frac{\partial W}{\partial I_2} \quad \text{and} \quad \phi_2 = \frac{\partial W}{\partial I_3}.$$

Thus, a requirement for $W$ to exist, and to be a perfect differential of $\sigma_{ij} dD_{ij}$ as is required to write Equation (21), is that

$$\frac{\partial^2 W}{\partial I_2 \partial I_3} = \frac{\partial^2 W}{\partial I_3 \partial I_2},$$

which implies that

$$\frac{\partial \phi_1}{\partial I_3} = \frac{\partial \phi_2}{\partial I_2},$$

just as was found in Equation (20).

A class of Reiner-Rivlin fluids that always satisfies the condition in Equation (20) are those for which $\phi_1$ is a function of $I_2$ solely and $\phi_2 = 0$. This is the rheological structure of Glen's law in glaciology and, therefore, a flow potential $W$ will always exist.