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On the determinacy of repetitive structures

S.D. Guest[a,*], J.W. Hutchinson[b]

[a]Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK
[b]Department of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138, USA

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Abstract

This paper shows that repetitive, infinite structures cannot be simultaneously statically, and kinematically, determinate.

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1. Introduction

Foams and other materials with a lattice-like structure are often considered as pin-jointed frames that are essentially infinite and repetitive, see e.g. Deshpande et al. (2001). A natural question for these structures is: what is the minimum number of bars that will lead to a rigid structure? A closely related question is whether it is possible to build a pin-jointed structure where changing the length of any bar leads only to a change of geometry of the structure, and not to self-stress—of interest if trying to design an adaptive structure.

For *finite* structures, these question may be answered straightforwardly using Maxwell's rule (Maxwell, 1864,1890). If an unsupported, two-dimensional frame composed of rigid bars connected by frictionless joints is statically and kinematically determinate, the number of bars b must be $2j-3$, where j is the number of joints, although these bars must be properly positioned (Laman, 1970). More generally (Calladine, 1978), the frame will support s states of self-stress (bar tensions in the absence of load) and m mechanisms (joint displacements without bar extension), where

$$s-m=b-2j+3. \tag{1}$$

* Corresponding author. Tel.: +44-1223-332708; fax: +44-1223-332662.
E-mail address: sdg@eng.cam.ac.uk (S.D. Guest).

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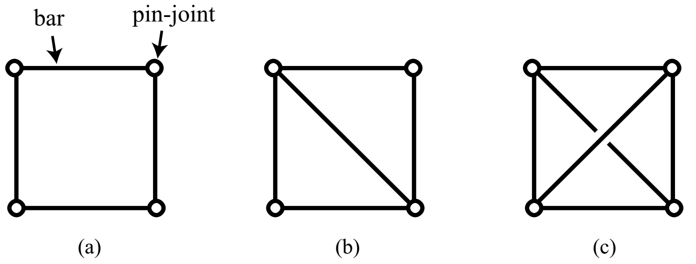


Fig. 1. (a) A *kinematically indeterminate* structure, with one mechanism, (b) a *statically and kinematically determinate* structure and (c) a *statically indeterminate* structure, with one state of self-stress.

Fig. 1 shows three simple finite 2D structures demonstrating these ideas. Fig. 1(b) is simultaneously statically and kinematically determinate. Adding bars to this structure, Fig. 1(c), leads to the possibility of states of self-stress, and hence static indeterminacy; removing bars, Fig. 1(a), leads to possible mechanisms, and kinematic indeterminacy. Maxwell's rule can be considered as ensuring that the equilibrium/compatibility matrices for the structure are square (Pellegrino and Calladine, 1986). Equilibrium and compatibility matrices will be discussed in Sections 2 and 3.

This paper deals with infinite, repetitive structures. All the examples and calculations will be for 2D structures, for simplicity, but the conclusion will be equally valid for 3D. We will define a unit cell for the structure, and assume that loads and deformations of the structure repeat with this unit cell. For this assumption, to ensure that there is a square equilibrium and compatibility matrix requires (in 2D) that for every node in the unit cell there are two bars. Indeed, this is the only way that the overall structure will be able to satisfy Maxwell's rule. Because every bar joins two nodes, this implies that the average number of bars for each joint (the *valency*) is four; there are an infinite variety of structures that will satisfy this (Grunbaüm and Shephard, 1987). This paper will show that no such structure can be rigid.

A very simple example of a 2D repetitive structure with a valency of four is the simple square grid shown in Fig. 2. This is clearly neither statically *nor* kinematically determinate. Indeed, Deshpande et al. (2001) show that this structure (or any other with *similarly situated* nodes, where the smallest translational unit cell contains a single node) would require a valency of six to be rigid. None of the other examples in this paper have similarly situated nodes.

This paper will examine two different approaches to answer the questions of when repetitive structures are rigid, one based on statics and the other on kinematics. For structures which overall satisfy Maxwell's rule, they come to incompatible conclusions, and this paradox is resolved by showing that repetitive structures cannot be both statically, and kinematically determinate.

2. A statics approach

In this approach, we write equilibrium equations relating internal tensions in the members to loads applied at the nodes. This leads to an *equilibrium matrix* **A** relating

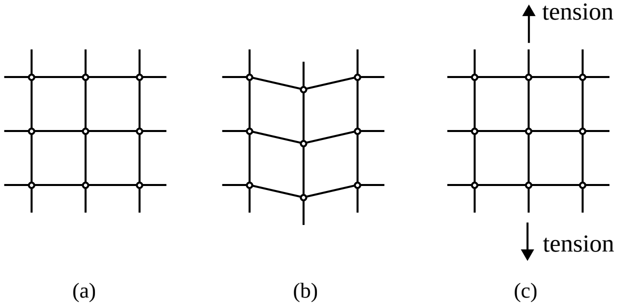


Fig. 2. (a) An infinite repetitive square grid structure showing; (b) one possible mechanism; and (c) one possible state of self-stress.

a vector of bar forces **t** to a vector of nodal loads, **p**, **At** = **p**. If a repetitive structure is able to carry any loading, however, the matrix **A** will not be of full rank. There are three possible average stresses, corresponding to loads at infinity (e.g. tension in the x-direction, tension in the y-direction, and shear loading in the xy-plane) that the structure must be able to carry, and that will appear in this formulation as states of self-stress—bar forces **t** in equilibrium with zero nodal loading, i.e. **t** lies in the nullspace of **A**. Thus, the $n \times n$ equilibrium matrix for a structure that can carry these loads can be at most rank $n - 3$, where n is the number of bars, or twice the number of joints, in the unit cell.

An example of a structure that is compatible with this static condition is the kagome structure shown in Fig. 3 (Hyun and Torquato, 2002). The 6×6 equilibrium matrix generated from the node and element numbering scheme shown is of rank 3, and it is clear that this structure will be able to carry three independent sets of loads at infinity.

3. A kinematics approach

In this approach, we write kinematic equations relating node displacements and bar extensions. This leads to a *compatibility* matrix **C** relating a vector of nodal displacements **d** to a vector of bar extensions **e**. It is clear that the resultant matrix cannot be of full rank, as rigid-body displacements in e.g. the x- and y-directions will cause no internal deformation (a rigid-body *rotation* is not allowed because of the basic periodicity assumption). Thus, if no other *internal* mechanisms are possible, the resultant $n \times n$ compatibility matrix will be of rank $n - 2$.

An example of a structure that fits this kinematic condition is the perturbed square grid shown in Fig. 4. The 8×8 compatibility matrix generated from the node and element numbering scheme shown is of rank 6, and the only possible mechanisms are the rigid-body displacements.

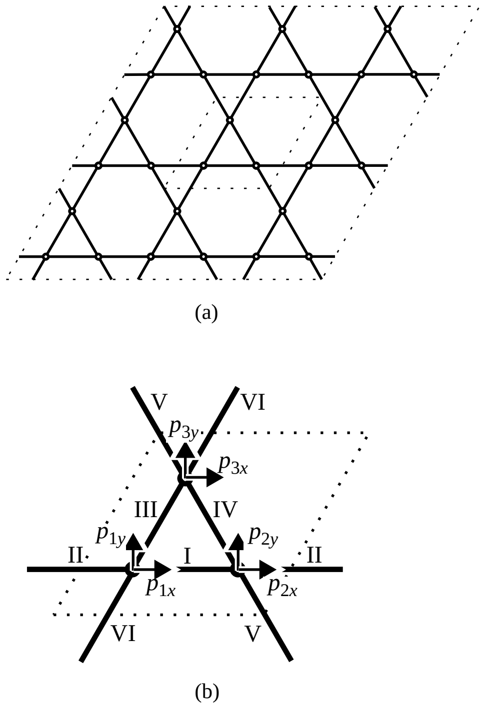


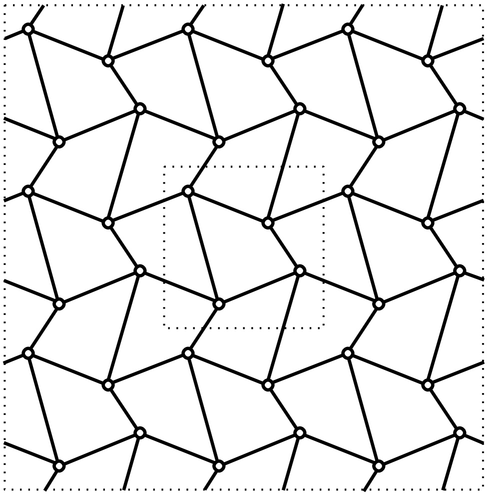
Fig. 3. (a) The kagome structure, showing a possible unit cell and (b) a possible numbering scheme for bar forces (I–VI) and applied loads (p_{1x}–p_{3y}) in the unit cell.

4. The paradox

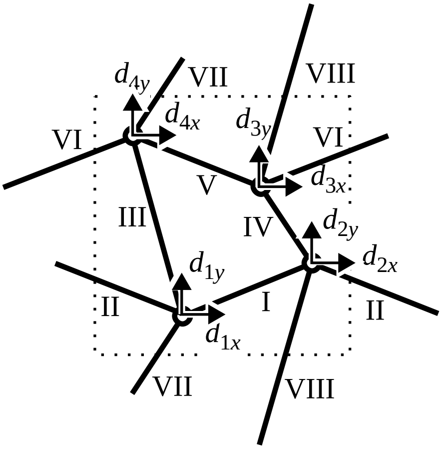
It is well known, and may be easily shown by e.g. a virtual work argument, that the compatibility and equilibrium matrices for a structure are related by $\mathbf{C} = \mathbf{A}^{\mathrm{T}}$, and hence must have equal rank. Thus, any repetitive structure with a square equilibrium/compatibility matrix, *cannot* satisfy both the static, and the kinematic, condition for rigidity. A structure that satisfies the static condition that rank equals $n-3$ must have an internal mechanism when the kinematics are considered. Similarly, a structure that satisfies the kinematic condition that rank equals $n-2$ will not be able to carry all possible average stresses, and must have a deformation mode that is allowed by distorting the unit cell. This is demonstrated for the example structures in Figs. 5 and 6.

Fig. 5(a) shows the periodic internal mechanism of the kagome structure. Alternating triangles rotate in opposite directions. This is the one finite mechanism of the structure, but in this case there are infinitely many other infinitesimal mechanisms; an example is shown in Fig. 5(b).

Fig. 6 shows the internal mechanism of the perturbed square grid. The figure also shows the deformation of the unit cell. This deformation can easily be calculated



(a)



(b)

Fig. 4. (a) A perturbed square grid structure, showing a possible unit cell and (b) a possible numbering scheme for bar extensions (I–VIII) and displacements of nodes (d_{1x}–d_{4y}) in the unit cell.

by augmenting the allowed displacement in the compatibility equations. Three extra deformations (of the unit cell) are allowed, as shown in Fig. 7, and this leads to an augmented compatibility matrix $\mathbf{C}'$ that is now $n \times (n+3)$. For the perturbed square grid, the 8×11 $\mathbf{C}'$ is now of rank 8, and the nullspace, defining possible mechanisms, contains two rigid-body translations, plus the mechanism shown in Fig. 6. A similar calculation for a structure that satisfies the static condition yields no new information, as the ability to carry all average stresses is equivalent to saying that the unit cell cannot

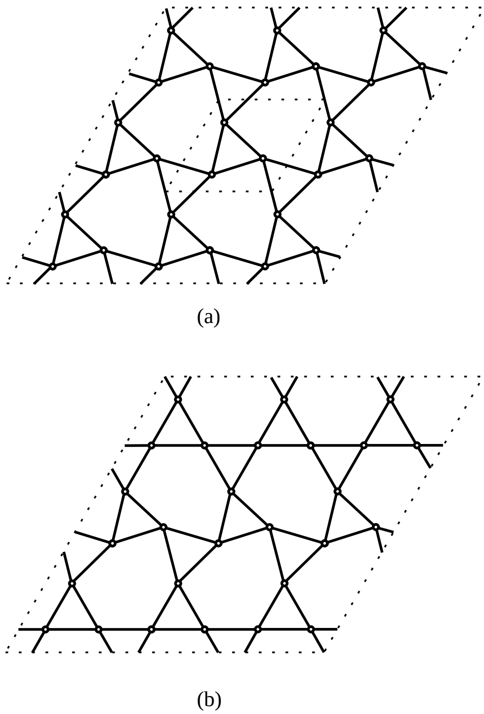


Fig. 5. Two mechanisms of the kagome structure: (a) the finite mechanism; and (b) one possible infinitesimal mechanism.

be distorted. For example, the kagome structure has an augmented compatibility matrix that is 6×9 and of rank 6, and the nullspace contains two rigid-body translations, plus the mechanism that had already been calculated, and shown in Fig. 5(a).

5. Definition of determinacy

The meaning of 'static determinacy' and 'kinematic determinacy' is not always well defined, and this is further complicated in this paper by the nature of repetitive structures: this section will more carefully define what we mean by these terms.

We define a *kinematically determinate* structure to be one where the only solutions to the compatibility equations for zero bar extensions, $\mathbf{Cd} = \mathbf{0}$, are rigid-body motions. For 2D structures, our assumption of repetitive behaviour leaves two such motions, and hence the nullspace of $\mathbf{C}$ (equal to the left nullspace of $\mathbf{A}$) will be two dimensional.

For finite structures, we would define a *statically determinate* structure to be one where the only solution to the equilibrium equations for zero external load, $\mathbf{Ar} = \mathbf{0}$ is $\mathbf{r} = \mathbf{0}$: the nullspace of $\mathbf{A}$ (equal to the left nullspace of $\mathbf{C}$) is empty. That definition clearly needs amending for the case of repetitive structures, as loadings of the structure at infinity that cause overall average stress in the structure correspond to zero nodal

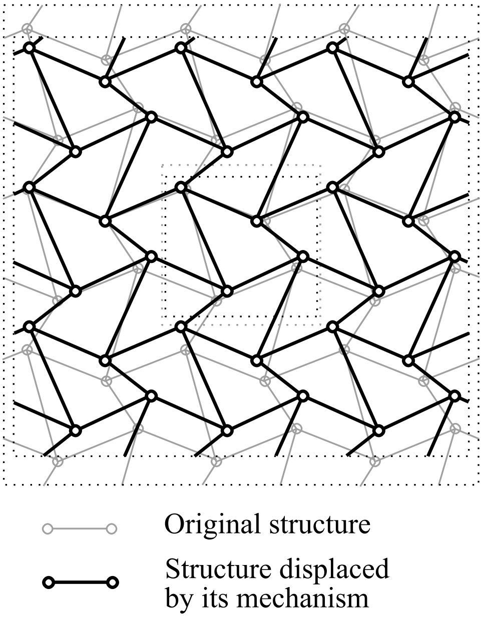


Fig. 6. The mechanism of the perturbed grid shown in Fig. 4; note the deformation of the unit cell.

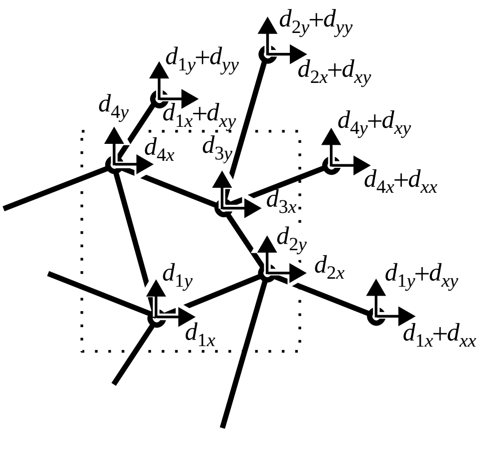


Fig. 7. An augmented nodal displacement numbering scheme for the perturbed square grid, allowing displacements of the nodes (d_{1x}–d_{4y}) and deformation of the unit cell (d_{xx}, d_{yy}, d_{xy}).

loading. Thus, we define a statically determinate repetitive structure to be one where there *are* non-zero solutions to the equilibrium equations for zero nodal loading, $\mathbf{Ar} = \mathbf{0}$, but these independent solutions correspond to independent average stresses. For a repetitive 2D structure, there are three independent average stresses, and hence the nullspace of $\mathbf{A}$ (equal to the left nullspace of $\mathbf{C}$) will be three dimensional.

6. Discussion

This paper has shown that, in 2D, infinite repetitive structures cannot be simultaneously statically, and kinematically, determinate. The results from the paper can be easily extended to three dimensions. Here, the average valency of the structure must be six; the static condition requires that the structure can support six loads at infinity, and hence the equilibrium matrix must be rank-deficient by six; the kinematic condition, allowing for three rigid-body motions, requires that the compatibility matrix be rank-deficient by three. Again, both conditions cannot be satisfied by a square equilibrium/compatibility matrix.

The arguments in this paper have been developed using pin-jointed bar models, but the result would be equally valid for any other structural assumption, e.g. assuming that a structure consists of rigid-bodies pinned together, as in the calculation of ‘rigid-unit modes’ for determining displacive phase transitions in crystal structures (Giddy et al., 1993).

Real structures are, of course, never infinite, and must eventually reach a boundary. Correctly adding bars at the boundary would make it possible for the complete, finite, structure to be formally both kinematically and statically determinate, but in practice this is not likely to greatly affect the overall behaviour, as discussed in Deshpande et al. (2001).

7. Conclusion

The results from this paper are applicable to any large-scale, repetitive system. If any such structure is rigid, it must be redundant.

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